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AN OPERATOR-VALUED YEH-WIENER INTEGRAL AND A KAC-FEYNMAN WIENER INTEGRAL EQUATION

CHULL PARK AND DAVID SKOUG

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ABSTRACT. Let $C[0, T]$ denote Wiener space, i.e., the space of all continuous functions $\eta(t)$ on $[0, T]$ such that $\eta(0) = 0$. For $Q = [0, S] \times [0, T]$ let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all $\mathbb{R}$-valued continuous functions $x(s, t)$ on Q such that $x(0, t) = x(s, 0) = 0$ for all (s, t) in Q. For $h \in L_2(Q)$ let $Z(x; s, t)$ be the Gaussian process defined by the stochastic integral

$$Z(x; s, t) = \int_0^t \int_0^s h(u, v)\, dx(u, v).$$

Then for appropriate functionals F and ψ we show that the operator-valued function space integral

$$(I_\lambda^h(F)\psi)(\eta(\cdot)) = E_x[F(\lambda^{-1/2}Z(x; \cdot, \cdot) + \eta(\cdot))\psi(\lambda^{-1/2}Z(x; S, \cdot) + \eta(\cdot))]$$

is the unique solution of a Kac-Feynman Wiener integral equation. We also use this integral equation to evaluate various Yeh-Wiener integrals.

1. INTRODUCTION

For $Q = [0, S]\times[0, T]$ let $C(Q)$ denote Yeh-Wiener space. Yeh [12] defined a Gaussian measure m_y on $C(Q)$ (later modified in [13]) such that as a stochastic process $\{x(s, t): (s, t) \in Q\}$ has mean $E[x(s, t)] = \int_{C(Q)} x(s, t)m_y(dx) = 0$ and covariance $E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$. Let $C_w \equiv C[0, T]$ denote the standard Wiener space on $[0, T]$ with Wiener measure m_w. For $0 \le s < S$, let $Q_s = [s, S] \times [0, T]$. The corresponding Yeh-Wiener space is denoted by $C(Q_s)$ and we use m_y for the Yeh-Wiener measure on $C(Q)$ as well as on $C(Q_s)$. The same convention holds in the one time-parameter case. We also need $C^*(Q)$, the space of $\mathbb{R}$-valued continuous functions $x(s, t)$ on Q such that $x(\cdot, 0) = 0$.

Consider the operator-valued integral in function space $I_\lambda(F) \equiv I_{\lambda,[0,T]}(F)$

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which takes the function ψ into the function $I_\lambda(F)\psi$ whose value at $\xi \in \mathbb{R}$ is

(1.1)
$$\begin{aligned}(I_\lambda(F)\psi)(\xi) &\equiv \int_{C[0,T]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(T)+\xi)m_w(dx) \\ &\equiv E_x[F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(T)+\xi)] \qquad (\lambda > 0).\end{aligned}$$

A very important class of functions in quantum mechanics consists of functions on $C[0, T]$ of the type

$$F(x) = \exp\left\{\int_0^T \theta(s, x(s))\,ds\right\},$$

where $\theta\colon [0, T]\times\mathbb{R} \to \mathbb{C}$.

In fact, it is well known that if $\theta(s, u)$ is bounded and continuous on $[0, T]\times\mathbb{R}$, then the operator $I_\lambda(F)$ transforms $L_2[0, T]$ into $L_2[0, T]$. Furthermore, if we define

$$F_t(x) \equiv \exp\left\{\int_t^T \theta(s, x(s))\,ds\right\} \qquad (0 \le t < T),$$

then the function

$$G(t, \xi, \lambda) = (I_{\lambda,[t,T]}(F_t)\psi)(\xi)$$

satisfies the Kac-Feynman integral equation

$$\begin{aligned}&G(t, \xi, \lambda) \\ &\quad= \left[\frac{\lambda}{2\pi(T-t)}\right]^{1/2} \int_{\mathbb{R}} \psi(v)\exp\left\{\frac{-\lambda(v-\xi)^2}{2(T-t)}\right\} dv \\ &\qquad+ \left(\frac{\lambda}{2\pi}\right)^{1/2} \int_t^T (s-t)^{-1/2} \int_{\mathbb{R}} \theta(s, v)G(s, v, \lambda)\exp\left\{\frac{-\lambda(v-\xi)^2}{2(s-t)}\right\} dv\,ds.\end{aligned}$$

If θ is sufficiently smooth, then this equation implies the backwards diffusion equation

$$\frac{1}{2\lambda}\frac{\partial^2 G}{\partial\xi^2} + \frac{\partial G}{\partial t} + \theta G = 0$$

with the boundary condition

$$\lim_{t\to T^-} G(t, \xi, \lambda) = \psi(\xi).$$

The celebrated paper [4] by Cameron and Storvick considered the operator-valued function space integral $I_\lambda(F) \equiv I_{\lambda,Q}(F)$ which maps the functional ψ into the functional $I_\lambda(F)\psi$ whose value at $\eta(\cdot) \in C[0, T]$ is given by

(1.2)
$$\begin{aligned}(I_\lambda(F)\psi)(\eta(\cdot)) &\equiv \int_{C(Q)} F(\lambda^{-1/2}x(\cdot,\cdot)+\eta(\cdot))\psi(\lambda^{-1/2}x(S,\cdot)+\eta(\cdot))m_y(dx) \\ &\equiv E_x[F(\lambda^{-1/2}x(\cdot,\cdot)+\eta(\cdot))\psi(\lambda^{-1/2}x(S,\cdot)+\eta(\cdot))] \qquad (\lambda > 0).\end{aligned}$$

