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A COLLARING THEOREM FOR CODIMENSION ONE MANIFOLDS

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ABSTRACT. The chief result implies that an n-manifold S embedded in the interior of an $(n+1)$-manifold M as a closed, separating subset is locally flatly embedded if the embedding is well behaved in a locally peripheral sense and if S has arbitrarily close neighborhoods Q such that the fundamental groups of appropriate components of $Q \setminus S$ admit a uniform finite upper bound on the number of generators.

Let M denote a compact $(n+1)$-manifold, possibly with boundary, and let $S \subset \operatorname{Int} M$ denote a closed n-manifold such that $M \setminus S$ has two components U and U'. The main theorem establishes that S has a collar in $S \cup U$ if the fundamental group of the end of U is finitely generated and S is locally peripherally collared from U. As a corollary, S has a collar in $S \cup U$ if S is locally flat except at the points of a Cantor set $C \subset S$ standardly embedded in S and if U is an open collar (that is, U is homeomorphic to the product of $(0, 1]$ with another closed n-manifold).

Interest in the issue emerged from our constructions [DT1, DT2] of $(n+1)$-manifolds M formed as a union of copies of $S \times [-1, 0]$ and $S' \times (0, 1]$, where S, S' are closed n-manifolds of different homotopy types and the level corresponding to $S \times 0$ is locally flat modulo a Cantor set. In these constructions one observes that the Cantor set turns out to be wildly embedded in $S \times 0$. Is wildness necessary? The result here gives an affirmative answer, for otherwise a bicollar on $S \times 0$ would force S and S' to be homotopically equivalent.

Influential in these developments was Kirby's prototype flatness result [K], which promises that an n-sphere S in $\mathbb{R}^n$ $(n > 3)$ is bicollared if it is locally flat modulo a Cantor set that is twice tame—tame both in $\mathbb{R}^n$ and in S. Another more direct influence was the paper of Burgess [B], which seems to be the first to combine conditions on local peripheral structure, defined in the next paragraph, with global conditions near the codimension one submanifold, to derive conclusions about local flatness, in the case of [B], for embeddings in 3-manifolds.

Let M be a connected $(n+1)$-manifold, possibly with boundary, $S \subset \operatorname{Int} M$

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an n-manifold embedded as a closed, separating subset of M, and U a component of $M\backslash S$. Then S is *locally peripherally collared from* U if, for each $\varepsilon > 0$ and each $s \in S$, there exists a neighborhood N_s of s in S with $\operatorname{diam}(N_s) < \varepsilon$ and ∂N_s a simply connected $(n-1)$-manifold that is collared from U. The latter means that there exists an embedding $\lambda\colon \partial N_s \times [0, 1] \to \operatorname{Cl}(U)$ with $\partial N_s = S \cap \operatorname{image}(\lambda) = \lambda(\partial N_s \times 0)$. (*Remark*: When $n = 2$, delete the impossible requirement that ∂N_s be 1-connected.)

Furthermore, S is said to have an *open collar neighborhood in* U if S has a neighborhood P such that $P \cap U$ is topologically equivalent to $S' \times (0, 1)$, for some closed n-manifold S'.

For completeness, here are definitions of the key older terms. A codimension k submanifold S in the interior of an m-manifold M is *locally flat at* $s \in S$ if s has a neighborhood U_s such that $(U_s, U_s \cap S)$ is homeomorphic as a pair to $(\mathbb{R}^m, \mathbb{R}^{m-k})$. A Cantor set $C \subset M$ is *tame* in M if there is a 1-manifold S such that $C \subset S \subset M$ and S is locally flat at every point $s \in S$.

Local Flatness Theorem. *Suppose M is a compact $(n+1)$-manifold with boundary, $n \geq 3$, S is a component of ∂M, $c\colon S \times [0, 1] \to S \cup \operatorname{Int} M$ is a collar on S, $U = M \setminus \operatorname{image}(c)$, and $c(S \times 1)$ is locally peripherally collared from U and has an open collar neighborhood in U. Then $c(S \times 1)$ is collared from U.*

Proof. Assume $c(S \times 1)$ has a neighborhood P such that there is a homeomorphism ψ of $S' \times (0, 1)$ onto $P \cap U$. We begin by claiming that $c(S \times 1)$ contains at most finitely many points at which U fails to be locally 1-connected. Fix $\varepsilon > 0$, and let k be the number of generators for $\pi_1(S')$. The claim will be proved by showing that, for any set of $k+1$ points $s_0, \dots, s_k$ in $c(S \times 1)$, at least one s_i has a neighborhood W_i such that all loops in $W_i \cap U$ are contractible in an ε-subset of U.

First, identify neighborhoods N_i of s_i in $c(S \times 1)$ fulfilling the definition of locally peripherally collared for this choice of ε, subject to the additional requirement that $N_i \cap N_j = \varnothing$ when $i \neq j$, and let $\lambda_i(\partial N_i \times [0, 1])$ denote the boundary collar. Build a neighborhood Q_1 of $c(S \times 1)$ in $\operatorname{Cl} U$ such that the components V_i of $Q_1 \setminus \bigcup(\lambda_s(\partial N_s \times [0, 1]))$ with $N_i \subset \operatorname{Cl} V_i$ $(i = 0, 1, \dots, k)$ satisfy:

(1) $\operatorname{diam} V_i < \varepsilon$,
(2) $V_i \cap V_j = \varnothing$ when $i \neq j$, and
(3) $Q_1 \cap \lambda_i(\partial N_i \times [0, 1]) \cap \operatorname{Cl}(V_i)$ is simply connected.

