PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 3, March 1994

EXISTENCE OF SMOOTH INVARIANT MEASURES FOR GEODESIC FLOWS OF FOLIATIONS OF RIEMANNIAN MANIFOLDS

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(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We construct a nontrivial smooth finite measure invariant under the geodesic flow of a foliation $\mathscr{F}$ of a compact Riemannian manifold M assuming that the transverse mean curvature of $\mathscr{F}$ is distributed "nicely" along some leaf geodesics.

In [W], we proved that if a foliation $\mathscr{F}$ of a Riemannian manifold M is transversely minimal (i.e., if $H^{\perp} \equiv 0$, where $H^{\perp}$ is the trace of the second fundamental form $\alpha^{\perp}$ of the orthogonal complement $N\mathscr{F}$ of $\mathscr{F}$), then the volume form Ω on the unit tangent bundle $S\mathscr{F}$ of $\mathscr{F}$ equipped with the so-called Sasaki metric defines a smooth measure λ invariant under the geodesic flow (g_t) of $\mathscr{F}$.

This is an analogue of the standard result of Riemannian geometry (and classical mechanics) saying that the Liouville measure is invariant under the geodesic flow of a Riemannian manifold [K].

In this note, we show the existence of a smooth (g_t)-invariant measure under a weaker assumption of a "nice" distribution of the transverse mean curvature $H^{\perp}$ of $\mathscr{F}$ along the leaf geodesics.

The existence of smooth invariant measures for dynamical systems is of some importance because of the following: Pesin's theory [P] allows one to estimate the topological entropy of a smooth dynamical system from below if such a measure exists (compare Remark 3 below).

Theorem. *If M is compact and the smooth functions $h_t : S\mathscr{F} \to \mathbb{R}$, $t \geq 0$, given by*

$$h_t(v) = \exp \int_0^t \langle H^{\perp}(\pi g_s v), g_s v \rangle \, ds,$$

are uniformly bounded from below by a constant $c > 0$ on an open subset $U \subset S\mathscr{F}$, then the flow (g_t) admits a smooth invariant measure.

Received by the editors November 1, 1991.
1991 *Mathematics Subject Classification.* Primary 53C12, 58F11, 28D10, 28D20.
Key words and phrases. Foliation, Riemannian manifold, geodesic flow, invariant measure, entropy.
This paper was written during the author's stay in St. Louis, where he enjoyed the hospitality of the faculty and staff of the Department of Mathematics of the Washington University.

Remark 1. The conditions of our Theorem are satisfied, for example, if there exists $t_0 \geq 0$ and a wandering leaf geodesic γ such that $H^\perp \equiv 0$ on the set $\pi(\bigcup_{t \geq t_0} g_t(V))$ for some neighbourhood V of $\dot\gamma(0)$ in $S\mathscr{F}$. This is the case if $\mathscr{F}$ admits a bunch of wandering simply-connected leaves L of nonpositive curvature for which $H^\perp \equiv 0$ outside compact sets $K \subset L$.

More particular examples of the foliation satisfying the assumptions can be obtained in the following way.

Take a smooth convex function $f : (-1, 1) \to \mathbb{R}$ such that $f(x) = f(-x)$ and $f^{(k)}(x) \to \infty$ when $x \to 1^-$ for $k = 0, 1, \ldots$. Construct the standard Reeb foliations $\tilde{\mathscr{F}}$ in $\tilde V = D^2 \times \mathbb{R}$ and $\mathscr{F}$ in the solid torus $V = D^2 \times S^1$ [CN]. Let $\tilde W = \{(r, \theta, z) \in \tilde V;\ r \geq \frac{1}{2}\}$. Equip the leaves of $\tilde{\mathscr{F}} | \tilde W$ with the flat metric induced from the standard flat metric $dz^2 + d\theta^2$ on the cylinder $C = \{r = 1\}$. Extend this metric first to the Riemannian metric g on $\tilde W$ requiring that the field $\frac{\partial}{\partial r}$ has norm 1 and is orthogonal to $\tilde{\mathscr{F}}$ and then to the whole set $\tilde V$ in an arbitrary way.