In particular, if

(1.3)
$$F_s(x(\cdot,\cdot)) \equiv \exp\left\{\int_s^S \theta(\sigma, x(\sigma,\cdot))\,d\sigma\right\},$$

then

$$(1.4)\quad \begin{aligned}&(I_{\lambda,Q_s}(F_s)\psi)(\eta(\cdot))\\ &\equiv \int_{C(Q_s)} F_s(\lambda^{-1/2}x(\cdot\,,\cdot)+\eta(\cdot))\psi(\lambda^{-1/2}x(S\,,\cdot)+\eta(\cdot))m_y(dx)\\ &\equiv E_x[F_s(\lambda^{-1/2}x(\cdot\,,\cdot)+\eta(\cdot))\psi(\lambda^{-1/2}x(S\,,\cdot)+\eta(\cdot))|x(s\,,\cdot)=0].\end{aligned}$$

One of the main results of Cameron and Storvick in [4] can be stated as follows:

Theorem A (Cameron and Storvick). *Let $\theta(\sigma\,, x(\sigma\,,\cdot))$ be bounded and continuous on $[0\,,S]\times[0\,,T]$, and let $\psi(\gamma w(\cdot)+\eta(\cdot))$ be Wiener integrable in w for each $\gamma>0$ and each $\eta\in C[0\,,T]$. For $\lambda>0$ and $\rho=\lambda^{-1/2}$, let $F_s(x)$ and $G(s\,,\eta(\cdot))\equiv(I_{\lambda,Q_s}(F_s)\psi)(\eta(\cdot))$ be defined by (1.3) and (1.4), respectively. Then $G(s\,,\eta(\cdot))$ satisfies the Kac-Feynman Wiener integral equation*

(1.5)

$$\begin{aligned}G(s\,,\eta(\cdot)) &= E_w[\psi(\rho\sqrt{S-s}\,w(\cdot)+\eta(\cdot))]\\ &\quad+\int_s^S E_w[\theta(\sigma\,,\rho\sqrt{\sigma-s}\,w(\cdot)+\eta(\cdot))G(\sigma\,,\rho\sqrt{\sigma-s}\,w(\cdot)+\eta(\cdot))]\,d\sigma.\end{aligned}$$

Remark. Equation (1.5) is slightly different from the equation given by Cameron and Storvick. The reason is that while they used the original "old" Yeh-Wiener process with covariance $E[x(s\,,t)x(u\,,v)]=\frac{1}{2}\min\{s\,,u\}\min\{t\,,v\}$, we used the standard Yeh-Wiener process given earlier in the introduction.

In this paper we generalize and extend considerably the results in [4]. In particular we replace the Yeh-Wiener process $x(s\,,t)$ with the Gaussian process $Z(x\,;s\,,t)$. Moreover our proofs are much simpler because we employ probabilistic techniques extensively.

2. The operator-valued integral $I_\lambda^h(F)$

For each $h\in L_2(Q)$ and $\lambda>0$, consider the operator-valued integral $I_\lambda^h(F)\equiv I_{\lambda,Q}^h(F)$ which transforms the functional ψ into the functional $I_\lambda^h(F)\psi$ whose value at $\eta(\cdot)\in C[0\,,T]$ is given by

$$(2.1)\quad\begin{aligned}&(I_\lambda^h(F)\psi)(\eta(\cdot))\\ &\equiv\int_{C(Q)}F\left(\lambda^{-1/2}\int_0^{\cdot}\int_0^{\cdot}h(u\,,v)\,dx(u\,,v)+\eta(\cdot)\right)\\ &\qquad\cdot\psi\left(\lambda^{-1/2}\int_0^{\cdot}\int_0^{S}h(u\,,v)\,dx(u\,,v)+\eta(\cdot)\right)m_y(dx)\\ &\equiv E_x\left[F\left(\lambda^{-1/2}\int_0^{\cdot}\int_0^{\cdot}h(u\,,v)\,dx(u\,,v)+\eta(\cdot)\right)\right.\\ &\qquad\left.\cdot\psi\left(\lambda^{-1/2}\int_0^{\cdot}\int_0^{S}h(u\,,v)\,dx(u\,,v)+\eta(\cdot)\right)\right].\end{aligned}$$

Since $\int_0^* \int_0^\cdot 1\, dx(u, v) = x(\cdot, *)$, the Cameron-Storvick operator-valued integral $I_\lambda(F)$ given by (1.2) is a special case of $I_\lambda^h(F)$ with $h \equiv 1$ on Q, i.e., $I_\lambda(F) = I_\lambda^1(F)$. For notational simplicity, we define

$$Z(x; s, t) \equiv Z_h(x; s, t) = \int_0^t \int_0^s h(u, v)\, dx(u, v), \qquad x \in C(Q), (s, t) \in Q.$$

Thus, $Z(x; \cdot, \cdot)$ is a stochastic integral which is a Gaussian process with mean 0 and covariance

$$E[Z(x; s, t) Z(x; s', t')] = \int_0^{t\hat{}t'} \int_0^{s\hat{}s'} h^2(u, v)\, du\, dv,$$

where $t\hat{}t' = \min\{t, t'\}$. Since the covariance function of $Z(x; \cdot, \cdot)$ is continuous, we may assume that almost every sample path of $Z(x; \cdot, \cdot)$ is in $C(Q)$ [7, p. 157]. Obviously $Z(x; \cdot, \cdot)$ has independent increments, and in terms of Z, (2.1) becomes

$$\text{(2.2)} \quad (I_\lambda^h(F)\psi)(\eta(\cdot)) = E_x[F(\lambda^{-1/2} Z(x; \cdot, \cdot) + \eta(\cdot))\psi(\lambda^{-1/2} Z(x; S, \cdot) + \eta(\cdot))].$$

