COUNTABLE METACOMPACTNESS IN $\Psi$-SPACES

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(Communicated by Andreas R. Blass)

Abstract. We prove under a variety of assumptions including $c = \aleph_2$ that, for every maximal almost disjoint family $\mathcal{A}$ of countable subsets of $\omega_1$, $\Psi(\mathcal{A})$ is not countably metacompact. In addition, a first countable, countably metacompact, regular space with a closed discrete set which is not a $G_\delta$ is constructed from the mutually consistent assumptions that $b = \omega_1$ and there can exist a Q-set.

1. Introduction

Recall that a space is perfect if each closed subset is a $G_\delta$. The easy but important result that all perfect spaces are countably metacompact raises the natural question: how perfect are countably metacompact spaces? In [Bu2] Burke proved that under PMEA closed discrete sets are $G_\delta$'s in first countable countably metacompact $T_1$ spaces.

Given a maximal almost disjoint (mad) family $\mathcal{A} \subseteq [\omega_1]^\omega$ we define the space $\Psi(\mathcal{A})$ as: $\omega_1 \cup \mathcal{A}$ is the underlying set. Every point in $\omega_1$ is isolated while a typical neighborhood of an $a \in \mathcal{A}$ looks like $\{a\} \cup a^{-1}y$ where $y$ is a finite subset of $a$. Then $\Psi(\mathcal{A})$ is a regular, first countable space and $\mathcal{A}$ is a closed discrete set which is not a $G_\delta$. So if there exists a mad $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is countably metacompact, there would be a nice counterexample to the PMEA result. In [Bu1] Burke raised this question and answered it negatively under the assumption $a = c$. In this note we again answer the question in the negative under a number of different assumptions, including $c = \aleph_2$.

There are only two consistent counterexamples to Burke's PMEA theorem in the literature. In [Sh] Shelah forced a normal countably metacompact ladder system space with a closed discrete set which is not a $G_\delta$, while in [BBu] Balogh and Burke constructed a regular counterexample in a ccc forcing extension. Assuming $b = \omega_1$ and there can exist a Q-set, we construct a regular first countable countably metacompact space with a closed discrete set which is not a $G_\delta$. This is the only known regular counterexample to the PMEA result which is not a forcing construction.


1991 Mathematics Subject Classification. Primary 03E05, 54D18, 54G20.

Key words and phrases. Countably metacompact, perfect, mad, unbounded, Q-set.

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0002-9939/94 $1.00 + .25$ per page

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Our terminology and notation are fairly standard. \([\omega_1]^\omega\) = the collection of countably infinite subsets of \(\omega_1\). \(\mathcal{A}\) indicates an infinite mad family on \([\omega_1]^\omega\). \(a\) is defined as the minimal cardinality of an infinite mad family on \(\omega\).

For \(f, g \in \omega_1^\omega\), \(f \preceq g\) means \(g(n) > f(n)\) for at most finitely many \(n\). \(b\) is the minimal cardinality of an unbounded family in \((\omega_1^\omega, \preceq^*)\), and \(\theta\) is the minimal cardinality of a dominating family in \((\omega_1^\omega, \preceq^*\). \(X \subseteq Y\) means that \(X\setminus Y\) is finite. Given \(Y \subseteq \omega_1\), \(\mathcal{A} \upharpoonright Y = \{a \cap Y : a \in \mathcal{A}\text{ s.t. } a \cap Y = \aleph_0\}\) and \(\Psi(\mathcal{A} \upharpoonright Y)\) is the subspace of \(\Psi(\mathcal{A})\) determined by \(Y \cup (\mathcal{A} \upharpoonright Y)\). For more on \(b, \delta, a,\) and \(\Psi(\mathcal{A})\) see [vD].

2. Preliminaries

We will use the following formulation of countable metacompactness.

**Definition 2.1.** A space \(X\) is countably meta\-compact iff for every decreasing sequence \(\{D_n : n < \omega\}\) of closed subsets of \(X\) such that \(\bigcap_{n<\omega} D_n = \emptyset\) there exist open \(U_n \supseteq D_n\) such that \(\bigcap_{n<\omega} U_n = \emptyset\).

The following lemma follows directly from the definitions.

