A COHOMOLOGICAL CLASS OF VECTOR BUNDLES

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ABSTRACT. The goal of this paper is to give a cohomological characterization of $F_{n,t}$, where $F_{n,t}:=\text{Ker}((n+t; n)\mathcal{O}_{P^n}(-t) \to \mathcal{O}_{P^n})$.

0. Introduction

Fix an algebraically closed ground field $k$ of characteristic zero. We set $S = k[X_0, \ldots, X_n]$, $m = (X_0, \ldots, X_n) \subset S$, and $\mathbb{P}^n = \text{Proj}(S)$. For all positive integers $a, b$ with $a \geq b$, $((a; b))$ will denote the binomial coefficient $((a; b)) = (a!)/(b!(a-b)!)$.

Choose a basis $v_1, \ldots, v_{((n+t); t)}$, $a_0(n,t) := ((n+t; n))$ of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t))$, $t > 0$. Let $\Phi(n,t) : \mathcal{O}_{\mathbb{P}^n} \to a_0(n,t)\mathcal{O}_{\mathbb{P}^n}(t)$ be the morphism defined by $\Phi(n,t)(c) := (cv_1, \ldots, cv_{((n+t); t)})$. Set $E_{n,t} := \text{Coker}(\Phi(n,t))$ and $F_{n,t} := E_{n,t}^\ast$ its dual. Note that $E_{n,t}$, $F_{n,t}$ are homogeneous and uniform vector bundles on $\mathbb{P}^n$. Furthermore, $E_{n,1} = T_{\mathbb{P}^n}$ and $F_{n,1} = \Omega_{\mathbb{P}^n}^1$, while $F_{n,t}$ and $E_{n,t}$ for $t > 1$, for instance, are as defined in [G, MM]. In [G] they are used to give a new proof of the explicit Noether-Lefschetz Theorem and [MM] (see also [B]) stressed their importance for studying the Hartshorne-Rao module of a space curve.

However, not only the cotangent bundles $\Omega_{\mathbb{P}^n}^1$ are important but so are their exterior powers. So, we define $F_{r,n,t} := \Lambda^r F_{n,t}$ for all $r \geq 1$ with the hope that they will also play an important role in the study of the cohomology groups of the ideal sheaf of closed subschemes of $\mathbb{P}^n$.

In §1 we will compute the cohomology groups and the order of $F_{r,n,t}$ and prove that $F_{n,t}$ are simple vector bundles on $\mathbb{P}^n$. In §2 we restrict our attention to the case $r = 1$ and give the main theorem of this paper. Concretely, given a vector bundle $E$ on $\mathbb{P}^n$, we find sufficient conditions involving only suitably chosen cohomological groups in order that $E$ be the direct sum of $F_{n,t}$ and line bundles. Our essential tool will be the Beilinson spectral sequence.

Notation. For a coherent sheaf $F$ on $\mathbb{P}^n$ we use the abbreviation $sF = F \oplus \cdots \oplus F$ for the $s$-fold direct sum of $F$, $H^i F(d) = H^i(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(d))$, and $h^i F(d) = \dim_k H^i F(d)$.

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First of all, we recall the definitions and basic facts that will be needed throughout this paper.

**Definition 1.1** [E]. Let $E$ be a rank $r$ vector bundle on $\mathbb{P}^n$. We set $o(j)(E) = \inf\{t|im((\bigoplus_j H^jE(l)) = 0\}$. In other words, $o(j)(E) = r$ means that the morphism $H^jE(l) \to H^jE(l+r)$ given by multiplication by any homogeneous form of degree $r$ is zero. The order of $E$ is $o(E) = \max\{o(j)(E)|1 \leq j \leq n-1\}$.

**Proposition 1.2** [Ei, Proposition 1.1]. Let $E$ be a rank $r$ vector bundle on $\mathbb{P}^n$. Assume that $E$ is generated by its global sections. If $H^nE(-n-1) \neq 0$, then $E \cong \mathcal{O}_{\mathbb{P}^n} \oplus F$ for some vector bundle $F$ of rank $r-1$ on $\mathbb{P}^n$.

