PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 3, March 1994

INFINITE CYCLIC NORMAL SUBGROUPS OF FUNDAMENTAL GROUPS OF NONCOMPACT 3-MANIFOLDS

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(Communicated by James West)

ABSTRACT. It is shown that an end-irreducible 3-manifold each of whose boundary components is compact and whose fundamental group contains an infinite cyclic normal subgroup is Seifert fibered.

In this paper, it is shown that if W is a noncompact, orientable, irreducible, end-irreducible 3-manifold such that each component of ∂W is compact and $\pi_1(W)$ contains an infinite cyclic normal subgroup, then W is Seifert fibered.

This was inspired by the work of Dave Gabai [G] and Andrew Casson [CJ], who showed independantly that if M is a compact, orientable, irreducible 3-manifold such that $\pi_1(M, *)$ contains an infinite cyclic normal subgroup, then M is a Seifert fibered space. This paper, however, will not need the full strength of this result. Instead Corollary II.6.4 of [JS], in which M is assumed to be sufficiently large, is used.

In this paper, W will always be a noncompact, orientable, irreducible 3-manifold such that $\pi_1(W, *)$ contains an infinite cyclic normal subgroup C, where $* \in W$ is fixed.

If for every compact $K \subset W$ and every noncompact component U of $\mathrm{cl}(W - K)$ there is a loop λ_U in U that is freely homotopic in W to a generator of C, then we say that C is *end-peripheral.* If for every compact $K \subset W$ there is a compact 3-manifold $M_K \subset W$ with $K \subset M_K - \mathrm{Fr}(M_K)$ and $\mathrm{Fr}(M_K)$ incompressible in W, we say that W is *end-irreducible.*

In order to prove the result stated in the first paragraph, it is proven that W is Seifert fibered whenever C is end-peripheral and W is end-irreducible. Then it is shown that C is end-peripheral whenever W is end-irreducible and each component of ∂W is compact.

The assumption of C being end-peripheral is not unnecessarily strong. In fact it is not difficult to see that if Σ is a noncompact Seifert fibered space and Γ is the infinite cyclic normal subgroup of $\pi_1(\Sigma, *)$ generated by a regular fiber, then Γ is end-peripheral.

Similarly, the assumptions of irreducibility and end-irreducibility are reasonable. By Lemma 2.2 of [W], it follows that $\mathbf{R}^2 \times S^1$ is the only noncompact Seifert fibered space that is not end-irreducible. It is shown in Lemma 2.1 of

Received by the editors February 13, 1992.
1991 *Mathematics Subject Classification.* Primary 57N10.

[W] that every noncompact Seifert fibered space is irreducible.

There do exist homotopy open solid tori with end-peripheral fundamental groups that are not homeomorphic to $\mathbf{R}^2 \times S^1$. One can be constructed as follows. Let H be an open manifold that is a union of solid tori $\{V_n | n \geq 0\}$ such that, for $n \geq 1$, $V_n \subset V_{n+1} - \partial V_{n+1}$ and the core of V_n is homotopic in V_{n+1} to the core of V_{n+1}. It is easy to show that $\pi_1(H, *) = \mathbf{Z}$ and that $\pi_1(H, *)$ is end-peripheral. However, it is possible to construct H so that H is not an open solid torus. (See M_2 of [ST], for example.) Such an H is not Seifert fibered because H is not end-irreducible. However, it is possible that another hypothesis could replace end-irreducibility. For instance, Mess has shown in [M] that if $\hat{M}$ is a noncompact 3-manifold such that $\pi_1(\hat{M}, *)$ is an abelian group of rank 1 which is a regular cover of a closed, irreducible 3-manifold, then $\hat{M}$ is an open solid torus and, therefore, Seifert fibered. In Lemma 3 of Theorem 1 of his proof, Mess actually shows that $\pi_1(\hat{M}, *)$ is end-peripheral, though he does not use this terminology.