**Lemma 2.2.** Given a mad \(\mathcal{A} \subseteq [\omega_1]^\omega\), \(\Psi(\mathcal{A})\) is countably metacompact iff for every partition \(\{\mathcal{A}_n : n \in \omega\}\) of \(\mathcal{A}\) there are \(X_n \subseteq \omega_1\) such that \(\forall n \forall m \geq n \forall a \in \mathcal{A}_m\ (a \subseteq^* X_n\) and \(\bigcap_{n<\omega} X_n = \emptyset\).

**Theorem 2.3.** Suppose \(\mathcal{C}\) is regular. Given a mad \(\mathcal{A} \subseteq [\omega_1]^\omega\), if \(\exists Z \in [\omega_1]^\omega\) such that \(|\mathcal{A} \upharpoonright Y| = \mathcal{C}\) for each \(Y \in [Z]^\omega\), then \(\Psi(\mathcal{A})\) is not countably metacompact.

**Proof.** Enumerate \(\{x \in [\omega_1]^\omega : |\mathcal{A} \upharpoonright x| = \mathcal{C}\}\) as \(\{x_\alpha : \alpha \leq \mathcal{C}\}\). Notice that for each \(X \in [\omega_1]^\omega\) if \(|\mathcal{A} \upharpoonright X| = \mathcal{C}\) then there is an \(x \in [X]^\omega\) s.t. \(|\mathcal{A} \upharpoonright x| = \mathcal{C}\).

For each \(n \in \omega\) we construct \(\mathcal{A}_n \subseteq \mathcal{A}\) inductively on \(\alpha \leq \mathcal{C}\) as follows. Fix \(a_n \in \mathcal{A}\) such that \(\forall n \neq m \ a_n \neq a_m\), and \(\forall n \ |a_n \cap x_\alpha| = \aleph_0\). Let \(\mathcal{A}_0^n = \{a_n\}\).

Having defined \(\mathcal{A}_n^\beta\) for all \(\beta \leq \alpha\) s.t.

(i) \(|\mathcal{A}_n^\beta| = |\beta|\),
(ii) \(\forall n \neq m \ \mathcal{A}_n^\beta \cap \mathcal{A}_m^\beta = \emptyset\),
(iii) \(\forall \beta \in \alpha \exists a \in \mathcal{A}_n^\beta\ (a \cap x_\beta) = \aleph_0\).

Let \(\mathcal{A}' = \bigcup_{\beta \in \alpha} \{\mathcal{A}_n^\beta : n \in \omega, \ \beta \in \alpha\}\). For each \(n \in \omega\) choose \(a_n^\alpha \in \mathcal{A} - \mathcal{A}'\) distinct such that \(|a_n^\alpha \cap x_\alpha| = \aleph_0\). Let \(\mathcal{A}_0^n = \mathcal{A}_0^n \cup \{a_n^\alpha\}\). Finally let \(\mathcal{A}_n = \bigcup_{\alpha \in \mathcal{C}} \mathcal{A}_0^n\). Notice that if \(x \in [\omega_1]^\omega\) is such that \(|\mathcal{A} \upharpoonright x| = \mathcal{C}\) then for each \(n\) there is an \(a \in \mathcal{A}_n\) such that \(|a \cap x| = \aleph_0\). Fix \(n \in \omega\). If \(X\) is such that \(\forall m \geq n \forall a \in \mathcal{A}_m\ a \subseteq^* X\), then \(|X\setminus Z|\) is countable. Hence, the \(\mathcal{A}_n\) witness that \(\Psi(\mathcal{A})\) is not countably metacompact.

The following is a corollary to the proof of Theorem 2.3.

**Corollary 2.4 (Burke).** \(\mathcal{A} = \mathcal{C}\) \(\rightarrow\) \(\Psi(\mathcal{A})\) is not countably metacompact for every mad \(\mathcal{A} \subseteq [\omega_1]^\omega\).

3. Unbounded families and partitions of mad families

By Lemma 2.2, to prove that \(\Psi(\mathcal{A})\) is not countably metacompact for some mad \(\mathcal{A}\) we must exhibit a nasty (i.e., witnessing not countable metacompact-
ness) partition of \(\mathcal{A}\) into countably many pieces. We do this by indexing \(\mathcal{A}\)
with a family $\mathcal{F} \subseteq \omega_\omega$ and proving that if $\mathcal{F}$ has certain nice properties then we can build a nasty partition for $\mathcal{A}$. A similar technique was used by Simon in [S] to build a Frechet space whose square is not Frechet.

