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A COHOMOLOGICAL CLASS OF VECTOR BUNDLES

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ABSTRACT. The goal of this paper is to give a cohomological characterization of $F_{n,t}$, where $F_{n,t} := \mathrm{Ker}((n+t;n)\mathcal{O}_{\mathbf{P}^n}(-t) \to \mathcal{O}_{\mathbf{P}^n})$.

0. INTRODUCTION

Fix an algebraically closed ground field $\mathbf{k}$ of characteristic zero. We set $S = \mathbf{k}[X_0, \dots, X_n]$, $m = (X_0, \dots, X_n) \subset S$, and $\mathbf{P}^n = \mathrm{Proj}(S)$. For all positive integers a, b with $a \geq b$, $((a;b))$ will denote the binomial coefficient $((a;b)) = (a!)/(b!(a-b)!)$.

Choose a basis $v_1, \dots, v_{a_0(n,t)}$, $a_0(n,t) := ((n+t;n))$ of $H^0(\mathbf{P}^n \mathcal{O}_{\mathbf{P}^n}(t))$, $t > 0$. Let $\Phi(n,t)\colon \mathcal{O}_{\mathbf{P}^n} \to a_0(n,t)\mathcal{O}_{\mathbf{P}^n}(t)$ be the morphism defined by $\Phi(n,t)(c) := (cv_1, \dots, cv_{a_0(n,t)})$. Set $E_{n,t} := \mathrm{Coker}(\Phi(n,t))$ and $F_{n,t} := E^*_{n,t}$ its dual. Note that $E_{n,t}$, $F_{n,t}$ are homogeneous and uniform vector bundles on $\mathbf{P}^n$. Furthermore, $E_{n,1} = T_{\mathbf{P}^n}$ and $F_{n,1} = \Omega^1_{\mathbf{P}^n}$, while $F_{n,t}$ and $E_{n,t}$ for $t > 1$, for instance, are as defined in [G, MM]. In [G] they are used to give a new proof of the explicit Noether-Lefshetz Theorem and [MM] (see also [B]) stressed their importance for studying the Hartshorne-Rao module of a space curve.

However, not only the cotangent bundles $\Omega^1_{\mathbf{P}^n}$ are important but so are their exterior powers. So, we define $F^r_{n,t} := \Lambda^r F_{n,t}$ for all $r \geq 1$ with the hope that they will also play an important role in the study of the cohomology groups of the ideal sheaf of closed subschemes of $\mathbf{P}^n$.

In §1 we will compute the cohomology groups and the order of $F^r_{n,t}$ and prove that $F_{n,t}$ are simple vector bundles on $\mathbf{P}^n$. In §2 we restrict our attention to the case $r = 1$ and give the main theorem of this paper. Concretely, given a vector bundle E on $\mathbf{P}^n$, we find sufficient conditions involving only suitably chosen cohomological groups in order that E be the direct sum of $F_{n,t}$ and line bundles. Our essential tool will be the Beilinson spectral sequence.

Notation. For a coherent sheaf F on $\mathbf{P}^n$ we use the abbreviation $sF = F \oplus \cdots \oplus F$ for the s-fold direct sum of F, $H^iF(d) = H^i(\mathbf{P}^n, F \otimes \mathcal{O}_{\mathbf{P}^n}(d))$, and $h^iF(d) = \dim_k H^iF(d)$.

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First of all, we recall the definitions and basic facts that will be needed throughout this paper.

Definition 1.1 [E]. Let E be a rank r vector bundle on $\mathbf{P}^n$. We set $o(j)(E) = \inf\{t | m^t(\bigoplus_l H^j E(l)) = 0\}$. In other words, $o(j)(E) = r$ means that the morphism $H^j E(l) \to H^j E(l+r)$ given by multiplication by any homogeneous form of degree r is zero. The order of E is $o(E) = \max\{o(j)(E) | 1 \le j \le n-1\}$.