Let $\mathscr{A}(C[0, T])$ be the class of functionals ψ on $C[0, T]$ such that $\psi(\int_0^\cdot k(u)\, dw(u) + \eta(\cdot))$ is Wiener integrable in w over $C[0, T]$ for every $k \in L_2[0, T]$ and $\eta \in C[0, T]$. Our first objective is to show that under suitable assumptions on F, and for $\lambda > 0$,

$$I_\lambda^h(F): \mathscr{A}(C[0, T]) \to \mathscr{A}(C[0, T]).$$

As usual, two Gaussian processes are said to be equivalent if they have identical mean and covariance functions. We start with the following lemma.

Lemma 1. *Let $h \in L_2(Q)$ and $k \in L_2[0, T]$. Let x be the Yeh-Wiener process on Q, and let w be the standard Wiener process on $[0, T]$. If x and w are independent, then for each $\rho > 0$, the two processes $\rho Z(x; S, \cdot) + \int_0^\cdot k(u)\, dw(u)$ and $\int_0^\cdot p(v)\, dw_1(v)$ are equivalent, where w_1 is the standard Wiener process on $[0, T]$ and $p(v) = [\rho^2 \int_0^S h^2(u, v)\, du + k^2(v)]^{1/2}$.*

Proof. The two processes are Gaussian processes on $[0, T]$ with mean zero and identical covariance. Thus, they are equivalent Gaussian processes.

Theorem 1. *Let F be a bounded functional, continuous in the uniform topology on $C^*(Q)$. Then for each $\lambda > 0$ and $h \in L_2(Q)$, $I_\lambda^h(F)$ maps $\mathscr{A}(C[0, T])$ into $\mathscr{A}(C[0, T])$.*

Proof. Let $\psi \in \mathscr{A}(C[0, T])$ and $|F| \le M$. We need to show that $(I_\lambda^h(F)\psi) \cdot (\int_0^\cdot k(u)\, dw(u) + \eta(\cdot))$ is Wiener integrable in w over $C[0, T]$ for every $k \in$

$L_2[0, T]$ and $\eta \in C[0, T]$. Now, with $\rho = \lambda^{-1/2}$,

$$\begin{aligned}
E_w &\left[\left\|(I_\lambda^h(F)\psi)\left(\int_0^\cdot k(u)\,dw(u) + \eta(\cdot)\right)\right\|\right] \\
&\le E_w\left(E_x\left[\left\|F\left(\rho Z(x;\cdot,\cdot) + \int_0^\cdot k(u)\,dw(u) + \eta(\cdot)\right)\right.\right.\right. \\
&\qquad \left.\left.\left.\cdot\,\psi\left(\rho Z(x;S,\cdot) + \int_0^\cdot k(u)\,dw(u) + \eta(\cdot)\right)\right\|\right]\right) \\
&\le M E_w\left(E_x\left[\left\|\psi\left(\rho Z(x;S,\cdot) + \int_0^\cdot k(u)\,dw(u) + \eta(\cdot)\right)\right\|\right]\right) \\
&= M E_{w_1}\left[\left\|\psi\left(\int_0^\cdot\left(\rho^2\int_0^S h^2(u,v)\,du + k^2(v)\right)^{1/2} dw_1(v) + \eta(\cdot)\right)\right\|\right],
\end{aligned}$$

where the last equality follows from Lemma 1. Since $\psi \in \mathscr{A}(C[0, T])$, the last expression is finite, and hence the proof is complete.

Next we compute $I_\lambda^h(F)\psi$ for a given function F which is continuous but is not bounded.

Example 1. Let $(S_1, T_1) \in (0, S) \times (0, T]$ and let $F(x) = \exp\{x(S_1, T_1)\}$. Then for $\psi \in \mathscr{A}(C[0, T])$, $h \in L_2(Q)$, and $\lambda > 0$ we obtain that

$$\begin{aligned}
(I_\lambda^h(F)\psi)(\eta(\cdot)) &= \int_{C(Q)} \exp\left\{\lambda^{-1/2}\int_0^{T_1}\int_0^{S_1} h\,dx + \eta(T_1)\right\} \\
&\qquad \cdot\,\psi\left(\lambda^{-1/2}\int_0^{T_1}\int_0^{S} h\,dx + \eta(T_1)\right) m_y(dx) \\
&= (2\pi)^{-1}\int_{\mathbf{R}^2} \exp\left\{\lambda^{-1/2}\left(\int_0^{T_1}\int_0^{S_1} h^2\right)^{1/2} u_1 + \eta(T_1) - \frac{u_1^2}{2} - \frac{u_2^2}{2}\right\} \\
&\quad \cdot\,\psi\left(\lambda^{-1/2}\left(\int_0^{T_1}\int_0^{S_1} h^2\right)^{1/2} u_1\right. \\
&\qquad\qquad \left. + \lambda^{-1/2}\left(\int_0^{T_1}\int_{S_1}^{S} h^2\right)^{1/2} u_2 + \eta(T_1)\right) du_2\,du_1 \\
&= \left(\frac{\lambda}{2\pi\int_0^{T_1}\int_0^{S_1} h^2}\right)^{1/2}\left(\frac{\lambda}{2\pi\int_0^{T_1}\int_{S_1}^{S} h^2}\right)^{1/2} \\
&\quad \cdot\int_{\mathbf{R}^2} \psi(v_2)\exp\left\{v_1 - \frac{\lambda(v_1 - \eta(T_1))^2}{2\int_0^{T_1}\int_0^{S_1} h^2} - \frac{\lambda(v_2 - v_1)^2}{2\int_0^{T_1}\int_{S_1}^{S} h^2}\right\} dv_2\,dv_1.
\end{aligned}$$