**Beilinson Theorem** [Be]. Let $F$ be a coherent sheaf on $\mathbb{P}^n$. There is a spectral sequence $E_\infty^{pq}$ with $E_1^{pq} = H^q(\mathbb{P}^n, F(p)) \otimes \mathcal{O}_{\mathbb{P}^n}(-p)$ such that $E_\infty^{pq} = 0$ for $p + q \neq 0$ and $\bigoplus_{p=0}^n E_{\infty}^{-pq}$ is the associated graded sheaf of a filtration of $F$.

**Definition 1.3.** For all integers $r \geq 1$, $n \geq 2$, set $a_0 = a_0(n, r) := ((n + r; n))$ and $F_{n,r} := \ker(a_0\mathcal{O}_{\mathbb{P}^n}(-r) \to \mathcal{O}_{\mathbb{P}^n})$. $F_{n,r}$ are homogeneous and uniform vector bundles of rank $a_0 - 1$ on $\mathbb{P}^n$. Note that $F_{n,1} = \Omega_{\mathbb{P}^n}$, $F_{n,r}\mathbb{P}^{n-1} = F_{n-1,r} \oplus ((n+r-1; r-1))\mathcal{O}_{\mathbb{P}^{n-1}}(-r)$ where $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a hyperplane, and the splitting type of $F_{n,r}$ is $(-r-1, \ldots, -r-1, -r, \ldots, -r)$.

**Proposition 1.4.** For all integers $r \geq 1$, $n \geq 2$ the following hold:

1. $H^iF_{n,r}(t) = 0$ for all $t$, for all $i = 1, 2, \ldots, n-1$.
2. $h^1F_{n,r}(t) = \begin{cases} ((t+n; n)) & \text{if } 0 \leq t \leq r-1, \\ 0 & \text{otherwise}. \end{cases}$
3. $F_{n,r}$ has order $r$.
4. $F_{n,r}$ is $(r+1)$-regular. In particular, $F_{n,r}$ is globally generated for all $t \geq r+1$.

**Proof.** The proof follows from the exact sequence

$(\ast) \quad 0 \to F_{n,r} \to a_0\mathcal{O}_{\mathbb{P}^n}(-r) \to \mathcal{O}_{\mathbb{P}^n} \to 0.$

In [G], Green proves that $F_{n,r}(r)$ is 1-regular. We will compute the precise graded Betti numbers appearing in a minimal free resolution of $F_{n,r}(r)$.

**Corollary 1.5.** For all integers $r \geq 1$, $n \geq 2$, $F_{n,r}$ has a resolution of the following kind:

$0 \to a_n(n, r)\mathcal{O}_{\mathbb{P}^n}(-n-r) \to \cdots \to a_i(n, r)\mathcal{O}_{\mathbb{P}^n}(-r-i) \to \cdots \to a_2(n, r)\mathcal{O}_{\mathbb{P}^n}(-r-2) \to a_1(n, r)\mathcal{O}_{\mathbb{P}^n}(-r-1) \to F_{n,r} \to 0$

where $a_i(n, r) = \sum_{j=1}^i(-1)^{i-j}((n+j; j))a_{i-j}(n, r) + (-1)^i((n+r+i; n))$.

**Proof.** The proof follows after a tedious computation.
Definition 1.6. For all integers $r, p \geq 1$, $n \geq 2$, we define $F_{n,r}^p$ as the $p$th exterior power of the vector bundle $F_{n,r}$; thus, $F_{n,r}^p := \Lambda^p F_{n,r}$.

Fact 1.7. Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then we have the following exact sequences involving alternating and symmetric powers (Eagon-Northcott complexes):

$$0 \to \Lambda^q E \to \Lambda^q F \to \Lambda^q F \otimes G \to \cdots \to F \otimes S^{q-1} G \to S^q G \to 0$$

and

$$0 \to S^q E \to S^{q-1} E \otimes F \to \cdots \to E \otimes \Lambda^{q-1} F \to \Lambda^q F \to \Lambda^q G \to 0.$$ 

Proposition 1.8. For all integers $r \geq 1$, $n \geq 2$, and $a_0 - 1 \geq p \geq 2$, the following hold:

1. $F_{n,r}^p$ is $(p + 1)$-regular. In particular, $F_{n,r}^p$ is globally generated for all $t \geq p(r + 1)$.
2. $H^i F_{n,r}^p(t) = 0$ for all $t$, for all $i = p + 1, \ldots, n - 1$.
3. $F_{n,r}^p$ has order less or equal to $r + p - 2$.