Proposition 1.2 [Ei, Proposition 1.1]. *Let E be a rank r vector bundle on $\mathbf{P}^n$. Assume that E is generated by its global sections. If $H^n E(-n-1) \ne 0$, then $E \cong \mathcal{O}_{\mathbf{P}^n} \oplus F$ for some vector bundle F of rank $r-1$ on $\mathbf{P}^n$.*

Beilinson Theorem [Be]. *Let F be a coherent sheaf on $\mathbf{P}^n$. There is a spectral sequence E_r^{pq} with E_1-term $E_1^{pq} = H^q(\mathbf{P}^n, F(p)) \otimes \Omega^p_{\mathbf{P}^n}(-p)$ such that $E_\infty^{pq} = 0$ for $p+q \ne 0$ and $\bigoplus_{p=0}^n E_\infty^{-pp}$ is the associated graded sheaf of a filtration of F.*

Definition 1.3. For all integers $r \ge 1$, $n \ge 2$, set $a_0 = a_0(n, r) := ((r+n; n))$ and $F_{n,r} := \operatorname{Ker}(a_0\mathcal{O}_{\mathbf{P}^n}(-r) \to \mathcal{O}_{\mathbf{P}^n})$. $F_{n,r}$ are homogeneous and uniform vector bundles of rank $a_0 - 1$ on $\mathbf{P}^n$. Note that $F_{n,1} = \Omega^1_{\mathbf{P}^n}$, $F_{n,r}|\mathbf{P}^{n-1} \cong F_{n-1,r} \oplus ((n+r-1; r-1))\mathcal{O}_{\mathbf{P}^{n-1}}(-r)$ where $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ is a hyperplane, and the splitting type of $F_{n,r}$ is $(\overbrace{-r-1, \dots, -r-1}^{r \text{ times}}, \underbrace{-r, \dots, -r}_{a_0-r-1 \text{ times}})$.

Proposition 1.4. *For all integers $r \ge 1$, $n \ge 2$ the following hold:*

(1) $H^i F_{n,r}(t) = 0$ *for all* t, *for all* $i = 1, 2, \dots, n-1$.

(2)
$$h^1 F_{n,r}(t) = \begin{cases} ((t+n; n)) & \text{if } 0 \le t \le r-1, \\ 0 & \text{otherwise}. \end{cases}$$

(3) $F_{n,r}$ *has order* r.

(4) $F_{n,r}$ *is* $(r+1)$*-regular. In particular,* $F_{n,r}$ *is globally generated for all* $t \ge r+1$.

Proof. The proof follows from the exact sequence

$$(*) \qquad 0 \to F_{n,r} \to a_0\mathcal{O}_{\mathbf{P}^n}(-r) \to \mathcal{O}_{\mathbf{P}^n} \to 0.$$

In [G], Green proves that $F_{n,r}(r)$ is 1-regular. We will compute the precise graded Betti numbers appearing in a minimal free resolution of $F_{n,r}(r)$.

Corollary 1.5. *For all integers $r \ge 1$, $n \ge 2$, $F_{n,r}$ has a resolution of the following kind:*

$$0 \to a_n(n, r)\mathcal{O}_{\mathbf{P}^n}(-n-r) \to \cdots \to a_i(n, r)\mathcal{O}_{\mathbf{P}^n}(-r-i)$$
$$\to \cdots \to a_2(n, r)\mathcal{O}_{\mathbf{P}^n}(-r-2) \to a_1(n, r)\mathcal{O}_{\mathbf{P}^n}(-r-1) \to F_{n,r} \to 0$$

where $a_i(n, r) = \sum_{j=1}^i (-1)^{j-1}((n+j; j))a_{i-j}(n, r) + (-1)^i((n+r+i; n))$.

Proof. The proof follows after a tedious computation.

Definition 1.6. For all integers $r, p \geq 1$, $n \geq 2$, we define $F^p_{n,r}$ as the pth exterior power of the vector bundle $F_{n,r}$; thus, $F^p_{n,r} := \Lambda^p F_{n,r}$.