Example 2. Choosing $F(x) = \exp\{x(S, T_1)\}$ and proceeding as above we obtain that

$$(I_\lambda^h(F)\psi)(\eta(\cdot)) = \left(\frac{\lambda}{2\pi \int_0^{T_1}\int_0^S h^2}\right)^{1/2} \int_{\mathbb{R}} \psi(v) \exp\left\{v - \frac{\lambda(v-\eta(T_1))^2}{2\int_0^{T_1}\int_0^S h^2}\right\} dv.$$

3. The operator $I_{\lambda, Q_s}^h(F_s)$ and an integral equation

Let $Z(x; s, t)$ be as before, and let F_s be given by (1.3) so that

$$F_s(Z(x)) \equiv F_s(Z(x; *, \cdot)) = \exp\left\{\int_s^S \theta(\sigma, Z(x; \sigma, \cdot))\, d\sigma\right\} \tag{3.1}$$

$(0 \le s < S)$. Define the operator $I_{\lambda, Q_s}^h(F_s)$ with $\rho = \lambda^{-1/2}$, $\lambda > 0$, by

$$\begin{aligned}(I_{\lambda, Q_s}^h(F_s)\psi)(\eta(\cdot)) \\ \equiv E[F_s(\rho Z(x; *, \cdot) + \eta(\cdot))\psi(\rho Z(x; S, \cdot) + \eta(\cdot)) | Z(x; s, \cdot) = 0].\end{aligned} \tag{3.2}$$

We start with the following lemma whose proof is immediate by checking the mean and covariance functions.

Lemma 2. *Let $h \in L_2(Q)$ and let x be a Yeh-Wiener process on Q. Let $0 \le s \le \sigma \le S$. Then $Z(x; \sigma, \cdot) - Z(x; s, \cdot)$ and $\int_0^{\cdot}(\int_s^\sigma h^2(u, v)\, du)^{1/2} dw(v)$ are equivalent processes, where w is a standard Wiener process on $[0, T]$.*

We are now ready to establish the following theorem.

Theorem 2. *Let $\theta(s, w(\cdot))$ be a bounded function on $[0, S] \times C[0, T]$, and continuous in $(s, w(\cdot))$ in the uniform topology for w. Let $\psi \in \mathscr{A}(C[0, T])$, $\lambda > 0$, and $\rho = \lambda^{-1/2}$. Let $F_s(Z(x))$ and $I_{\lambda, Q_s}^h(F_s)\psi$ be defined by (3.1) and (3.2), respectively. Then, for each $\eta \in C[0, T]$, the function*

$$G(s, \eta(\cdot)) \equiv (I_{\lambda, Q_s}^h(F_s)\psi)(\eta(\cdot)) \tag{3.3}$$

satisfies the Kac-Feynman Wiener integral equation

(3.4)

$$\begin{aligned}G(s, \eta(\cdot)) = E_w\left[\psi\left(\rho\int_0^{\cdot}\left(\int_s^S h^2(u, v)\, du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right] \\ + \int_s^S E_w\left[(\theta \cdot G)\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^\sigma h^2(u, v)\, du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right] d\sigma,\end{aligned}$$

where $(\theta \cdot G)(u, v) = \theta(u, v)G(u, v)$.

Proof. Since $F_s(Z(x)) = \exp\{\int_s^S \theta(\sigma, Z(x; \sigma, \cdot))\, d\sigma\}$, it follows that

$$\frac{d}{ds}F_s(\rho Z(x) + \eta) = -\theta(s, \rho Z(x; s, \cdot) + \eta(\cdot))F_s(\rho Z(x) + \eta).$$

Integrate over $[s, S]$ to obtain

$$1 - F_s(\rho Z(x) + \eta) = -\int_s^S \theta(\sigma, \rho Z(x; \sigma, \cdot) + \eta(\cdot))F_\sigma(\rho Z(x) + \eta)\, d\sigma.$$

Multiply each member by $\psi(\rho Z(x; S, \cdot) + \eta(\cdot))$ and take the conditional expectation with conditioning $Z(x; s, 0) = 0$ to obtain

$$\begin{aligned}
&E_x[\psi(\rho Z(x; S, \cdot) + \eta(\cdot))|Z(x; s, \cdot) = 0] - G(s, \eta(\cdot)) \\
&\quad = -\int_s^S E_x[\theta(\sigma, \rho Z(x; \sigma, \cdot) + \eta(\cdot))F_\sigma(\rho Z(x) + \eta) \\
&\qquad\qquad \cdot \psi(\rho Z(x; S, \cdot) + \eta(\cdot))|Z(x; s, \cdot) = 0]\, d\sigma.
\end{aligned} \tag{3.5}$$

Noting that for $0 \le s \le \sigma \le S$, $Z(x; \sigma, \cdot) - Z(x; s, \cdot)$ and $Z(x; s, \cdot)$ are independent processes, it follows from (3.5) that