**Definition 3.1.** Given $\mathcal{F} \subseteq \omega_\omega$, we say $\mathcal{F}$ is fully unbounded if $\forall S \in [\mathcal{F}]^{\mathcal{F}}$ $S$ is unbounded under $\leq^*$.

Recall that if $f, g \in \omega_\omega$, then $f \preceq^* g \Leftrightarrow \{n : g(n) > f(n)\}$ is finite. Clearly no family of size $\kappa$ where $\kappa < b$ or $\kappa$ is regular and $> c$ can be fully unbounded. However, if $b \leq \kappa \leq c$ then we have positive, consistent, and independent results.

**Theorem 3.2.** (i) There are fully unbounded families of size $b$ and $c$.

(ii) Let $\kappa < \delta < \lambda$ be regular uncountable cardinals. Then $\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + (b = \kappa) + (\delta = \lambda) \text{ and there is no fully unbounded family of size } \delta)$.

(iii) Let $\kappa$ be a regular uncountable cardinal. Then $\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + (b = \aleph_1) + (\delta = c = \kappa) \text{ and there is a fully unbounded family of size } \delta \text{ for each uncountable } \delta < c)$.

**Proof.** (i) Fix a well-ordered unbounded family of type $b$ and a dominating family $\{f_\alpha : \alpha < \delta\}$ such that $\alpha < \beta \rightarrow f_\beta \preceq^* f_\alpha$. Then both families are fully unbounded.

(ii) We start with a model $M$ of CH and iterate the dominating real poset along the well-founded poset $((\kappa \times \lambda, \leq), \{\alpha, \beta\} \leq (\gamma, \eta) \iff \alpha \leq \gamma \text{ and } \beta \leq \eta$. This is Hechler's model [H] for cofinally embedding $\kappa \times \lambda$ into $(\omega_\omega, \preceq^*)$. If $f_{(\alpha, \beta)}$ is the $(\alpha, \beta)$th function added, then $(\alpha, \beta) \leq (\gamma, \eta)$ and $(\alpha, \beta) \neq (\gamma, \eta)$ implies that $f_{(\alpha, \beta)} \preceq^* f_{(\gamma, \eta)}$. Let $\mathcal{F} \subseteq \omega_\omega$ be of size $\delta$. For each $f \in \mathcal{F}$ there is $(\alpha_f, \beta_f)$ s.t. $f \in M[G_{(\alpha_f, \beta_f)}]$. Fix $\alpha$ such that $\mathcal{F}^0 = \{f : \alpha_f = \alpha\}$ has size $\delta$. Then there is a $\beta$ above $\beta_f : f \in \mathcal{F}^0$. Therefore, $\mathcal{F}^0 \subseteq M[G_{(\alpha, \beta)}]$, which implies that $\mathcal{F}^0 \preceq^* f_{(\alpha, \beta)}$. Therefore, $\mathcal{F}$ is not fully unbounded.

(iii) Start with a model of $\text{MA} + c = \kappa$ and add $\kappa$ Cohen reals. It is straightforward to prove that $b = \aleph_1$, $c = \kappa$, and for any $\delta \leq \kappa$ uncountable the family consisting of the first $\delta$ Cohen reals is fully unbounded.

**Theorem 3.3.** Let $\mathcal{A} \subseteq [\omega_1]^{\omega_1}$ be mad such that

$$\forall x \in [\omega_1]^{\omega_1} \quad |\mathcal{A} \setminus x| \geq \aleph_0 \rightarrow |\mathcal{A} \setminus x| = |\mathcal{A}| = \kappa.$$ 

Assume further that there exists an $\mathcal{F} \subseteq \omega_\omega$ such that $|\mathcal{F}| = \kappa$ and $\mathcal{F}$ is fully unbounded; then $\mathcal{W}(\mathcal{A})$ is not countably metacompact.

**Proof.** Fix $\mathcal{F}$ as in the hypothesis of the theorem and index $\mathcal{A}$ as $\{a_f : f \in \mathcal{F}\}$. For each $n, m \in \omega$, let $\mathcal{A}_m = \{a_f : f(n) = m\}$; then, for each $n$, $\mathcal{A} = \bigcup_{m<\omega}\mathcal{A}_m^* n$ is a partition of $\mathcal{A}$. Assume $\mathcal{W}(\mathcal{A})$ is countably metacompact; then for each $n$ there is $(U^*_n)_{m<\omega} \subseteq [\omega_1]^{\omega_1}$, such that $\bigcap_{m<\omega} U^*_n = \varnothing$ and, $\forall m \forall k > m \forall a \in \mathcal{A}_k \cap a \preceq^* U^*_n$.

