EXISTENCE OF SMOOTH INVARIANT MEASURES
FOR GEODESIC FLOWS OF FOLIATIONS
OF RIEMANNIAN MANIFOLDS

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Abstract. We construct a nontrivial smooth finite measure invariant under
the geodesic flow of a foliation $\mathcal{F}$ of a compact Riemannian manifold $M$
assuming that the transverse mean curvature of $\mathcal{F}$ is distributed "nicely" along
some leaf geodesics.

In [W], we proved that if a foliation $\mathcal{F}$ of a Riemannian manifold $M$ is
transversely minimal (i.e., if $H^\perp \equiv 0$, where $H^\perp$ is the trace of the second
fundamental form $\alpha^\perp$ of the orthogonal complement $N\mathcal{F}$ of $\mathcal{F}$), then the
volume form $\Omega$ on the unit tangent bundle $S\mathcal{F}$ of $\mathcal{F}$ equipped with the so-
called Sasaki metric defines a smooth measure $\lambda$ invariant under the geodesic
flow $(g_t)$ of $\mathcal{F}$.

This is an analogue of the standard result of Riemannian geometry (and
classical mechanics) saying that the Liouville measure is invariant under the
godesic flow of a Riemannian manifold [K].

In this note, we show the existence of a smooth $(g_t)$-invariant measure under
a weaker assumption of a "nice" distribution of the transverse mean curvature
$H^\perp$ of $\mathcal{F}$ along the leaf geodesics.

The existence of smooth invariant measures for dynamical systems is of some
importance because of the following: Pesin's theory [P] allows one to estimate
the topological entropy of a smooth dynamical system from below if such a
measure exists (compare Remark 3 below).

Theorem. If $M$ is compact and the smooth functions $h_t : S\mathcal{F} \to \mathbb{R}$, $t \geq 0$,
given by

$$h_t(v) = \exp \int_0^t \langle H^\perp(\pi g_s v), g_s v \rangle \, ds,$$

are uniformly bounded from below by a constant $c > 0$ on an open subset $U \subset
S\mathcal{F}$, then the flow $(g_t)$ admits a smooth invariant measure.

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Remark 1. The conditions of our Theorem are satisfied, for example, if there exists \( t_0 \geq 0 \) and a wandering leaf geodesic \( \gamma \) such that \( H^\perp \equiv 0 \) on the set \( \pi(\bigcup_{t \geq t_0} \mathcal{g}_t(V)) \) for some neighbourhood \( V \) of \( \gamma(0) \) in \( S\mathcal{F} \). This is the case if \( \mathcal{F} \) admits a bunch of wandering simply-connected leaves \( L \) of nonpositive curvature for which \( H^\perp \equiv 0 \) outside compact sets \( K \subset L \).

More particular examples of the foliation satisfying the assumptions can be obtained in the following way.

Take a smooth convex function \( f : (-1, 1) \to \mathbb{R} \) such that \( f(x) = f(-x) \) and \( f^{(k)}(x) \to \infty \) when \( x \to 1^- \) for \( k = 0, 1, \ldots \). Construct the standard Reeb foliations \( \mathcal{F}_r \) in \( \hat{V} = D^2 \times \mathbb{R} \) and \( \mathcal{F} \) in the solid torus \( V = D^2 \times S^1 \) [CN]. Let \( W = \{(r, \theta, z) \in \hat{V}; \ r \geq \frac{1}{2}\} \). Equip the leaves of \( \mathcal{F}|W \) with the flat metric induced from the standard flat metric \( dz^2 + d\theta^2 \) on the cylinder \( C = \{r = 1\} \). Extend this metric first to the Riemannian metric \( g \) on \( W \) requiring that the field \( \frac{\partial}{\partial r} \) has norm 1 and is orthogonal to \( \mathcal{F} \) and then to the whole set \( \hat{V} \) in an arbitrary way.

An elementary calculation shows that

\[
H^\perp(r, \theta, z) = \frac{f''(r)}{f'(r)^2} \frac{\partial}{\partial r}
\]

in \( \hat{W} \).