(3.6)

$$\begin{aligned}
&E_x[\psi(\rho\{Z(x; S, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot))] - G(s, \eta(\cdot)) \\
&\quad = -\int_s^S E_x[\theta(\sigma, \rho\{Z(x; \sigma, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot)) \\
&\qquad\qquad \cdot F_\sigma(\rho\{F(x; *, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot)) \\
&\qquad\qquad \cdot \psi(\rho\{Z(x; S, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot))]\, d\sigma \\
&\quad = -\int_s^S E_x[\theta(\sigma, \rho\{Z(x; \sigma, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot)) \\
&\qquad \cdot F_\sigma(\rho\{Z(x; *, \cdot) - Z(x; \sigma, \cdot) + Z(x; \sigma, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot)) \\
&\qquad \cdot \psi(\rho\{Z(x; S, \cdot) - Z(x; \sigma, \cdot) + Z(x; \sigma, \cdot) - Z(x; s, \cdot)\} + \eta(\cdot))]\, d\sigma.
\end{aligned}$$

Checking the definition of $F_s(Z(x; *, \cdot))$ given by (3.1), we see that the value of '$*$' ranges between s and S. Accordingly, the value of $*$ in the expression $F_\sigma(\rho Z(x; *, \cdot) + \eta(\cdot))$ ranges between σ and S. Thus, $Z(x; *, \cdot) - Z(x; \sigma, \cdot)$ and $Z(x; \sigma, \cdot) - Z(x; s, \cdot)$ appearing in the last expression of (3.6) are independent. Upon applying Lemma 2, we may rewrite (3.6) in the form

(3.7)

$$\begin{aligned}
&E_w\left[\psi\left(\rho\int_0^\cdot\left(\int_s^S h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right] - G(s, \eta(\cdot)) \\
&\quad = -\int_s^S E_x\Bigg(E_w\Bigg[\theta\left(\sigma, \rho\int_0^\cdot\left(\int_s^\sigma h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right) \\
&\qquad\qquad \cdot F_\sigma\Bigg(\rho\{Z(x; *, \cdot) - Z(x; \sigma, \cdot)\} \\
&\qquad\qquad\qquad + \rho\int_0^\cdot\left(\int_s^\sigma h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\Bigg) \\
&\qquad\qquad \cdot \psi\Bigg(\rho\{Z(x; S, \cdot) - Z(x; \sigma, \cdot)\} \\
&\qquad\qquad\qquad + \rho\int_0^\cdot\left(\int_s^\sigma h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\Bigg)\Bigg]\Bigg)\, d\sigma.
\end{aligned}$$

Using the Fubini theorem to switch the order of the inner integrations on the right-hand side of (3.7), and then recognizing the resulting expression as a conditional expectation, we may write (3.7) in the form

(3.8)

$$E_w\left[\psi\left(\rho\int_0^{\cdot}\left(\int_s^S h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right]-G(s,\eta(\cdot))$$
$$=-\int_s^S E_w\left\{\theta\left(\sigma,\rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right.$$
$$\cdot E_x\left[F_\sigma\left(\rho Z(x;*,\cdot)+\rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right.$$
$$\cdot\psi\left(\rho Z(x;S,\cdot)+\rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)$$
$$\left.\left.|Z(x;\sigma,\cdot)=0\right]\right\}d\sigma.$$

Using (3.2) and (3.3), (3.8) becomes

$$E_w\left[\psi\left(\rho\int_0^{\cdot}\left(\int_s^S h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right]-G(s,\eta(\cdot))$$
$$=-\int_s^S E_w\left[\theta\left(\sigma,\rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right.$$
$$\left.\cdot G\left(\sigma,\rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right]d\sigma,$$

and hence the proof is complete.

We now give two corollaries to Theorem 2.

Corollary 1. *If $h\equiv 1$ on Q, then* (3.4) *becomes*

$$G(s,\eta(\cdot))=E_w[\psi(\rho\sqrt{S-s}\,w(\cdot)+\eta(\cdot))]$$
$$+\int_s^S E_w[(\theta\cdot G)(\sigma,\rho\sqrt{\sigma-s}\,w(\cdot)+\eta(\cdot))]\,d\sigma,$$

which agrees with Theorem A given by Cameron and Storvick.

Corollary 2. *Let $\phi(\sigma,\tau,y(\cdot))$ be bounded and continuous on $Q\times C[0,T]$ (in the uniform topology for y). Let $\xi(u)$ be defined on $\mathbb{R}$ such that $\xi(u)\exp(-\gamma u^2)$ is integrable on $\mathbb{R}$ for every $\gamma>0$. Let G be given by*

$$G(s,\eta(\cdot))=E_x\left[\exp\left\{\int_0^T\int_s^S\phi(\sigma,\tau,\rho Z(x;\sigma,\tau)+\eta(\tau))\,d\sigma\,d\tau\right\}\right.$$
$$\left.\cdot\xi(\rho Z(x;S,T)+\eta(T))|x(s,\cdot)=0\right].$$

Then G *satisfies the following integral equation*:

$$G(s, \eta(\cdot)) = E_w\left[\xi\left(\rho\int_0^T\left(\int_s^S h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(T)\right)\right]$$
$$+ E_w\left[\int_s^S\int_0^T \phi\left(\sigma, \tau, \rho\int_0^\tau\left(\int_s^\sigma h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\tau)\right) d\tau\right.$$
$$\left. d\cdot G\left(\sigma, \rho\int_0^\cdot\left(\int_s^\sigma h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right) d\sigma\right].$$

Proof. This corollary follows from Theorem 2 by setting $\sigma(s, y) = \int_0^T \phi(s, v, y(v))\,dv$ and $\psi(y(\cdot)) = \xi(y(T))$.