An elementary calculation shows that

$$H^\perp(r, \theta, z) = \frac{f''(r)}{f'(r)^2} \cdot \frac{\partial}{\partial r}$$

in $\tilde W$.

Let $v_0 = \frac{\partial}{\partial z} + \frac{1}{f'(r_0)} \frac{\partial}{\partial r}$ for some $r_0 > \frac{1}{2}$. Any leaf geodesic γ with the initial vector v in a neighbourhood U of v_0 stays in $\tilde W$ for ever. Also, the function $r \circ \gamma$ increases with a bounded derivative. Therefore,

$$\begin{aligned} h_t(v) &\geq \exp \int_0^t \frac{f''(r(\gamma(s)))}{f'(r(\gamma(s)))^2}\, ds \\ &= \exp \int_{r(\gamma(0))}^{r(\gamma(t))} \frac{f''(u)}{f'(u)^2} \frac{du}{(r \circ \gamma)'(u)} \geq C \exp\left(\frac{1}{f'(\frac{1}{2})}\right) \end{aligned}$$

for a constant $C > 0$, all v in U, and $t \geq 0$.

Making all the extensions invariant under the translation $(r, \theta, z) \mapsto (r, \theta, z + 1)$ we can obtain an appropriate foliation in V. Embedding some copies of V into a closed 3-manifold M and extending $\mathscr{F}$ and g to M yield a suitable example. (For the results concerned with the existence of such extensions we refer to the series of Gabai's papers [G].)

Since any foliation of any closed 3-manifold with finite fundamental group admits a Reeb component [N], we get the following.

Corollary. *For any foliation $\mathscr{F}$ of a closed 3-manifold M with finite fundamental group there exists a Riemannian metric on M for which the geodesic flow of $\mathscr{F}$ admits a nontrivial smooth invariant measure.*

Proof of the Theorem. Let X be the vector field of the flow (g_t) on $S\mathscr{F}$. The calculation in the proof of Theorem 1 in [W] shows that

$$\mathscr{L}_X \Omega = \langle \pi_* \circ X, H^\perp \circ \pi \rangle \Omega.$$

Since $\pi_*(X(v)) = v$ for all v, it follows that

$$g_t^* \Omega = h_t \Omega \qquad (t \in \mathbb{R}).$$

Let

$$\mu_t = \frac{1}{t}\int_0^t h_s\, ds \cdot \Omega.$$

μ_t is a smooth measure on $S\mathcal{F}$ of finite total mass $\lambda(S\mathcal{F})$. There exists a sequence (t_n) such that $t_n \nearrow \infty$ and $\mu_{t_n} \to \mu$ for some finite Borel measure μ on $S\mathcal{F}$ [Wa]. Consider the Lebesgue decomposition [R]

$$\mu = \mu_a + \mu_s$$

of μ relative to λ with $\mu_a \ll \lambda$ and $\mu_s \perp \lambda$.

The measure μ_a is finite, smooth, and (g_t)-invariant. (In fact, $\mu = g_t^*\mu = g_t^*\mu_a + g_t^*\mu_s$ is another Lebesgue decomposition of μ and this is unique.) It remains to prove that $\mu_a \neq 0$.

Assume that $\mu_a = 0$ and let $A = \operatorname{supp}\mu$. Then the volume of A equals 0, so we can find a nonnegative, continuous function f on $S\mathcal{F}$ such that $f \equiv 0$ on A and $f(v) > 0$ for some v in U. Applying the Fubini and Mean Value Theorems we get

$$\begin{aligned} 0 = \int_{S\mathcal{F}} f\, d\mu &= \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n}\left(\int_{S\mathcal{F}} f h_s\, d\lambda\right) ds \\ &= \lim_{n\to\infty}\int_{S\mathcal{F}} h_{\theta_n(v)t_n}(v) f(v)\, d\lambda(v) \geq c \cdot \int_U f\Omega > 0 \end{aligned}$$

for some numbers $\theta_n(v) \in [0, 1]$.