For each $n$, choose $h(n)$ inductively such that $|\bigcap_{k \leq n} \omega_1 \setminus U^*_h(k)| = \aleph_1$. Clearly this can be done since $\forall n \bigcup_{m<\omega} \omega_1 \setminus U^*_n = \omega_1$. Construct $x \in [\omega_1]^{\omega_1}$ such that $\forall k x \preceq^* \omega_1 \setminus U^*_h(k)$ and such that $|\mathcal{A} \setminus x| = |\mathcal{A}|$ as follows:

Let $y_0$ be a pseudo-intersection of $\{\omega_1 \setminus U^*_h(j)\}_{j<\omega}$ and pick $a_0 \in \mathcal{A}$, such that $|a_0 \cap y_0| = \aleph_0$. Having constructed $\{y_k : k < n\}$ and $\{a_k : k < n\}$, such
that

(i) $y_k \subseteq U_{h(k)}^k$,

(ii) $y_k$ is a pseudo-intersection of $\{\omega_1 \setminus U_{h(j)}^j\}_{j<\omega}$,

(iii) $|a_k \cap y_k| = \aleph_0$, and

(iv) $i \neq k$ implies $a_i \neq a_k$,

let $y_n$ be a pseudo-intersection of $\{\omega_1 \setminus U_{h(n)}^n\}$ such that every ordinal in $y_n$ is above every ordinal in $\bigcup_{k<n} \alpha_k$. Pick $a_n \in \mathcal{A}$ such that $|a_n \cap y_n| = \aleph_0$. Let $x = \bigcup_{n<\omega} y_n$. Then $\forall k < \omega \ |a_k \cap x| = \aleph_0$. Therefore, $|\mathcal{A} \uparrow x| \geq \aleph_0$; hence, by assumption $|\mathcal{A} \uparrow x| = \kappa$.

Since $\mathcal{F}$ is fully unbounded, fix $f \in \mathcal{F}$ and $n \in \omega$ such that $|a_f \cap x| = \aleph_0$ and $h(n) < f(n)$. Therefore, $a_f \subseteq U_{h(n)}^n$, contradicting that $a_f \cap x$ is infinite and $x \cap U_{h(n)}^n$ is finite.

Corollary 3.4. If $\mathfrak{c} = \aleph_2$ or $\mathfrak{b}^+ = \mathfrak{c}$, then, for each $\mathcal{A} \subseteq [\omega_1]^{\omega}$ mad, $\Psi(\mathcal{A})$ is not countably metacompact.

Proof. Let $\mathcal{A}$ be mad. Then, since $\mathfrak{b} \leq \omega$, either $\forall Y \in [\omega_1]^{\omega} \ |\mathcal{A} \uparrow Y| = \mathfrak{c}$, or $\exists X \in [\omega_1]^{\omega}$ such that $\forall Y \in [X]^{\omega} \ |\mathcal{A} \uparrow Y| = \mathfrak{b}$. In the first case Theorem 2.3 implies $\Psi(\mathcal{A})$ is not countably metacompact, while the second case follows from Theorem 3.3 and the existence of a fully unbounded family of size $\mathfrak{b}$.

Theorem 3.3 suggests the following question: Does the existence of a mad family of size $\kappa$ imply the existence of a fully unbounded family of size $\kappa$? Since there are no fully unbounded families of regular size $> \omega$, the question is only interesting for mad families of singular cardinality or of size $\leq \omega$.

4. A CONSISTENT COUNTEREXAMPLE

We present in this section the construction of a regular, first countable, countably metacompact space $X$ with a closed discrete subset which is not a $G_\delta$.

The space is constructed under the consistent assumption that $\mathfrak{b} = \omega_1$ and there exists a $Q$-set.

Definition 4.1. An uncountable subset of the reals is called a $Q$-set if every subset is a relative $G_\delta$.

The following lemma was proved by Todorcevic (see [T, Lemma 2.5]). The set function $H$ was used there to construct, among other things, a compact $S$-space from the assumption that $\mathfrak{b} = \omega_1$ and $\mathfrak{b} = \aleph_1$.

