PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 3, March 1994

A CHARACTERIZATION OF NORMAL EXTENSIONS FOR SUBFACTORS

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(Communicated by Palle E. T. Jorgensen)

Dedicated to Professor Tsuyosi Ando on his sixtieth birthday

ABSTRACT. Let $N \subset M \subset L$ be a tower of factors. If L is a crossed product $N \rtimes_\alpha G$ of N by an outer action α of a finite group G on N then it is well known that there exists a subgroup H of G such that $M = N \rtimes_{\alpha|_H} H$. We prove in this paper that H is a normal subgroup of G if and only if there exist a finite group F and an outer action β of F on M such that $L = M \rtimes_\beta F$.

INTRODUCTION

Let $N \subset M \subset L$ be a tower of finite factors. In [NT] Nakamura and Takeda studied a Galois theory for finite factors. If N is a fixed-point algebra L^G of L by an outer action α of a finite group G on L then there exists a subgroup H of G such that M is a fixed-point algebra L^H. In [T] Takeda proved that H is a normal subgroup of G if and only if N is a fixed point algebra M^F of M by an outer action β of some finite group F on M. We present in this paper a simple proof for arbitrary factors. To be more concrete, suppose that L is a crossed product $N \rtimes_\alpha G$ of a factor N by an outer action α of a finite group G on N and M is a crossed product $N \rtimes_{\alpha|_H} H$ of N by the induced outer action $\alpha|_H$ (by restriction of α) of a subgroup H of G on N. We prove in this case that H is a normal subgroup of G if and only if there exist a finite group F and an outer action β of F on M such that L is a crossed product $M \rtimes_\beta F$. Using Kosaki's characterization of crossed product [K] and examining the restriction-induction graph (for example, see [KY] or [O]), one can also obtain the result. But we give a simple proof without examining higher relative commutants of $M \subset L$.

THEOREM AND COROLLARIES

Theorem. *Let $N \subset M \subset L$ be a tower of factors. If L is a crossed product $N \rtimes_\alpha G$ of N by an outer action α of a finite group G on N then the subgroup H of G associated with M is a normal subgroup of G if and only if L is a*

Received by the editors June 16, 1992.
1991 *Mathematics Subject Classification.* Primary 46L37.
Key words and phrases. Factor, outer action, crossed product, Galois theory, fixed point algebra, conditional expectation.

crossed product $M \rtimes_\beta F$ of M by an outer action β of some finite group F on M. Furthermore, F is the quotient group G/H.

Proof. Suppose that H is a normal subgroup of G and $\{u_g\}_{g\in G}$ are canonical unitaries of a crossed product. It is known that a 2-cocycle is a coboundary. (See [J1] or [S].) So we can choose the section $\theta : F = G/H \to G$ such that $\{u_{\theta(\sigma)}\}_{\sigma\in F}$ is a unitary representation of F. Observing that

$$L = (M \cup \{u_{\theta(\sigma)}\}_{\sigma\in F})''$$

and $\beta_\sigma = Adu_\sigma$ is an outer action of F on M, we have L is a crossed product $M \rtimes_\beta F$ of M by an outer action β of F.

Conversely suppose that L is a crossed product $M \rtimes_\beta F$ of M by an outer action β of some finite group F on M. Then we have

$$M = (N \cup \{u_g | g \in H\})'', \qquad L = (N \cup \{u_g | g \in G\})'',$$

and

$$L = (M \cup \{z_\sigma | \sigma \in F\})''$$

where u_g and z_σ are canonical unitaries of crossed products. We set $E_M : L \to M$ as the unique conditional expectation such that

$$E_M(X) = x_e \quad \text{for } X = \sum_{\sigma\in F} x_\sigma z_\sigma \in L$$

where x_σ is a element of M and e is the unit element of F. Then we get

$$u_g = \sum_{\sigma\in F} E_M(u_g z_\sigma^*) z_\sigma ,$$

i.e., $\{z_\sigma | \sigma \in F\}$ are Pimsner-Popa type bases in L with respect to M (see [PP] or [W]). Therefore, there is $\sigma_0 \in F$ with $E_M(u_g z_{\sigma_0}^*) \neq 0$. Since, for any $x \in N$, $x u_g z_{\sigma_0}^* = u_g z_{\sigma_0}^* z_{\sigma_0} u_g^* x u_g z_{\sigma_0}^*$ and $z_{\sigma_0} u_g^* x u_g z_{\sigma_0}^* \in M$, we have

$$x E_M(u_g z_{\sigma_0}^*) = E_M(u_g z_{\sigma_0}^*) z_{\sigma_0} u_g^* x u_g z_{\sigma_0}^* \quad \text{for } x \in N$$

and

$$E_M(u_g z_{\sigma_0}^*) z_{\sigma_0} u_g^* \in N' \cap L = \mathbb{C};$$

hence, $u_g z_{\sigma_0}^* \in M$. Therefore,

$$u_g M u_g^* = u_g z_{\sigma_0}^* M z_{\sigma_0} u_g^* = M$$

for any $g \in G$. This means, for any $g \in G$ and any $h \in H$, $u_g u_h u_g^* = u_{ghg^{-1}} \in M$, so $ghg^{-1} \in H$. Thus H is a normal subgroup of G, and by the first part of this proof we get $F = G/H$. □

Using this theorem, we immediately obtain the next corollaries.

Corollary. *Let M be a factor and K a fixed-point algebra M^G by an outer action α of a finite group G on M. If H is a subgroup of G and N is a fixed-point algebra M^H then H is a normal subgroup of G if and only if there exist a finite group F such that K is a fixed-point algebra N^F of N by an outer action β of F on N. Furthermore, F is the quotient group G/H.*

This corollary is an extension of Takeda's theorem in [T] to arbitrary factors.

Proof. Suppose that M acts on a Hilbert space $\mathscr{H}$ in a standard way. By the duality for an inclusion of factors, $K' = M' \rtimes G$ and $N' = M' \rtimes H$. (For example, see [GHJ].) By the previous theorem, H is a normal subgroup of G if and only if there exists a finite group F such that $K' = N' \rtimes F$; i.e., K is a fixed-point algebra N^F. □

Corollary. *With the same notation as the previous theorem, if H is not a normal subgroup of G then for any factor K and any outer action β of a finite group F on K, neither $K \subset K \rtimes_\beta F$ nor $K^F \subset K$ is conjugate to $M \subset L$.*

Proof. By the previous theorem, it is obvious that $M \subset L$ and $K \rtimes_\beta F$ are not conjugate. Suppose that $M \subset L$ and $K^F \subset K$ are conjugate. Let N_1 and M_1 be Jones's basic constructions [J2] of L by N and M respectively. Since $N' \cap N_1$ is isomorphic to $l^\infty(G)$ and $N' \cap N_1 \supset M' \cap M_1$, $M' \cap M_1$ is commutative. However, $M' \cap M_1$ is isomorphic to a group von Neumann algebra $R(F)$. Therefore, F is abelian. So we get

$$K = K^F \rtimes_{\tilde{\beta}} F$$

where $\tilde{\beta}$ is a dual action of β. It is a contradiction to the theorem. □

Acknowledgment

I am greatly indebted to Professor Y. Watatani for his many useful suggestions and encouragement.

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