There do exist examples of end-irreducible 3-manifolds whose fundamental groups contain non-end-peripheral infinite cyclic normal subgroups. Let F be a compact, connected 2-manifold with $\partial F \neq \varnothing$ that is not an annulus or a disk. Let J be a component of ∂F. Let $M = F \times S^1$ and $T = J \times S^1$. Let $x \in J$, $l = x \times S^1$, and $m = J \times 1$. Let L be a simple closed curve in T which is equal to $l^2 m^3$ in $\pi_1(T, *)$. Let $M' = M - L$. Then M' is irreducible. Let K be a compact subset of M'. Let V be a regular neighborhood of L in $M - K$. Then V is a solid torus whose core is homotopic in V to L. Let $M_K = \mathrm{cl}(M - V)$. Then $\mathrm{Fr}(M_K)$ is incompressible in M', so M' is end-irreducible. Note that $\pi_1(M', *_1)$ contains an infinite cyclic normal subgroup that is generated by $*_1 \times S^1$, where $*_1 \in F - \partial F$. However, M' is not Seifert fibered because $\langle *_1 \times S^1 \rangle$ is not end-peripheral. This is because any loop in $V \cap M'$ is homotopic in V to a power of $l^2 m^3$ and, therefore, cannot be freely homotopic in M' to $*_1 \times S^1$ or its inverse.

Infinite cyclic normal subgroups

Let $p : \tilde{W} \to W$ be the cover of W such that $p_*(\pi_1(\tilde{W}, \tilde{*})) = C$, where $\tilde{*} \in \tilde{W}$ with $p(\tilde{*}) = *$. By [S] there is a compact 3-manifold $L \subset \tilde{W}$ such that $j_* : \pi_1(L, \tilde{*}) \to \pi_1(\tilde{W}, \tilde{*})$ is an isomorphism where $j : L \to \tilde{W}$ is the inclusion map. We shall say that a compact 3-submanifold M of W is *C-aware* if $p(L) \subset M$.

Lemma 1. *Let $M \subset W$ be a C-aware compact 3-manifold, and let $i : M \to W$ be the inclusion map.*

(1) *C is a subgroup of $i_* \pi_1(M, *)$.*

(2) *If i_* is injective and no component of ∂M is a sphere, then M has a Seifert fibering in which each fiber is freely homotopic in M to a generator of C.*

Proof. To prove (1), let $q : L \to M$ be defined by $q(x) = p(x)$ for every $x \in L$. Then $iq = pj$. So $i_* q_* = p_* j_*$. Therefore, $C = i_* q_* \pi_1(L, \tilde{*})$.

To prove (2), suppose that i_* is injective. Then $\pi_1(M, *)$ contains an infinite cyclic normal subgroup. It is not difficult to show that M is Haken. Therefore, by Corollary II.6.4 of [JS], it follows that M is Seifert fibered in the way described in (2). □

For the rest of the section, suppose that every compact $K \subset W$ is contained in a compact 3-submanifold M_K of W such that the inclusion induced map $\pi_1(M_K, *) \to \pi_1(W, *)$ is injective.

Proposition 2. *Every compact component of ∂W is a torus.*

Proof. Let F be a compact component of ∂W. Let M_F be a compact 3-submanifold of W such that $F \subset M_F - \mathrm{Fr}(M_F)$ and the inclusion induced map $\pi_1(M_F, *) \to \pi_1(W, *)$ is injective. We may assume that M_F is C-aware and that no component of $\mathrm{cl}(W - M_F)$ is compact. By Lemma 1, it follows that M_F is Seifert fibered. Since F is a component of ∂M_F, it follows that F is a torus. □

Proposition 3. *If W is not end-irreducible and ∂W is compact, then for every compact $K \subset W$ there is a solid torus V such that $K \subset V - \mathrm{Fr}(V)$. In particular, $\partial W = \varnothing$ and $\pi_1(W, *)$ is a subgroup of $(\mathbf{Q}, +)$.*

Proof. Let $K \subset W$ be compact. We may assume that $\partial W \subset K$. Let V be a compact, C-aware 3-manifold in W with $K \subset V - \mathrm{Fr}(V)$ such that no component of $\mathrm{cl}(W - V)$ is compact and the inclusion induced map $\pi_1(V, *) \to \pi_1(W, *)$ is injective. It follows easily that V must be Haken. By Lemma 1, it follows that V is Seifert fibered.