Let \( v_0 = \frac{\partial}{\partial z} + \frac{1}{f'(r_0)} \frac{\partial}{\partial r} \) for some \( r_0 > \frac{1}{2} \). Any leaf geodesic \( \gamma \) with the initial vector \( v \) in a neighbourhood \( U \) of \( v_0 \) stays in \( \hat{W} \) for ever. Also, the function \( r \circ \gamma \) increases with a bounded derivative. Therefore,

\[
h_t(v) = \exp \int_0^t \frac{f''(r(\gamma(s)))}{f'(r(\gamma(s)))^2} ds
= \exp \int_{r(\gamma(0))}^{r(\gamma(t))} \frac{f''(u)}{f'(u)^2} \frac{du}{(r \circ \gamma)'(u)} \geq C \exp \left( \frac{1}{f'(r_0)} \right)
\]

for a constant \( C > 0 \), all \( v \) in \( U \), and \( t \geq 0 \).

Making all the extensions invariant under the translation \( (r, \theta, z) \mapsto (r, \theta, z + 1) \) we can obtain an appropriate foliation in \( V \). Embedding some copies of \( V \) into a closed 3-manifold \( M \) and extending \( \mathcal{F} \) and \( g \) to \( M \) yield a suitable example. (For the results concerned with the existence of such extensions we refer to the series of Gabai's papers [G].)

Since any foliation of any closed 3-manifold with finite fundamental group admits a Reeb component [N], we get the following.

**Corollary.** For any foliation \( \mathcal{F} \) of a closed 3-manifold \( M \) with finite fundamental group there exists a Riemannian metric on \( M \) for which the geodesic flow of \( \mathcal{F} \) admits a nontrivial smooth invariant measure.

**Proof of the Theorem.** Let \( X \) be the vector field of the flow \( (g_t) \) on \( S\mathcal{F} \). The calculation in the proof of Theorem 1 in [W] shows that

\[
\mathcal{L}_X \Omega = \langle \pi_* \circ X, H^\perp \circ \pi \rangle \Omega.
\]

Since \( \pi_*(X(v)) = v \) for all \( v \), it follows that

\[
g_t^* \Omega = h_t \Omega \quad (t \in \mathbb{R}).
\]
Let
\[ \mu_t = \frac{1}{t} \int_0^t h_s \, ds \cdot \Omega. \]
\( \mu_t \) is a smooth measure on \( S\mathcal{F} \) of finite total mass \( \lambda(S\mathcal{F}) \). There exists a sequence \( (t_n) \) such that \( t_n \not\to \infty \) and \( \mu_{t_n} \to \mu \) for some finite Borel measure \( \mu \) on \( S\mathcal{F} \) [Wa]. Consider the Lebesgue decomposition [R]
\[ \mu = \mu_a + \mu_s \]
of \( \mu \) relative to \( \lambda \) with \( \mu_a \ll \lambda \) and \( \mu_s \perp \lambda \).

The measure \( \mu_a \) is finite, smooth, and \( (g_t) \)-invariant. (In fact, \( \mu = g_t^* \mu = g_t^* \mu_a + g_t^* \mu_s \) is another Lebesgue decomposition of \( \mu \) and this is unique.) It remains to prove that \( \mu_a \neq 0 \).

Assume that \( \mu_a = 0 \) and let \( A = \text{supp} \mu \). Then the volume of \( A \) equals 0, so we can find a nonnegative, continuous function \( f \) on \( S\mathcal{F} \) such that \( f \equiv 0 \) on \( A \) and \( f(v) > 0 \) for some \( v \) in \( U \). Applying the Fubini and Mean Value Theorems we get
\[ 0 = \int_{S\mathcal{F}} f \, d\mu = \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \left( \int_{S\mathcal{F}} fh_s \, d\lambda \right) \, ds \]
\[ = \lim_{n \to \infty} \int_{S\mathcal{F}} h_{\theta_n(v), t_n}(v) f(v) \, d\lambda(v) \geq c \cdot \int_U f \Omega > 0 \]
for some numbers \( \theta_n(v) \in [0, 1] \).

