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HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN SPHERES

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ABSTRACT. Let M^n be a compact hypersurface of a sphere with constant mean curvature H. We introduce a tensor ϕ, related to H and to the second fundamental form, and show that if $|\phi|^2 \leq B_H$, where $B_H \neq 0$ is a number depending only on H and n, then either $|\phi|^2 \equiv 0$ or $|\phi|^2 \equiv B_H$. We also characterize all M^n with $|\phi|^2 \equiv B_H$.

1. INTRODUCTION

(1.1) Let M^n be an n-dimensional orientable manifold and let $f\colon M^n \to S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an immersion of M into the unit $(n+1)$-sphere $S^{n+1}(1)$ of the euclidean space $\mathbf{R}^{n+2}$. Choose a unit normal field η along f, and denote by $A\colon T_pM \to T_pM$ the linear map of the tangent space T_pM, at the point $p \in M$, associated to the second fundamental form of f along η, i.e.,

$$\langle AX, Y\rangle = \langle \overline{\nabla}_X Y, \eta\rangle,$$

where X and Y are tangent vector fields on M and $\overline{\nabla}$ is the connection of $S^{n+1}(1)$. A is a symmetric linear map and can be diagonalized in an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM, i.e., $Ae_i = k_i e_i$, $i = 1, \dots, n$. We will denote by $H = \frac{1}{n}\sum_i k_i$ the mean curvature of f and by $|A|^2 = \sum_i k_i^2$.

When f is minimal $(H = 0)$ the following gap theorem is well known.

(1.2) **Theorem.** *Let M^n be compact and $f\colon M^n \to S^{n+1}(1)$ be a minimal hypersurface. Assume that $|A|^2 \leq n$, for all $p \in M$. Then:*

(i) *Either $|A|^2 \equiv 0$ (and M^n is totally geodesic) or $|A|^2 \equiv n$.*
(ii) *$|A|^2 \equiv n$ if and only if M^n is a Clifford torus in $S^{n+1}(1)$, i.e., M^n is a product of spheres $S^{n_1}(r_1) \times S^{n_2}(r_2)$, $n_1 + n_2 = n$, of appropriate radii.*

(1.3) *Remark.* The sharp bound (i) is due to Simons [S]. The characterization given in (ii) was obtained independently by Chern, do Carmo, and Kobayashi [CdCK] and Lawson [L]. The result in (ii) is local.

Attempts have been made to extend the above result to hypersurfaces with constant mean curvature H (see, e.g., Okumura [O]), but as far as we know no

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sharp bound has yet been found. The purpose of this paper is to describe such a sharp bound and characterize the hypersurfaces that appear when the bound is reached.

For that, it is convenient to define a linear map $\phi\colon T_pM \to T_pM$ by

$$\langle \phi X, Y\rangle = H\langle X, Y\rangle - \langle AX, Y\rangle.$$

It is easily checked that trace $\phi = 0$ and that

$$|\phi|^2 = \frac{1}{2n}\sum_{i,j}(k_i - k_j)^2, \qquad i, j = 1, \dots, n,$$

so that $|\phi|^2 = 0$ if and only if M is totally umbilic.

It turns out that ϕ is the natural object to use when extending the above theorem to constant mean curvature. In fact, Theorem 1.5 below can be proved.

We need some notation. An $H(r)$-torus in $S^{n+1}(1)$ is obtained by considering the standard immersions $S^{n-1}(r) \subset \mathbf{R}^n$, $S^1(\sqrt{1-r^2}) \subset \mathbf{R}^2$, $0 < r < 1$, where the value within the parentheses denotes the radius of the corresponding sphere, and taking the product immersion $S^{n-1}(r) \times S^1(\sqrt{1-r^2}) \hookrightarrow \mathbf{R}^n \times \mathbf{R}^2$. By the choices made, the $H(r)$-torus turns out to be contained in $S^{n+1}(1)$ and has principal curvatures given, in some orientation, by

$$k_1 = \cdots = k_{n-1} = \frac{\sqrt{1-r^2}}{r}, \qquad k_n = -\frac{r}{\sqrt{1-r^2}}, \tag{1.4}$$

or the symmetric of these values for the opposite orientation.