Second, produce a neighborhood $Q_2 \subset Q_1$ of $c(S \times 1)$ in $\operatorname{Cl} U$ such that $Q_2 \cap U = \psi(S' \times (0, r))$.

Third, construct a neighborhood $Q_3 \subset Q_2$ of $c(S \times 1)$ in $\operatorname{Cl} U$ such that the components W_i of $Q_3 \setminus \bigcup(\lambda_s(\partial N_s \times [0, 1]))$ with $N_i \subset \operatorname{Cl} W_i$ again satisfy:

(3′) $Q_3 \cap \lambda_i(\partial N_i \times [0, 1]) \cap \operatorname{Cl}(W_i)$ is simply connected.

Analogs of (1) and (2) also hold, simply because $W_i \subset V_i$ for $i = 0, 1, \dots, k$.

Now the crucial observation is that one of the inclusions $U \cap \operatorname{Cl} W_i \to U \cap \operatorname{Cl} V_i$ induces a trivial π_1-homomorphism. To check, apply the Seifert-van Kampen Theorem and (3′) to see that $\pi_1(U \cap Q_3)$ has a free product decomposition involving factors $\pi_1(\operatorname{Cl} W_i)$ and, similarly, $\pi_1(U \cap Q_1)$ has a free product decomposition involving factors $\pi_1(\operatorname{Cl} V_i)$; furthermore, the inclusion-induced homomorphism $\Theta\colon \pi_1(U \cap Q_3) \to \pi_1(U \cap Q_1)$ sends $\pi_1(\operatorname{Cl} W_i)$ into $\pi_1(\operatorname{Cl} V_i)$.

Since Θ factors through $\pi_1(S' \times (0, r))$, image(Θ) can be generated by a set with k elements, and Grushko's Theorem (cf. [M, pp. 224ff]) certifies that at least one of the images $\pi_1(U \cap \mathrm{Cl}\, V_i) \to \pi_1(U \cap \mathrm{Cl}\, W_i)$ is trivial.

Based on this verification of the initial claim, one has by [Ce, D2] for $n > 3$ and [Q] for $n = 3$ that $c(S \times 1)$ is locally flat except possibly at the points of some finite set. Finally, since $n + 1 \geq 4$, $c(S \times 1)$ has no isolated points of wildness [Ca] (or [K]), so it is locally flat everywhere.

The argument actually establishes the following:

Theorem. *Suppose M is an $(n+1)$-manifold with boundary, $n \geq 3$, S is a compact component of ∂M, $c\colon S \times [0, 1] \to S \cup \mathrm{Int}\, M$ is a collar on S, $U = M \setminus \mathrm{image}(c)$, and $c(S \times 1)$ is locally peripherally collared from U. Suppose there exists a positive integer k such that for every neighborhood Q of $c(S \times 1)$ there is another neighborhood $Q' \subset Q$ of $c(S \times 1)$ for which the image of the inclusion-induced $\pi_1(Q' \cap U) \to \pi_1(Q \cap U)$ is generated by k elements. Then $c(S \times 1)$ is collared from U.*

Corollary 1. *Suppose M is a connected $(n+1)$-manifold, $n \geq 3$, $S \subset \mathrm{Int}\, M$ is a compact n-manifold, $M \setminus S$ has two components U, U', and S has an open collar neighborhood in U. Suppose also S is locally flat except possibly at the points of some Cantor set standardly embedded in S. Then S is collared from U.*

Proof. According to arguments like those in [D3] or [D5], there exists an embedding $e\colon \mathrm{Cl}\, U \to M$ such that $e(S)$ is collared from $M \setminus e(U)$.

When $n = 2$ one encounters obvious exceptions to the Local Flatness Theorem. For instance, there is a classical example due to Fox and Artin [FA] of a wild 3-cell $C \subset \mathbb{R}^3$ such that $\mathbb{R}^3 \setminus C$ is homeomorphic to $S^2 \times (0, 1)$, yet ∂C is locally peripherally collared from $\mathbb{R}^3 \setminus C$, since ∂C is locally flat except at one point. Next we detail a sense in which all exceptions are of this type.

Three-Dimensional Local Flatness Theorem. *Suppose M is a compact 3-manifold with boundary and $S \subset M$ is a closed, connected 2-manifold such that $M \setminus S$ has components U, U', where S is collared from U'. Suppose S has an open collar neighborhood in U and S is locally peripherally collared from U. Then S is locally flat at each point, provided S is not a 2-sphere, in which case S is locally flat except possibly at one point.*

Proof. Restrict M, if necessary, so $\mathrm{Cl}\, U' = S \times (-1, 0]$ and $U \approx S' \times (0, 1)$. As then M is partitioned into mutually exclusive copies of S and S', these two manifolds have the same homology type [D4], which in this dimension implies they are homeomorphic. The conclusions follow from [B, D1].

We close by stating the application to crumpled laminations.

Corollary 2. *Suppose $p\colon M \to (-1, 1)$ is a proper map defined on a connected, orientable $(n+1)$-manifold M such that each $p^{-1}t$ is a closed, orientable n-manifold, with $p^{-1}t$ bicollared in M for all $t \neq 0$ and $p^{-1}0$ locally flat except possibly at the points of some Cantor set standardly embedded in $p^{-1}0$. Then all pairs of manifolds $p^{-1}s$, $p^{-1}t$ have the same homotopy type.*

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