The contradiction just obtained ends the proof.

**Remark 2.** The above proof shows that \( U \subset \text{supp} \mu_a \).

**Remark 3.** Our Theorem together with the Lyapunov exponent estimates of [W] and the Pesin entropy estimate [P] (see also [M]) yield the following estimate for the topological and \( \mu_a \)-entropy of \( (g_t) \) provided that the sectional curvature \( K \) of \( M \) is strictly negative and \( \mathcal{F} \) is close to being totally geodesic in the sense that the norms of the second fundamental form \( \alpha \) of \( M \) and its covariant derivative are small enough:
\[ h_{\text{top}}(g_t) \geq h_{\mu_a}(g_t) \geq m \int_{S\mathcal{F}} \Psi \, d\mu_a \geq m \inf_U \sqrt{\Psi} \cdot c \cdot \text{vol}(U), \]
where \( m \) is the minimal dimension of the space of horizontal vectors of \( TTM \) tangent to \( S\mathcal{F} \), \( \Psi : S\mathcal{F} \to \mathbb{R} \) is given by
\[ \Psi(v) = (1 - |\alpha(v)|)(K(v)(d - 1) - |\alpha(v)| - |\nabla \alpha(v), v|), \]
\( K(v) = \max\{K(v, w); w \perp v\} \) and \( d \) is a positive constant such that all the quadratic forms \( Q_w, \ w \in S\mathcal{F}, \)
\[ Q_w(x, y) = x^2((d^2 - d)K(w) - (d + 1)|\nabla \alpha(w, w)| - |\alpha(w)|^2) \]
\[ + xy(2d - 1)|\alpha(w)| + y^2d, \]
are nonnegative on \( \mathbb{R} \). (The existence of such \( d \) was established in [W].)

In particular, for transversely minimal close to geodesic foliations of compact negatively curved manifolds we get \( c = 1 \) and
\[ h_{\text{top}}(g_t) \geq h_\lambda(g_t) \geq m \int_{S\mathcal{F}} \sqrt{\Psi} \, d\Omega. \]
If $\mathcal{F}$ is totally geodesic and transversely minimal, then
\[
h_{\text{top}}(g_t) \geq h_{\mathcal{F}}(g_t) \geq m \int_{S\mathcal{F}} \sqrt{-K(u)} \, d\Omega(u).
\]
This last estimate could be compared with that of [OS].

Note also that $m + 1 \geq p = \dim \mathcal{F}$. In fact, the equality
\[
\nabla_v X = (\nabla_v X)^\top + \alpha(v, X)
\]
defining the second fundamental form $\alpha$ of $\mathcal{F}$ for $v \in TM$ and sections $X$ of $T\mathcal{F}$ shows that the space of horizontal vectors tangent to $S\mathcal{F}$ at any point $w$ of $S_x\mathcal{F}$ coincides with the intersection $H^\top_w \cap \pi^{-1}_x \ker \tilde{\alpha}(w, \cdot)$, where $H^\top_w$ is the horizontal subspace of the bundle $T_S\mathcal{F}$ with respect to the natural connection $\nabla^\top$ in $S\mathcal{F}$, and $\tilde{\alpha}$ is the restriction of $\alpha$ to the orthogonal complement $\{w\}^\perp$ of $w$ in $T_xM$. Also, $\pi_x$ maps $H^\top_w$ onto $\{w\}^\perp$ so $\dim H^\top_w = n - 1$ and $\dim \pi^{-1}_x \ker \tilde{\alpha}(w, \cdot) = p + \dim \ker \tilde{\alpha}(w, \cdot) \geq 2p - 1$. This yields our estimate for $m$. (In [W], the author claimed that $m = \dim M - 1$ but it was pointed out to us by A. Zeghib that in general this is not true.)

References


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