Let M^n be compact and orientable, and let $f\colon M^n \to S^{n+1}(1)$ have constant mean curvature H; choose an orientation for M such that $H \geq 0$. For each H, set

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2+1),$$

and let B_H be the square of the positive root of $P_H(x) = 0$. Notice that for $H = 0$, $B_0 = n$.

(1.5) **Theorem.** *Assume that $|\phi|^2 \leq B_H$ for all $p \in M$. Then:*

(i) *Either $|\phi|^2 \equiv 0$ (and M is totally umbilic) or $|\phi|^2 \equiv B_H$.*

(ii) *$|\phi|^2 \equiv B_H$ if and only if:*

(a) *$H = 0$ and M^n is a Clifford torus in $S^{n+1}(1)$.*

(b) *$H \neq 0$, $n \geq 3$, and M^n is an $H(r)$-torus with $r^2 < \frac{n-1}{n}$.*

(c) *$H \neq 0$, $n = 2$, and M^n is an $H(r)$-torus with $r^2 \neq \frac{n-1}{n}$.*

(1.6) *Remark.* As it will be seen in the proof, part (ii) of Theorem (1.5) is again a local result.

(1.7) *Remark.* It is an interesting fact that not all $H(r)$-tori appear in the equality case for $n \geq 3$, but only those for which $r^2 < (n-1)/n$ (it can be checked that if we orient those $H(r)$-tori for which $r^2 > (n-1)/n$ in such a way that $H \geq 0$, then $|\phi|^2 > B_H$). This has to do with the fact that the term which contains H in the equation $P_H(x) = 0$ vanishes when $n = 2$. Thus, if $H \neq 0$, the equation defining B_H is invariant by a change of orientation if and only if $n = 2$.

(1.8) *Remark.* In the minimal case, Theorem (1.2) can be extended to higher codimensions (see [CdCK]). In her doctoral dissertation of IMPA, Walcy Santos has also been able to extend Theorem (1.5) to higher codimensions (for the precise statement in this case, see [Sa]).

2. Proof of Theorem (1.5)

(2.1) We first compute the Laplacian $\Delta\phi$ of ϕ. We first observe that given a Riemannian manifold M and a symmetric linear map on the tangent spaces of M that satisfy formally the Codazzi equation, Cheng and Yau [CY] have already computed such a Laplacian. This turns out to be the case for ϕ, and the result of [CY] in our context can be described as follows.

Let $\{e_1, \dots, e_n\}$ be an orthonormal frame which diagonalizes ϕ at each point of M, i.e., $\phi e_i = \mu_i e_i$, and let ∇ be the induced connection on M. Then [CY, p. 198]

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \sum_i \mu_i (tr\phi)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\mu_i - \mu_j)^2, \tag{2.2}$$

where R_{ijij} is the sectional curvature of the plane $\{e_i, e_j\}$.

We first compute the last term on the right-hand side of (2.2). By the definition of ϕ, $\mu_i = H - k_i$ and, by Gauss's formula,

$$R_{ijij} = 1 + k_i k_j = 1 + \mu_i\mu_j - H(\mu_i + \mu_j) + H^2.$$

We now use a result of Nomizu and Smyth [NS, p. 372] which implies, since $tr\phi = 0$, that

$$\frac{1}{2}\sum_{i,j}(1 + \mu_i\mu_j)(\mu_i - \mu_j)^2 = n\sum_i \mu_i^2 - \left(\sum_i \mu_i^2\right)^2.$$

Therefore, since $\sum_{i,j}(\mu_i - \mu_j)^2 = 2n|\phi|^2$, we obtain

(2.3)

$$\begin{aligned}\frac{1}{2}\sum_{i,j} R_{ijij}(\mu_i - \mu_j)^2 &= n\sum_i \mu_i^2 - \left(\sum_i \mu_i^2\right)^2 \\ &\quad - \frac{H}{2}\sum_{i,j}(\mu_i + \mu_j)(\mu_i - \mu_j)^2 + \frac{H^2}{2}\sum_{i,j}(\mu_i - \mu_j)^2 \\ &= n|\phi|^2 - |\phi|^4 + nH^2|\phi|^2 - \frac{H}{2}\sum_{i,j}(\mu_i + \mu_j)(\mu_i - \mu_j)^2.\end{aligned}$$

On the other hand, since $\sum_i \mu_i = 0$, it is easily checked that

$$\frac{1}{2}\sum_{i,j}(\mu_i + \mu_j)(\mu_i - \mu_j)^2 = n\sum_i \mu_i^3. \tag{2.4}$$

It follows from (2.3) and (2.4) that (2.2) can be written as

$$\tfrac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + n|\phi|^2 + nH^2|\phi|^2 - nH\sum_i \mu_i^3. \tag{2.5}$$

We want to estimate $\sum_i \mu_i^3$. For that, we use the following lemma, the inequality case of which is stated in Okumura [O].