Fact 1.7. Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then we have the following exact sequences involving alternating and symmetric powers (Eagon-Northcott complexes):

$$0 \to \Lambda^q E \to \Lambda^q F \to \Lambda^{q-1} F \otimes G \to \cdots \to F \otimes S^{q-1} G \to S^q G \to 0$$

and

$$0 \to S^q E \to S^{q-1} E \otimes F \to \cdots \to E \otimes \Lambda^{q-1} F \to \Lambda^q F \to \Lambda^q G \to 0.$$

Proposition 1.8. *For all integers* $r \geq 1$, $n \geq 2$, *and* $a_0 - 1 \geq p \geq 2$, *the following hold*:

(1) $F^p_{n,r}$ *is* $p(r+1)$*-regular. In particular,* $F^p_{n,r}$ *is globally generated for all* $t \geq p(r+1)$.
(2) $H^i F^p_{n,r}(t) = 0$ *for all* t, *for all* $i = p+1, \dots, n-1$.
(3) $F^p_{n,r}$ *has order less or equal to* $r + p - 2$.

Proof. (1) By Proposition 1.4, $F_{n,r}$ is $(r+1)$-regular. Since we are working in characteristic zero, $F^p_{n,r} := \Lambda^p F_{n,r}$ are direct summands of the p-fold tensor product $T^p F_{n,r}$ of $F_{n,r}$ which are $p(r+1)$-regular.

From the pth exterior power of the exact sequence $(*)$ taking into account Fact 1.7 we get the exact sequence

$$(**) \qquad 0 \to F^p_{n,r}(pr) \to ((a_0\,;\,p))\mathcal{O}_{\mathbf{P}^n} \to F^{p-1}_{n,r}(pr) \to 0.$$

Now, (2) and (3) easily follows from the exact sequence $(**)$.

It seems not easy to decide whether the vector bundles $F_{n,r}$ are stable or not, however, we can prove that they are at least simple.

Proposition 1.9. *For all integers* $r \geq 1$, $n \geq 2$, $F_{n,r}$ *are simple.*

Proof. We tensor the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to a_0 \mathcal{O}_{\mathbf{P}^n}(r) \to F^*_{n,r} \to 0$$

with $F_{n,r}$ and obtain

$$0 \to F_{n,r} \to a_0 F_{n,r}(r) \to F_{n,r} \otimes F^*_{n,r} \to 0.$$

The cohomology sequence is as follows:

$$\cdots \to H^0(\mathbf{P}^n, a_0 F_{n,r}(r)) \to H^0(\mathbf{P}^n, F_{n,r} \otimes F^*_{n,r}) \\ \to H^1(\mathbf{P}^n, F_{n,r}) \to H^1(\mathbf{P}^n, a_0 F_{n,r}(r)) \to .$$

From Proposition 1.4, it follows that $H^0(\mathbf{P}^n, F_{n,r} \otimes F^*_{n,r}) \cong H^1(\mathbf{P}^n, F_{n,r}) \cong k$. Thus, $F_{n,r}$ is simple.

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Now, using Beilinson's theorem, we will give sufficient conditions involving only a finite number of suitably chosen cohomology groups in order that a vector bundle E on $\mathbf{P}^n$ be the direct sum of $F_{n,r}$ and line bundles.

Theorem 2.1. *Let E be a rank ρ vector bundle on $\mathbf{P}^n$ such that:*

(1) $H^iE(t) = 0$ *for all* t, *for all* $i = 2, \ldots, n-1$;

(2)
$$h^1E(t) = \begin{cases} ((t+r+n; m)) & \text{if } -r \le t \le -1, \\ 0 & \text{otherwise}; \end{cases}$$

(3) $H^0E = 0$.

Then E has order r and $E \cong F_{n,r}(r) \oplus$ (*line bundles*).