The contradiction just obtained ends the proof.

Remark 2. The above proof shows that $U \subset \operatorname{supp}\mu_a$.

Remark 3. Our Theorem together with the Lyapunov exponent estimates of [W] and the Pesin entropy estimate [P] (see also [M]) yield the following estimate for the topological and μ_a-entropy of (g_t) provided that the sectional curvature K of M is strictly negative and $\mathcal{F}$ is close to being totally geodesic in the sense that the norms of the second fundamental form α of M and its covariant derivative are small enough:

$$h_{\text{top}}(g_t) \geq h_{\mu_a}(g_t) \geq m \int_{S\mathcal{F}} \Psi\, d\mu_a \geq m \inf_U \sqrt{\Psi} \cdot c \cdot \operatorname{vol}(U),$$

where m is the minimal dimension of the space of horizontal vectors of TTM tangent to $S\mathcal{F}$, $\Psi : S\mathcal{F} \to \mathbb{R}$ is given by

$$\Psi(v) = (1 - |\alpha(v)|)(K(v)(d-1) - |\alpha(v)| - |\nabla\alpha(v, v)|),$$

$K(v) = \max\{K(v, w); w \perp v\}$ and d is a positive constant such that all the quadratic forms Q_w, $w \in S\mathcal{F}$,

$$\begin{aligned} Q_w(x, y) = x^2((d^2 - d)K(w) &- (d+1)|\nabla\alpha(w, w)| - |\alpha(w)|^2) \\ &+ xy(2d-1)|\alpha(w)| + y^2 d, \end{aligned}$$

are nonnegative on $\mathbb{R}$. (The existence of such d was established in [W].)

In particular, for transversely minimal close to geodesic foliations of compact negatively curved manifolds we get $c = 1$ and

$$h_{\text{top}}(g_t) \geq h_\lambda(g_t) \geq m \int_{S\mathcal{F}} \sqrt{\Psi}\, d\Omega.$$

If $\mathscr{F}$ is totally geodesic and transversely minimal, then

$$h_{\mathrm{top}}(g_t) \geq h_\lambda(g_t) \geq m \int_{S\mathscr{F}} \sqrt{-K(u)}\, d\Omega(u).$$

This last estimate could be compared with that of [OS].

Note also that $m+1 \geq p = \dim \mathscr{F}$. In fact, the equality

$$\nabla_v X = (\nabla_v X)^\top + \alpha(v, X)$$

defining the second fundamental form α of $\mathscr{F}$ for $v \in TM$ and sections X of $T\mathscr{F}$ shows that the space of horizontal vectors tangent to $S\mathscr{F}$ at any point w of $S_x\mathscr{F}$ coincides with the intersection $\mathscr{H}_w^\top \cap \pi_x^{-1} \ker \tilde{\alpha}(w, \cdot)$, where $\mathscr{H}_w^\top$ is the horizontal subspace of the bundle $TS\mathscr{F}$ with respect to the natural connection $\nabla^\top$ in $S\mathscr{F}$, and $\tilde{\alpha}$ is the restriction of α to the orthogonal complement $\{w\}^\perp$ of w in T_xM. Also, π_x maps $\mathscr{H}_w^\top$ onto $\{w\}^\perp$ so $\dim \mathscr{H}_w^\top = n-1$ and $\dim \pi_*^{-1} \ker \tilde{\alpha}(w, \cdot) = p + \dim \ker \tilde{\alpha}(w, \cdot) \geq 2p-1$. This yields our estimate for m. (In [W], the author claimed that $m = \dim M - 1$ but it was pointed out to us by A. Zeghib that in general this is not true.)

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