4. Solution of the integral equation

In §3 it was shown that for each $\eta \in C[0, T]$ and $\psi \in \mathscr{A}(C[0, T])$, the function

$$G(s, \eta(\cdot)) \equiv (I^h_{\lambda, Q_s}(F_s)\psi)(\eta(\cdot))$$

satisfies the Kac-Feynman Wiener integral equation (3.4) under a suitable assumption on θ. We show here that the integral equation has a unique solution.

Our first lemma follows easily by checking the mean and covariance functions.

Lemma 3. *Let* $k_1, k_2 \in L_2[0, T]$, *and let* w *and* w_1 *be two independent standard Wiener processes on* $[0, T]$. *Then* $\int_0^\cdot k_1(u)\,dw(u) + \int_0^\cdot k_2(u)\,dw_1(u)$ *and* $\int_0^\cdot \sqrt{k_1^2(u) + k_2^2(u)}\,dw(u)$ *are equivalent processes on* $[0, T]$.

Lemma 4. *Let* $h \in L_2(Q)$, $\psi \in \mathscr{A}(C[0, T])$, *and let*

$$A(s, \eta(\cdot)) \equiv E_w\left[\left\|\psi\left(\rho\int_0^\cdot\left(\int_s^S h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right\|\right] \tag{4.1}$$

for $\eta \in C[0, T]$. *Then*

$$E_w\left[A\left(\sigma, \rho\int_0^\cdot\left(\int_s^S h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right] = A(s, \eta(\cdot)). \tag{4.2}$$

Proof. Using the definition of $A(\cdot, \cdot)$, we may write the left-hand side of (4.2) as

$$I \equiv E_w\left(E_{w_1}\left[\left\|\psi\left(\rho\int_0^\cdot\left(\int_\sigma^S h^2(u, v)\,du\right)^{1/2} dw_1(v)\right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \rho\int_0^\cdot\left(\int_s^\sigma h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right\|\right]\right).$$

An application of Lemma 3 to the last expression yields

$$I \equiv E_w\left[\left\|\psi\left(\rho\int_0^\cdot\left(\int_s^S h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right\|\right],$$

which completes the proof of the lemma.

Theorem 3. *Let $\theta(s, y(\cdot))$ be bounded by M on $[0, S] \times C[0, T]$ and be continuous there (in the uniform topology for y), let $\psi \in \mathscr{A}(C[0, T])$, and let $\lambda > 0$ with $\rho = \lambda^{-1/2}$. Then for $\eta \in C[0, T]$, the solution of the Kac-Feynman Wiener integral equation (3.4) is given by*

$$G(s, \eta(\cdot)) = \sum_{j=0}^{\infty} H_j(s, \eta(\cdot)), \tag{4.3}$$

where the sequence $\{H_j\}$ is given inductively by

$$H_0(s, \eta(\cdot)) = E_w\left[\psi\left(\rho\int_0^{\cdot}\left(\int_s^S h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right], \tag{4.4}$$

and

$$\begin{aligned} &H_{j+1}(s, \eta(\cdot)) \\ &\quad= \int_s^S E_w\left[(\theta\cdot H_j)\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u, v)\,du\right)^{1/2} dw(v) + \eta(\cdot)\right)\right] d\sigma. \end{aligned} \tag{4.5}$$

The solution $G(s, \eta(\cdot))$ satisfies the inequality

$$|G(s, \eta(\cdot))| \le A(s, \eta(\cdot))\exp\{MS\}, \tag{4.6}$$

and (4.3) is the only solution satisfying (4.6).

Proof. First, we establish that

$$\begin{aligned} |H_j(s, \eta(\cdot))| &\le \frac{[M(S-s)]^j}{j!}A(s, \eta(\cdot)) \\ &\le \frac{(MS)^j}{j!}A(s, \eta(\cdot)) \quad \text{for } j = 0, 1, \ldots. \end{aligned} \tag{4.7}$$

The case $j = 0$ follows from (4.1). Assume (4.7) holds for some j. Then from (4.5) it follows that

$$\begin{aligned} |H_{j+1}(s, \eta(\cdot))| \le \frac{M^{j+1}}{j!}\int_s^S (\sigma - s)^j E_w\bigg[A\bigg(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u, v)\,du\right)^{1/2} \\ \cdot dw(v) + \eta(\cdot)\bigg)\bigg]\, d\sigma. \end{aligned}$$

Next applying (4.2), we obtain

$$|H_{j+1}(s, \eta(\cdot))| \le \frac{M^{j+1}}{j!}A(s, \eta(\cdot))\int_s^S (\sigma - s)^j\, d\sigma = \frac{M^{j+1}(S-s)^{j+1}}{(j+1)!}A(s, \eta(\cdot)).$$

Thus,

$$\sum_{j=0}^{\infty} |H_j(s, \eta(\cdot))| \le A(s, \eta(\cdot))\exp\{MS\}, \tag{4.8}$$

and hence, by the dominated convergence theorem,

$$\begin{aligned}\int_s^S E_w\Bigg[\theta\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\\ \cdot\sum_{j=0}^{\infty} H_j\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\Bigg]\,d\sigma\\ =\sum_{j=0}^{\infty}\int_s^S E_w\Bigg[\theta\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\\ \cdot H_j\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\Bigg]\,d\sigma\\ =\sum_{j=0}^{\infty} H_{j+1}(s, \eta(\cdot)) = \sum_{j=0}^{\infty} H_j(s, \eta(\cdot)) - H_0(s, \eta(\cdot)).\end{aligned}$$

Therefore,

$$G(s, \eta(\cdot)) \equiv \sum_{j=0}^{\infty} H_j(s, \eta(\cdot))$$

is a solution of (3.4). Furthermore, using (4.8), $G(s, \eta(\cdot))$ satisfies the inequality (4.6).