Proof. From Definition 1.1 it follows that the order of E is r. Let $t = \max\{l \mid H^nE(l) \neq 0\}$. If $t \ge -n$, then $E(t+n+1)$ is generated by its global sections and $H^nE(t) \neq 0$. Hence, by Proposition 1.2, $E \cong E_0 \oplus \mathcal{O}_{\mathbf{P}^n}(-t-1-n)$ for some $(\rho-1)$-vector bundle E_0 on $\mathbf{P}^n$. Repeating this argument we may assume that $E \cong F \oplus (\bigoplus_i \mathcal{O}_{\mathbf{P}^n}(a_i))$ where $-t-n-1 \le a_1 < 0$ and F is a vector bundle on $\mathbf{P}^n$ such that:

(1) $H^0F = 0$;

(2) $H^iF(t) = 0$ for all t, for all $i = 2, \ldots, n-1$;

(3) $H^nF(t) = 0$ for all $t \ge -n$;

(4)
$$h^1F_{n,r}(t) = \begin{cases} ((t+r+n; n)) & \text{if } -r \le t \le -1, \\ 0 & \text{otherwise.} \end{cases}$$

To end the proof it is enough to see that $F \cong F_{n,r}(r)$. We apply Beilinson's spectral sequence with E_1-terms $E_1^{pq} = H^q(\mathbf{P}^n, F(p)) \otimes \Omega^p_{\mathbf{P}^n}(-p)$. The diagram of the E_1-terms is as follows:

$$\begin{array}{ccccc|l} 0 & \cdots & 0 & 0 & 0 & \uparrow q \\ \cdot & \cdots & \cdot & \cdot & \cdot & \\ \cdot & \cdots & \cdot & \cdot & \cdot & \\ \cdot & \cdots & \cdot & \cdot & \cdot & \\ 0 & \cdots & 0 & 0 & 0 & \\ \Omega^n(n)^{h^1F(-n)} & \cdots & \Omega^2(2)^{h^1F(-2)} & \Omega^1(1)^{h^1F(-1)} & 0 & \\ 0 & \cdots & 0 & 0 & 0 & \\ \hline & & & & & \rightarrow p \end{array}$$

Since $E_2^{pq} = E_\infty$, the only nonzero row is exact with only one exception $\Omega^1_{\mathbf{P}^n}(1)^{h^1F(-1)}$ where the cokernel is F. So, we have the exact sequence

$$(***) \qquad \begin{aligned} 0 \to \Omega^n_{\mathbf{P}^n}(n)^{h^1F(-n)} &\to \cdots \to \Omega^2_{\mathbf{P}^n}(2)^{h^1F(-2)} \\ &\to \Omega^1_{\mathbf{P}^n}(1)^{h^1F(-1)} \to F \to 0. \end{aligned}$$

In particular, we get that $c_i(F) = c_i(F_{n,r}(r))$ for $i = 1, \ldots, n$; and $\operatorname{rk}(F) = \operatorname{rk}(F_{n,r}(r))$. Hence, in order to prove that F and $F_{n,r}(r)$ are isomorphic it is enough to see that there is a monomorphism between $F_{n,r}$ and F. First of all, note that applying $\operatorname{Hom}(\cdot, F)$ to the exact sequence

$$0 \to F_{n,r}(r) \to a_0\mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(r) \to 0,$$

we get the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathcal{O}_{\mathbf{P}^n}(r), F) &\to a_0 \operatorname{Hom}(\mathcal{O}_{\mathbf{P}^n}, F) \to \operatorname{Hom}(F_{n,r}(r), F) \\ &\to \operatorname{Ext}^1(\mathcal{O}_{\mathbf{P}^n}(r), F) \to a_0 \operatorname{Ext}^1(\mathcal{O}_{\mathbf{P}^n}, F) \to \cdots. \end{aligned}$$