Proof. (1) By Proposition 1.4, $F_{n,r}^p$ is $(r + 1)$-regular. Since we are working in characteristic zero, $F_{n,r}^p := \Lambda^p F_{n,r}$ are direct summands of the $p$-fold tensor product $T^p F_{n,r}$ of $F_{n,r}$ which are $p(r + 1)$-regular.

From the $p$th exterior power of the exact sequence $(\ast)$ taking into account Fact 1.7 we get the exact sequence

$$(\ast \ast) \quad 0 \to F_{n,r}^p(pr) \to ((a_0 ; p))_{P^n} \to F_{n,r}^{p-1}(pr) \to 0.$$ 

Now, (2) and (3) easily follows from the exact sequence $(\ast \ast)$.

It seems not easy to decide whether the vector bundles $F_{n,r}$ are stable or not, however, we can prove that they are at least simple.

Proposition 1.9. For all integers $r \geq 1$, $n \geq 2$, $F_{n,r}$ are simple.

Proof. We tensor the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to a_0 \mathcal{O}_{\mathbb{P}^n}(r) \to F_{n,r}^* \to 0$$

with $F_{n,r}$ and obtain

$$0 \to F_{n,r} \to a_0 F_{n,r}(r) \to F_{n,r} \otimes F_{n,r}^* \to 0.$$ 

The cohomology sequence is as follows:

$$\cdots \to H^0(\mathbb{P}^n, a_0 F_{n,r}(r)) \to H^0(\mathbb{P}^n, F_{n,r} \otimes F_{n,r}^*) \to \cdots$$

$$H^1(\mathbb{P}^n, F_{n,r}) \to H^1(\mathbb{P}^n, a_0 F_{n,r}(r)) \to .$$

From Proposition 1.4, it follows that $H^0(\mathbb{P}^n, F_{n,r} \otimes F_{n,r}^*) \cong H^1(\mathbb{P}^n, F_{n,r}) \cong k$.

Thus, $F_{n,r}$ is simple.

Now, using Beilinson's theorem, we will give sufficient conditions involving only a finite number of suitably chosen cohomology groups in order that a vector bundle $E$ on $\mathbb{P}^n$ be the direct sum of $F_{n,r}$ and line bundles.
Theorem 2.1. Let $E$ be a rank $\rho$ vector bundle on $\mathbb{P}^n$ such that:

1. $H^iE(t) = 0$ for all $t$, for all $i = 2, \ldots, n - 1$;
2. 
   
   $$h^1E(t) = \begin{cases} ((t + r + n ; m)) & \text{if } -r \leq t \leq -1, \\
   0 & \text{otherwise}; \end{cases}$$
3. $H^0E = 0$.

Then $E$ has order $r$ and $E \cong F_{n,r}(r) \oplus (\text{line bundles})$.

Proof. From Definition 1.1 it follows that the order of $E$ is $r$. Let $t = \max\{l|H^nE(l) \neq 0\}$. If $t \geq -n$, then $E(t + n + 1)$ is generated by its global sections and $H^nE(t) \neq 0$. Hence, by Proposition 1.2, $E \cong E_0 \oplus \mathcal{O}_{\mathbb{P}^n}(-t - 1 - n)$ for some $(\rho - 1)$-vector bundle $E_0$ on $\mathbb{P}^n$. Repeating this argument we may assume that $E \cong F \oplus (\bigoplus_i \mathcal{O}_{\mathbb{P}^n}(a_i))$ where $-t - n - 1 \leq a_1 < 0$ and $F$ is a vector bundle on $\mathbb{P}^n$ such that:

1. $H^0F = 0$;
2. $H^iF(t) = 0$ for all $t$, for all $i = 2, \ldots, n - 1$;
3. $H^0F(t) = 0$ for all $t \geq -n$;
4. 
   
   $$h^1F_{n,r}(t) = \begin{cases} ((t + r + n ; n)) & \text{if } -r \leq t \leq -1, \\
   0 & \text{otherwise}. \end{cases}$$