We claim that V is a solid torus. Since W is not end-irreducible, we may choose V so that $\mathrm{Fr}(V)$ is compressible in W. Note that each component of $\mathrm{Fr}(V)$ is a component of ∂V because $\partial W \subset V - \mathrm{Fr}(V)$. Consequently each component of $\mathrm{Fr}(V)$ is a torus. Suppose that D is a compressing disk for $\mathrm{Fr}(V)$ in W.

We claim that $D \subset V$. Suppose that $D \subset \mathrm{cl}(W - V)$. Let T be the component of $\mathrm{Fr}(V)$ that contains ∂D. Then T is a torus. By compressing T along D, we can obtain a 2-sphere S which must bound a 3-cell $B \subset W$. Since no component of $\mathrm{cl}(W - V)$ is compact, it follows that $V \subset B$. Since V is C-aware, this is a contradiction.

Since $D \subset V$, it follows V is a solid torus. It is an exercise to show that $\pi_1(W, *)$ can be embedded in $(\mathbf{Q}, +)$. □

End-peripheral subgroups

In this section, we shall suppose that C is end-peripheral and W is end-irreducible.

Let M be a C-aware compact, connected 3-manifold in W such that $\mathrm{Fr}(M)$ is incompressible in W and no component of $\mathrm{cl}(W - M)$ is compact. Let N be a compact, connected 3-manifold in W with $M \subset N - \mathrm{Fr}(N)$ such that $\mathrm{Fr}(N)$ is incompressible in W, no component of $\mathrm{cl}(W - N)$ is compact, and each component of $\mathrm{cl}(W - M)$ meets only one component of $\mathrm{cl}(N - M)$. Let Q be a component of $\mathrm{cl}(N - M)$, and let $M_Q = M \cup Q$.

Lemma 4. *For some $*_1 \in Q \cap \mathrm{Fr}(M)$, $\pi_1(Q, *_1)$ contains an infinite cyclic subgroup G that is normal in $\pi_1(W, *_1)$ (and so in $\pi_1(Q, *_1)$) whose generator is freely homotopic in W to a generator of C.*

Proof. Let U be the component of $\mathrm{cl}(W - M)$ that contains Q. There is a loop $\lambda' : S^1 \to W$ with $\lambda'(S^1) \subset U$ that is freely homotopic in W to a generator of C. By the usual arguments involving the incompressibility of $\mathrm{Fr}(Q)$ in W, it follows that λ' is freely homotopic in U to a loop $\lambda : S^1 \to U$ with $\lambda(S^1)$ contained in $Q \cap \mathrm{Fr}(M)$. Let $*_1 = \lambda(1)$, and let G be the subgroup of $\pi_1(Q, *_1)$ that is generated by λ.

We claim that G is an infinite cyclic normal subgroup of $\pi_1(Q, *_1)$. Since $\mathrm{Fr}(Q)$ is incompressible in W, we may consider $\pi_1(Q, *_1)$ to be a subgroup of $\pi_1(W, *_1)$. Note that λ is freely homotopic in W to a generator of C. Let $\Lambda : S^1 \times I \to W$ be a map such that $\Lambda(z, 0) = \lambda(z)$ and $\Lambda|S^1 \times 1$ generates C. Let $\alpha : I \to W$ be the path from $*_1$ to $*$ defined by $\alpha(t) = \Lambda(1, t)$. Let $\phi_\alpha : \pi_1(W, *_1) \to \pi_1(W, *)$ be the change of base-point isomorphism along α. Then $\phi_\alpha(\lambda)$ generates C. Therefore, $\phi_\alpha(G) = C$. So $\phi_\alpha(G)$ is normal in $\phi_\alpha(\pi_1(W, *_1))$. Therefore, G is normal in $\pi_1(W, *_1)$. □

Lemma 5. *W is Seifert fibered.*

Proof. By Lemma 1, M is Seifert fibered with fiber freely homotopic in M to a generator of C.