(2.6) **Lemma.** *Let μ_i, $i = 1, \dots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = \text{const} \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3,$$

and equality holds in the right-hand (left-hand) side if and only if $(n-1)$ of the μ_i's are nonpositive and equal ($(n-1)$ of the μ_i's are nonnegative and equal).

Proof of the lemma. We can assume that $\beta > 0$, and use the method of Lagrange's multipliers to find the critical points of $g = \sum_i \mu_i^3$ subject to the conditions: $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = \beta^2$. It follows that the critical points are given by the values of μ_i that satisfy the quadratic equation

$$\mu_i^2 - \lambda\mu_i - \alpha = 0, \qquad i = 1, \dots, n.$$

Therefore, after reenumeration if necessary, the critical points are given by:

$$\mu_1 = \mu_2 = \cdots = \mu_p = a > 0, \qquad \mu_{p+1} = \mu_{p+2} = \cdots = \mu_n = -b < 0.$$

Since, at the critical points,

$$\beta^2 = \sum_i \mu_i^2 = pa^2 + (n-p)b^2,$$

$$0 = \sum_i \mu_i = pa - (n-p)b,$$

$$g = \sum_i \mu_i^3 = pa^3 - (n-p)b^3,$$

we conclude that

$$a^2 = \frac{n-p}{pn}\beta^2, \qquad b^2 = \frac{p}{(n-p)n}\beta^2, \qquad g = \left(\frac{n-p}{n}a - \frac{p}{n}b\right)\beta^2.$$

It follows that g decreases when p increases. Hence g reaches a maximum when $p = 1$, and the maximum of g is given by

$$\begin{aligned} a^3 - (n-1)b^3 &= ((n-1)b)^3 - (n-1)b^3 = (n-2)n(n-1)b^2 b \\ &= \frac{n-2}{\sqrt{n(n-1)}}\beta^3. \end{aligned}$$

Since g is symmetric, this proves the lemma.

(2.7) *Remark.* For later use, it is convenient to observe from the proof that the equality holds in the right-hand side if and only if $(n-1)$ μ_i's are of the form $-b = -(1/n(n-1))^{1/2}\beta$ and the remaining one is $a = ((n-1)/n)^{1/2}\beta$.

(2.8) We return to the proof of Theorem (1.5). By using Lemma (2.6) in (2.5), we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\phi|^2 &\geq |\nabla\phi|^2 - |\phi|^4 + n(H^2+1)|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 \\ &= |\nabla\phi|^2 + |\phi|^2\left(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(H^2+1)\right). \end{aligned}$$

Integrating both sides of the above inequality, using Stokes' theorem and the hypothesis, we conclude that

$$0 \geq \int_M |\nabla\phi|^2 + \int_M |\phi|^2 \left(-|\phi|^2 - \frac{n(n-2)}{\sqrt{(n(n-1)}} H|\phi| + n(H^2+1) \right) \geq 0.$$

Thus $|\nabla\phi|^2 \equiv 0$ and either $|\phi|^2 \equiv 0$ or $|\phi|^2 \equiv B_H$. This proves part (i) of Theorem (1.5).

We now consider part (ii). Notice first that if $|\phi|^2 \equiv B_H$, the right-hand side of inequality (2.8) vanishes identically irrespective of the compactness of M. Since this is all that we will use, the remaining part of the argument is local.

If $H = 0$, the theorem reduces to Theorem (1.2) which gives (ii)(a).

If $H \neq 0$, we conclude that $\nabla\phi = 0$ and that equality holds in the right-hand side of Lemma (2.6). It follows that $k_i = \text{const}$ and $(n-1)$ of the k_i's are equal (see, e.g., [CdCK, p. 67]). After reenumeration if necessary, we can assume that

$$k_1 = k_2 = \cdots = k_{n-1}, \qquad k_1 \neq k_n, \ k_i = \text{const}.$$

In this situation, if $n \geq 3$, a theorem of do Carmo and Dajczer [dCD, p. 701] implies that M^n is (contained in) a rotation hypersurfaces of $S^{n+1}(1)$ obtained by rotating a curve of constant curvature. It follows that M^n is an $H(r)$-torus.