To show that (4.3) is the only such solution, assume that $G_1(s, \eta(\cdot))$ is another such solution. Then $H \equiv G - G_1$ is a solution of the equation

$$H(s, \eta(\cdot)) = \int_s^S E_w\left[(\theta\cdot H)\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} \cdot dw(v)+\eta(\cdot)\right)\right]d\sigma, \tag{4.9}$$

satisfying the condition

$$|H(s, \eta(\cdot))| \le 2A(s, \eta(\cdot))\exp\{MS\}. \tag{4.10}$$

The use of (4.10) in (4.9) together with the fact that $|\theta(\cdot, \cdot)| \le M$ yields

(4.11)

$$\begin{aligned}|H(s, \eta(\cdot))| &\le 2M\exp\{MS\}\int_s^S E_w\left[A\left(\sigma, \rho\int_0^{\cdot}\left(\int_s^{\sigma} h^2(u,v)\,du\right)^{1/2} \cdot dw(v)+\eta(\cdot)\right)\right]d\sigma\\ &= 2M\exp\{MS\}A(s, \eta(\cdot))(S-s),\end{aligned}$$

where the last equality follows from (4.2). Next we use (4.11) in (4.9) and obtain that

$$|H(s, \eta(\cdot))| \le 2M^2\exp\{MS\}A(s, \eta(\cdot))\frac{(S-s)^2}{2}.$$

Repeating this procedure, it follows by induction that

$$|H(s, \eta(\cdot))| \le \exp\{MS\}A(s, \eta(\cdot))\frac{[M(S-s)]^n}{n!}, \qquad n = 1, 2, 3, \ldots.$$

Thus $H(s, \eta(\cdot)) = 0$, and hence the uniqueness of the solution is established.

The following corollary to Theorem 3 follows from the fact that

$$\begin{aligned}(I^h_{\lambda, Q_s}(F_s)\psi)(\eta(\cdot)) &= E_x\left[\exp\left\{\int_s^S \theta(\sigma, \rho Z(x;\sigma,\cdot)+\eta(\cdot))\,d\sigma\right\}\right.\\ &\qquad\left.\cdot\,\psi(\rho Z(x;S,\cdot)+\eta(\cdot))|Z(x;s,\cdot)=0\right]\\ &= G(s, \eta(\cdot)).\end{aligned}$$

Corollary 3. *Under the same hypotheses as in Theorem* 3,

$$\begin{aligned}&E_x\left[\exp\left\{\int_s^S \theta(\sigma, \rho Z(x;\sigma,\cdot)+\eta(\cdot))\,d\sigma\right\}\right.\\ \text{(4.12)}\qquad &\qquad\left.\cdot\,\psi(\rho Z(x;S,\cdot)+\eta(\cdot))|Z(x;s,\cdot)=0\right]\\ &= \sum_{n=0}^{\infty} H_n(s, \eta(\cdot)),\end{aligned}$$

where the sequence $\{H_n\}$ *is the same as in Theorem* 3. *In particular, setting* $s = 0$ *we have*

(4.13)

$$\begin{aligned}&E_x\left[\exp\left\{\int_0^S \theta(\sigma, \rho Z(x;\sigma,\cdot)+\eta(\cdot))\,d\sigma\right\}\psi(\rho Z(x;S,\cdot)+\eta(\cdot))\right]\\ &= E_w\left[\psi\left(\rho\int_0^{\cdot}\left(\int_0^S h^2(u,v)\,du\right)^{1/2} dw(v)+\eta(\cdot)\right)\right]\\ &+\sum_{n=1}^{\infty}\int_{\Delta_n(S)} E_{w_1,\dots,w_{n+1}}\left\{\left[\prod_{j=1}^n \theta\left(\sigma_j, \rho\sum_{k=1}^j\int_0^{\cdot}\left(\int_{\sigma_{k-1}}^{\sigma_k} h^2(u,v)\,du\right)^{1/2}\right.\right.\right.\\ &\qquad\left.\left.\cdot\, dw_k(v)+\eta(\cdot)\right)\right]\\ &\qquad\left.\cdot\,\psi\left(\rho\sum_{k=1}^{n+1}\int_0^{\cdot}\left(\int_{\sigma_{k-1}}^{\sigma_k} h^2(u,v)\,du\right)^{1/2} dw_k(v)+\eta(\cdot)\right)\right\}d\vec{\sigma},\end{aligned}$$

where

$$\Delta_n(S) = \{\vec{\sigma} = (\sigma_1, \dots, \sigma_n) : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \sigma_{n+1} = S\}.$$