By Lemma 4, $\pi_1(Q, *_1)$ contains an infinite cyclic normal subgroup G whose generator g is freely homotopic in W to a generator c of C. In particular, it follows from Corollary II.6.4 of [JS] that Q is Seifert fibered with fiber freely homotopic in Q to g.

Let $\Lambda : S^1 \times I \to W$ be a map such that $g = \Lambda|S^1 \times 0$ and $c = \Lambda|S^1 \times 1$. Let $\alpha : I \to W$ be defined by $\alpha(t) = \Lambda(1, t)$. Then α is a path in W from $*_1$ to $*$. Let $\phi_\alpha : \pi_1(W, *_1) \to \pi_1(W, *)$ be the change of base-point isomorphism along α. Let β be a path in M from $*_1$ to $*$, and let $\phi_\beta : \pi_1(W, *_1) \to \pi_1(W, *)$ be the change of base-point isomorphism along β.

Given a path $f : I \to W$, let $\hat{f} : I \to W$ be defined by $\hat{f}(t) = f(1-t)$. Therefore, $\beta\hat{\alpha}$ and $\alpha\hat{\beta}$ are loops in W based at $*_1$ and $(\beta\hat{\alpha})(\alpha\hat{\beta})$ is homotopic with $*_1$ fixed to $*_1$. Since G is normal in $\pi_1(W, *_1)$, it follows that $\beta\hat{\alpha}g\alpha\hat{\beta}$ is homotopic with $*_1$ fixed to g^ϵ, where $|\epsilon| = 1$. Hence $\phi_\beta(g^\epsilon) = \hat{\alpha}g\alpha$, and $\hat{\alpha}g\alpha$ is homotopic with $*$ fixed to c. Therefore, $\phi_\beta(g^\epsilon) = c$. Since $\phi_\beta(g^\epsilon)$ and c are loops in M and the inclusion induced map $\pi_1(M, *) \to \pi_1(W, *)$ is injective, $\phi_\beta(g^\epsilon)$ and c are homotopic in M.

It is now easy to see that we may isotop the fibering of Q by an isotopy fixed off a neighborhood of $\partial Q \cap M$ in such a way that each fiber of $Q \cap \partial M$ is a fiber of $M \cap \partial Q$. That is, we may extend the fibering of M to M_Q. Hence we may extend the fibering of M to N.

Since W is end-irreducible, it follows that W can be given a Seifert fibering. □

Compact boundary components

For the rest of the paper, it will be assumed that W is end-irreducible and that each component of ∂W is compact.

Lemma 6. *C is end-peripheral.*

Proof. Suppose that $K \subset W$ is compact. Let U be a noncompact component of $\mathrm{cl}(W - K)$. Let M be a compact, connected, C-aware 3-manifold such that $\mathrm{Fr}(M)$ is incompressible in W and such that $M - \mathrm{Fr}(M)$ contains K and every component of ∂W that meets K. Let F be the union of components of ∂M that do not meet K. Let $F' = F \cap U$. Since $M - \mathrm{Fr}(M)$ contains K and all of the components of ∂W that meet K, it follows that F' is a nonempty union of components of ∂M.

By Lemma 1, it follows that M is fibered with fibers that are freely homotopic to a generator of C. Let λ be a fiber of M that is contained in F'. This ends the proof. □

Theorem 7. *W is Seifert fibered.*

Proof. This follows by Lemmas 5 and 6. □

References

[G] David Gabai, *Convergence groups are Fuchsian groups*, preprint.

[CJ] Andrew Casson and Douglas Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, preprint.

[JS] William H. Jaco and Peter B. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc., vol. 21, no. 220, Amer. Math. Soc., Providence, RI, 1979.

[M] G. Mess, *Centers of 3-manifold groups and groups which are coarse quasiisometric to planes*, preprint.

[S] G. P. Scott, *Compact submanifolds of 3-manifolds*, J. London Math. Soc. (2) **7** (1973), 246–250.

[STu] Peter Scott and Thomas Tucker, *Some examples of exotic non-compact 3-manifolds*, Quart. J. Math. Oxford **40** (1989), 481–499.

[W] Bobby Neal Winters, *Planes in 3-manifolds of finite genus at infinity* (in preparation).

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