To identify which $H(r)$-tori do appear, we first observe that the equality case of Lemma (2.6) gives (with the enumeration above):

$$\mu_n = \sqrt{\frac{n-1}{n}}|\phi|, \qquad \mu_1 = \mu_2 = \cdots = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}}|\phi|.$$

Thus

$$k_n k_1 = \left(H - \sqrt{\frac{n-1}{n}}|\phi| \right) \left(H + \sqrt{\frac{1}{n(n-1)}}|\phi| \right),$$

hence, since $|\phi|^2 \equiv B_H$,

$$n k_n k_1 = nH^2 - |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| = -n,$$

that is, $k_n k_1 = -1$. On the other hand, from

$$k_n = H - \mu_n = \frac{k_n + (n-1)k_1}{n} - \mu_n,$$

we conclude that

$$(n-1)k_n - (n-1)k_1 = -n\mu_n,$$

and, since $\mu_n > 0$, we obtain that $k_n < k_1$. Because $k_n k_1 = -1$, this implies that $k_n < 0$. It follows that the oriented $H(r)$-torus selected by the equality case of Lemma (2.6) is given by (1.4). Since its mean curvature

$$H = \frac{(n-1) - nr^2}{nr\sqrt{1-r^2}}$$

is positive, we must have $r^2 < (n-1)/n$. This completes the proof of case (b) in (ii).

To prove finally the case (ii)(c), we observe that $M^2 \subset S^3(1)$ is an isoparametric surface in $S^3(1)$ which is known to be either totally umbilic or an $H(r)$-torus. Since $|\phi|^2 \neq 0$, M^2 is an $H(r)$-torus. By the above argument, we see that $k_2 k_1 = -1$. Now, however, because the equality case of Lemma (2.6) gives no additional information, we can have both cases: $k_2 > 0$, $k_1 < 0$ and $k_2 < 0$, $k_1 > 0$. Thus, the (positive) mean curvature can be either

$$H = \frac{(n-1) - nr^2}{nr\sqrt{1-r^2}} \quad \text{or} \quad H = \frac{nr^2 - (n-1)}{nr\sqrt{1-r^2}}, \qquad n = 2,$$

and all $r^2 \neq \frac{n-1}{n}$ will occur. This concludes the proof of (ii)(c) and of the theorem.

3. Further remarks

(3.1) Theorem (1.5) raises the following question: Consider the set of hypersurfaces of $S^{n+1}(1)$ with $H = \text{const}$ and $|\phi| = \text{const}$. Is the set of values of $|\phi|$ discrete? For minimal hypersurfaces, this question was raised in [CdCK], and even in this simple case it was shown to be a hard question. For $n = 3$ and $H = 0$, a significant contribution was given by Peng and Terng [PT] who showed that *if* $3 < |\phi|^2 \leq 6$, $|\phi| = \text{const}$, *then* $|\phi|^2 = 6$ *and* M^3 *is a minimal isoparametric hypersurface of* $S^4(1)$ *with three distinct principal curvatures.*

The result of Peng and Terng was extended to hypersurfaces of $S^4(1)$ with constant mean curvature H by Almeida and Brito [AB]. They proved that *if* $|\phi|^2 = \text{const}$ *and* $|\phi|^2 \leq 6 + 6H^2$, *then* M^3 *is an isoparametric hypersurface of* $S^4(1)$ *with constant mean curvature* H; *furthermore, if* $4 + 6H^2 \leq |\phi|^2 \leq 6 + 6H^2$, *then* $|\phi|^2 = 6 + 6H^2$ *and* M^3 *has three distinct principal curvatures.*

The result of Almeida and Brito solves the above question for $n = 3$ and $|\phi|^2 \leq 6 + 6H^2$ and also throws some light on what happens to the $H(r)$-tori when $H \neq 0$ and $r^2 > \frac{2}{3}$: they are all in the interval $B_H < |\phi|^2 < 4 + 6H^2$ (cf. Remark 1.7).

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