Example 3. Using Corollary 3, we can express the Yeh-Wiener integral

$$\text{(4.14)}\qquad I \equiv E_x\left[\exp\left\{\int_0^S\int_0^T \phi\left(\sigma, \tau, \int_0^{\tau}\int_0^{\sigma} h(u,v)\,dx(u,v)\right)d\tau\, d\sigma\right\}\right]$$

as a sum of Lebesgue integrals, where $\phi(s, t, u)$ is a bounded continuous function on $Q \times \mathbb{R}$. To do so we choose $\psi \equiv 1$, $\rho = 1$, $\eta \equiv 0$ and we let

$$\theta(s, y(\cdot)) = \int_0^T \phi(s, \tau, y(\tau))\,d\tau.$$

In addition for $n = 1, 2, \ldots$ let

$$\Delta_n(S) = \{\vec{\sigma} = (\sigma_1, \ldots, \sigma_n) : 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n < S\}.$$

Then using (4.13) we have

$$\begin{aligned}
I &\equiv E_x\left[\exp\left\{\int_0^S \phi\left(\sigma, \int_0^{\cdot}\int_0^{\sigma} h(u,v)\,dx(u,v)\right) d\sigma\right\}\right] \\
&= 1 + \sum_{n=1}^{\infty}\int_{\Delta_n(S)} E_{w_1,\ldots,w_n} \\
&\qquad \cdot\left\{\prod_{j=1}^{n}\left[\theta\left(\sigma_j, \sum_{k=1}^{j}\int_0^{\cdot}\left(\int_{\sigma_{k-1}}^{\sigma_k} h^2(u,v)\,du\right)^{1/2} dw_k(v)\right)\right]\right\} d\vec{\sigma} \\
&= 1 + \sum_{n=1}^{\infty}\int_{\Delta_n(S)} E_{w_1,\ldots,w_n} \\
&\qquad \cdot\left\{\prod_{j=1}^{n}\left[\int_0^T \phi\left(\sigma_j, \tau_j, \sum_{k=1}^{j}\int_0^{\tau_j}\left(\int_{\sigma_{k-1}}^{\sigma_k} h^2(u,v)\,du\right)^{1/2} dw_k(v)\right) d\tau_j\right]\right\} d\vec{\sigma} \\
&= 1 + \sum_{n=1}^{\infty}\int_{\Delta_n(S)}\int_{[0,T]^n} E_{w_1,\ldots,w_n} \\
&\qquad \cdot\left\{\prod_{j=1}^{n}\phi\left(\sigma_j, \tau_j, \sum_{k=1}^{j}\int_0^{\tau_j}\left(\int_{\sigma_{k-1}}^{\sigma_k} h^2(u,v)\,du\right)^{1/2} dw_k(v)\right)\right\} d\vec{\tau}\,d\vec{\sigma}
\end{aligned}$$

To evaluate the Wiener integrals involved above, we introduce the following notation: For $1 \le j \le n$, define

$$W_{n-j+1}(u_{j,j}, u_{j,j+1}, \ldots, u_{j,n}; \tau_j, \ldots, \tau_n) = \begin{cases} 0 & \text{if } \tau_1, \ldots, \tau_n \text{ are not all distinct}, \\ \displaystyle\prod_{k=j}^{n}\left[2\pi\int_{\tau_{\alpha_{k-1}}}^{\tau_{\alpha_k}} h^2\right]^{-1/2} \exp\left[-\frac{(u_{j,\alpha_k} - u_{j,\alpha_{k-1}})^2}{2\int_{\tau_{\alpha_{k-1}}}^{\tau_{\alpha_k}}\int_{\sigma_{j-1}}^{\sigma_j} h^2}\right] & \text{otherwise}, \end{cases}$$

where $(\alpha_j, \ldots, \alpha_n)$ is a permutation of $(j, \ldots, n)$ such that $\tau_{\alpha_j} < \tau_{\alpha_{j+1}} < \cdots < \tau_{\alpha_n}$ and $\alpha_{j-1} = 0$, $\tau_0 = 0$, and $u_{j,0} = 0$. Then, we have

$$I = 1 + \sum_{n=1}^{\infty}\int_{\Delta_n(S)}\int_{[0,T]^n}\int_{\mathbf{R}^{n(n+1)/2}}\prod_{j=1}^{n}\left\{W_{n-j+1}(u_{j,j}, \ldots, u_{j,n}; \tau_j, \ldots, \tau_n) \cdot\phi\left(\sigma_j, \tau_j, \sum_{k=1}^{j} u_{k,j}\right)\right\} d\vec{u}\,d\vec{\tau}\,d\vec{\sigma}.$$

Example 4. Let ϕ and θ be as in Example 3 above. Then

$$E_x\left[\int_0^S\int_0^T\phi\left(\sigma,\tau,\int_0^\tau\int_0^\sigma h(u,v)\,dx(u,v)\right)d\tau\,d\sigma\right]$$
$$=\int_0^S\int_0^T\left[2\pi\int_0^\tau\int_0^\sigma h^2\right]^{-1/2}\int_{\mathbb{R}}\phi(\sigma,\tau,u)\exp\left\{-\frac{u^2}{2\int_0^\tau\int_0^\sigma h^2}\right\}du\,d\tau\,d\sigma$$
$$=\int_0^S E_w\left[\theta\left(\sigma,\int_0^{\cdot}\left(\int_0^\sigma h^2(u,v)\,du\right)^{1/2}dw(v)\right)\right]d\sigma.$$

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Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056
E-mail address: `cpark@miavx1.acs.muohio.edu`

Department of Mathematics and Statistics, University of Nebraska, Lincoln, Nebraska 68588-0323
E-mail address: `dskoug@hoss.unl.edu`