To end the proof it is enough to see that $F \cong F_{n,r}(r)$. We apply Beilinson’s spectral sequence with $E_1$-terms $E_1^{pq} = H^q(\mathbb{P}^n, F(p)) \otimes \Omega_{\mathbb{P}^n}^p(-p)$. The diagram of the $E_1$-terms is as follows:

Since $E_2^{pq} = E_{\infty}$, the only nonzero row is exact with only one exception $\Omega_{\mathbb{P}^n}(1)^{h^1F(-1)}$ where the cokernel is $F$. So, we have the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n}(n)^{h^1F(-n)} \rightarrow \cdots \rightarrow \Omega_{\mathbb{P}^n}(2)^{h^1F(-2)} \rightarrow \Omega_{\mathbb{P}^n}(1)^{h^1F(-1)} \rightarrow F \rightarrow 0.$$

In particular, we get that $c_i(F) = c_i(F_{n,r}(r))$ for $i = 1, \ldots, n$; and rk($F$) = rk($F_{n,r}(r)$). Hence, in order to prove that $F$ and $F_{n,r}(r)$ are isomorphic it is enough to see that there is a monomorphism between $F_{n,r}$ and $F$. First of all, note that applying Hom$(\cdot, F)$ to the exact sequence

$$0 \rightarrow F_{n,r}(r) \rightarrow a_0 \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(r) \rightarrow 0,$$

we get the exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^n}(r), F) \rightarrow a_0 \text{Hom}(\mathcal{O}_{\mathbb{P}^n}, F) \rightarrow \text{Hom}(F_{n,r}(r), F)$$

$$\rightarrow \text{Ext}^1(\mathcal{O}_{\mathbb{P}^n}(r), F) \rightarrow a_0 \text{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, F) \rightarrow \cdots.$$
Since $\text{Hom}(\mathcal{O}_{\mathbb{P}^n}, F) = H^0(\mathbb{P}^n, F) = 0$ and $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, F) = H^1(\mathbb{P}^n, F) = 0$, we conclude that $\text{Hom}(F_{n,r}(r), F) \cong \text{Ext}^1(\mathcal{O}_{\mathbb{P}^n}(r), F) \cong (\mathbb{P}^n, F(-r)) \cong k$.

Similarly, applying $\text{Hom}(\cdot, F_{n,r}(r))$ to the exact sequence (***) we get that $\text{Hom}(F, F_{n,r}(r)) \neq 0$. Now, we choose a nontrivial morphism $\Phi: F_{n,r}(r) \to F$ and $\Psi: F \to F_{n,r}(r)$ and consider the composition $\Psi\Phi: F_{n,r}(r) \to F_{n,r}(r)$. Since $F_{n,r}(r)$ are simple, we have $\Psi\Phi = c\text{Id}_{F_{n,r}(r)}$ for some $c \in k$.

**Claim.** $c \neq 0$.

Since $c$ is a nonzero constant, we conclude that $\Phi$ is a monomorphism, which gives the desired result.

**Proof of the Claim.** Assume that $\Psi\Phi = 0$. Set $a_i = h^1F(-i)$. We have the exact sequences:

\[ 0 \to \Omega^2_{\mathbb{P}^n}(n)^{a_2} \to \cdots \to \Omega^2_{\mathbb{P}^n}(2)^{a_2} \to \Omega^1_{\mathbb{P}^n}(1)^{a_1} \to F_{n,r}(r) \to 0 \]

and

\[ 0 \to \Omega^2_{\mathbb{P}^n}(n)^{a_2} \to \cdots \to \Omega^2_{\mathbb{P}^n}(2)^{a_2} \to \Omega^1_{\mathbb{P}^n}(1)^{a_1} \to F \to 0. \]

Cutting (2) into short exact sequences, we prove that the morphism $\Phi\beta$ can be lifted to a nontrivial morphism $f: \Omega^1_{\mathbb{P}^n}(1)^{a_1} \to \Omega^1_{\mathbb{P}^n}(1)^{a_1}$ in order that the following square commutes:

\[ \begin{array}{ccc} & & \Omega^1_{\mathbb{P}^n}(1)^{a_1} \beta & \to & F_{n,r}(r) & \to & 0 \\ & f & \downarrow & \Phi & \downarrow & \Phi \\ \Omega^1_{\mathbb{P}^n}(1)^{a_1} \gamma & \to & F & \to & 0 \end{array} \]