Lemma 4.2 (Todorcevic). Assume $\mathfrak{b} = \omega_1$. Fix $Z \subseteq \omega^\omega$. There is a set function $H : Z \to [Z]^{\omega}$ such that, for each $z \in Z$, $H(z)$ is either finite or a sequence converging to $z$ with the property that, if $Y$ and $D$ are subsets of $Z$ such that $Y \subseteq D$ and $Y$ is uncountable, then $\{y \in Y : H(y) \cap D \text{ is finite}\}$ is countable.

Fix $Z \subseteq \omega^\omega$ a $Q$-set of size $\aleph_1$. We define a topology on $X = Z \times 2$ using $H$ by letting $Z \times \{0\}$ be isolated in $X$, and a typical neighborhood of $(z, 1)$ looks like $\{(z, 1)\} \cup [H(z) \setminus F] \times \{0\}$ for some finite set $F$. Clearly $X$ with this topology is first countable and regular. The fact that $Z$ is a $Q$-set implies that the space is countably metacompact while the set function $H$ assures that $Z \times \{1\}$ is not a $G_\delta$ in $X$.

Claim 4.3. $X$ is countably metacompact.
Proof. Suppose \( Z_0 \supseteq Z_1 \supseteq \cdots \) is a sequence of subsets of \( Z \) such that \( \bigcap_{i<\omega} Z_i = \emptyset \). So \( \{Z_i \times \{1\} : i < \omega\} \) is a typical decreasing sequence of closed sets in \( X \). We use the fact that each \( Z_i \) is a \( G_\delta \) in \( Z \) to construct the open fattenings \( U_i \) of \( Z_i \times \{1\} \) with \( \bigcap_{i<\omega} U_i = \emptyset \). For each \( i < \omega \) there are Euclidean openings \( V^i(n) \supseteq Z_i \) such that \( \bigcap_{n<\omega} V^i(n) \cap Z = Z_i \). Without loss of generality we may assume that, for each \( n \) and each \( i < j \), \( V^i(n) \subseteq V^j(n) \). Then, for each \( i < \omega \), \( V^i(n) \times \{0\} \cup Z_i \times \{1\} \) is open in \( X \).

Let \( U_i = V^i(i) \times \{0\} \cup Z_i \times \{1\} \). It is straightforward to verify that \( \bigcap_{i<\omega} U_i = \emptyset \).

**Claim 4.4.** \( Z \times \{1\} \) is not a \( G_\delta \) in \( X \).

**Proof.** It clearly suffices to prove that if \( U \) is an open neighborhood of \( Z \times \{1\} \) then \( Z \times \{0\} \setminus U \) is countable.

Suppose \( Y = \{z : (z, 0) \notin U\} \) is uncountable. Fix \( D \subseteq Y \) countable dense in the Euclidean topology on \( Y \). Then by Lemma 4.2 there is a \( y \in Y \) such that \( H(y) \cap D \) is infinite. Therefore \( (y, 1) \in D \times \{0\} \), which is a contradiction.

Unfortunately \( X \) is not normal. This follows from the next claim, that \( X \) is not countably paracompact, and from the fact that normal, countably metacompact spaces are countably paracompact.

**Claim 4.5.** \( X \) is not countably paracompact.

**Proof.** Let \( \{X_n : n < \omega\} \) be a decreasing sequence of subsets of \( Z \) such that in the Euclidean topology, each \( X_n \) is \( \aleph_1 \)-dense in \( Z \). Let \( A \) be countable and dense in \( Z \). Fix \( n < \omega \) and \( U_n \supseteq X_n \times \{1\} \) an open subset of \( X \). By Lemma 4.2, \( \{x \in X_n : H(x) \cap A \text{ is finite}\} \) is countable. Therefore, \( X'_n = \{x \in X_n : H(x) \cap A \text{ is infinite}\} \) is \( \aleph_1 \)-dense in \( Z \). Letting \( A' = A \times \{0\} \cap U_n \), Lemma 4.2 implies that \( (X \times \{0\}) \cap \overline{U_n} \) is cocountable. Therefore, for any sequence of open sets \( U_n \supseteq X_n \times \{1\} \), \( \bigcap_{n<\omega} U_n \neq \emptyset \).

In [FM] Fleissner and Miller prove the consistency of the existence of a Q-set concentrated on a countable set. As \( b = \omega_1 \) is equivalent to the existence of an uncountable set of reals concentrated on a countable set \([R]\), \( b = \omega_1 \) and the existence of a Q-set are mutually consistent with ZFC.

**References**


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