In the same way we get a commutative diagram:

\[ \begin{array}{ccc} \Omega^2_{\mathbb{P}^n}(2)^{a_2} & \to & \Omega^1_{\mathbb{P}^n}(1)^{a_1} \beta & \to & F_{n,r}(r) & \to & 0 \\ \downarrow & \Phi & \downarrow & \phi & \downarrow & \phi \\ \Omega^2_{\mathbb{P}^n}(2)^{a_2} & \to & \Omega^1_{\mathbb{P}^n}(1)^{a_1} \gamma & \to & F & \to & 0 \\ \downarrow & g & \downarrow & \psi & \downarrow & \psi \\ \Omega^2_{\mathbb{P}^n}(2)^{a_2} & \to & \Omega^1_{\mathbb{P}^n}(1)^{a_1} \beta & \to & F_{n,r}(r) & \to & 0 \end{array} \]

Hence, we have $0 = \Psi\Phi\beta = \beta gf$. Therefore, $\text{Im}(gf) \subset \text{Ker}(\beta) = \text{Im}(\rho_2)$ and $gf$ can be lifted to a nontrivial morphism $h: \Omega^1_{\mathbb{P}^n}(1)^{a_1} \to \text{Im}(\rho_2)$. Finally, applying the functor $\text{Hom}(\Omega^1_{\mathbb{P}^n}(1)^{a_1}, \cdot)$ to the short exact sequence

\[ 0 \to \text{Ker}(\rho_2) \to \Omega^2_{\mathbb{P}^n}(2)^{a_2} \to \text{Im}(\rho_2) = \text{Ker}(\beta) \to 0 \]

and taking into account that $\text{Ext}^1(\Omega^1_{\mathbb{P}^n}(1)^{a_1}, \text{Ker}(\rho_2)) = 0$, we get that $h$ and, hence, $fg$ can be lifted to a nontrivial morphism $\Omega^1_{\mathbb{P}^n}(1)^{a_1} \to \Omega^2_{\mathbb{P}^n}(2)^{a_2}$. This is a contradiction because $\text{Hom}(\Omega^1_{\mathbb{P}^n}(1), \Omega^2_{\mathbb{P}^n}(2)) = 0$. 

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As a corollary, we have the following well-known result:

**Corollary 2.2.** Let $E$ be a rank $\rho$ vector bundle on $\mathbb{P}^n$ such that $H^iE(*) = 0$ for $0 < i < n$ with the only exception $h^1E(-1) = 1$. Then, $E \cong \Omega^1(1) \oplus \text{(line-bundles)}$.

**Proof.** Set $t = \min\{l | H^0E(l) \neq 0\}$. If $t \leq 0$, then $H^iE(t-i-1) = 0$ for $0 \leq i < n$. Hence, by [AO, Theorem 2], $E \cong F \oplus \mathcal{O}(-t)^{h^0E(t)}$ where $F$ is a locally free sheaf on $\mathbb{P}^n$ such that $H^iF(*) = 0$ for $0 < i < n$ with only exception $h^1F(-1) = 1$ and $\min\{l | H^0E(l) \neq 0\} < \min\{l | H^0F(l) \neq 0\}$.

Repeating this process we may assume that $E \cong F \oplus \text{(line-bundles)}$ where $F$ is a locally free sheaf on $\mathbb{P}^n$ such that $H^iF(*) = 0$ for $0 < i < n$ with only exception $h^1F(-1) = 1$ and $\min\{l | H^0F(l) \neq 0\} > 0$. Now, applying Theorem 2.1, we have $F \cong \Omega^1(1) \oplus \text{(line-bundles)}$, which gives the desired result.

**Question 2.4.** Given a vector bundle $E$ on $\mathbb{P}^n$, are there sufficient conditions involving only a finite number of suitably chosen cohomology groups in order that the vector bundle $E$ be the direct sum of $F^*_{\rho,r}$ and line bundles?

**References**


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