Since $\operatorname{Hom}(\mathscr{O}_{\mathbf{P}^n}, F) = H^0(\mathbf{P}^n, F) = 0$ and $\operatorname{Ext}^1(\mathscr{O}_{\mathbf{P}^n}, F) = H^1(\mathbf{P}^n, F) = 0$, we conclude that $\operatorname{Hom}(F_{n,r}(r), F) \cong \operatorname{Ext}^1(\mathscr{O}_{\mathbf{P}^n}(r), F) \cong (\mathbf{P}^n, F(-r)) \cong k$. Similarly, applying $\operatorname{Hom}(\cdot, F_{n,r}(r))$ to the exact sequence $(***)$ we get that $\operatorname{Hom}(F, F_{n,r}(r)) \neq 0$. Now, we choose a nontrivial morphism $\Phi: F_{n,r}(r) \to F$ and $\Psi: F \to F_{n,r}(r)$ and consider the composition $\Psi\Phi: F_{n,r}(r) \to F_{n,r}(r)$. Since $F_{n,r}(r)$ are simple, we have $\Psi\Phi = c\,\mathrm{Id}_{F_{n,r}(r)}$ for some $c \in k$.

Claim. $c \neq 0$.

Since c is a nonzero constant, we conclude that Φ is a monomorphism, which gives the desired result.

Proof of the Claim. Assume that $\Psi\Phi = 0$. Set $a_i = h^1F(-i)$. We have the exact sequences:

(1) $$0 \to \Omega^n_{\mathbf{P}^n}(n)^{a_n} \xrightarrow[\rho_n]{} \cdots \xrightarrow[\rho_3]{} \Omega^2_{\mathbf{P}^n}(2)^{a_2} \xrightarrow[\rho_2]{} \Omega^1_{\mathbf{P}^n}(1)^{a_1} \xrightarrow[\beta]{} F_{n,r}(r) \longrightarrow 0$$

and

(2) $$0 \to \Omega^n_{\mathbf{P}^n}(n)^{a_n} \to \cdots \to \Omega^2_{\mathbf{P}^n}(2)^{a_2} \longrightarrow \Omega^1_{\mathbf{P}^n}(1)^{a_1} \xrightarrow[\gamma]{} F \longrightarrow 0.$$

Cutting (2) into short exact sequences, we prove that the morphism $\Phi\beta$ can be lifted to a nontrivial morphism $f: \Omega^1_{\mathbf{P}^n}(1)^{a_1} \to \Omega^1_{\mathbf{P}^n}(1)^{a_1}$ in order that the following square commutes:

$$\begin{array}{ccccccc}
\longrightarrow & \Omega^1_{\mathbf{P}^n}(1)^{a_1} & \xrightarrow[\beta]{} & F_{n,r}(r) & \longrightarrow & 0 \\
 & \downarrow f & & \downarrow \Phi & & \\
\longrightarrow & \Omega^1_{\mathbf{P}^n}(1)^{a_1} & \xrightarrow[\gamma]{} & F & \longrightarrow & 0
\end{array}$$

In the same way we get a commutative diagram:

$$\begin{array}{ccccccc}
\Omega^2_{\mathbf{P}^n}(2)^{a_2} & \longrightarrow & \Omega^1_{\mathbf{P}^n}(1)^{a_1} & \xrightarrow[\beta]{} & F_{n,r}(r) & \longrightarrow & 0 \\
\downarrow & & \downarrow f & & \downarrow \Phi & & \\
\Omega^2_{\mathbf{P}^n}(2)^{a_2} & \longrightarrow & \Omega^1_{\mathbf{P}^n}(1)^{a_1} & \xrightarrow[\gamma]{} & F & \longrightarrow & 0 \\
\downarrow & & \downarrow g & & \downarrow \Psi & & \\
\Omega^2_{\mathbf{P}^n}(2)^{a_2} & \xrightarrow[\rho_2]{} & \Omega^1_{\mathbf{P}^n}(1)^{a_1} & \xrightarrow[\beta]{} & F_{n,r}(r) & \longrightarrow & 0
\end{array}$$

Hence, we have $0 = \Psi\Phi\beta = \beta g f$. Therefore, $\operatorname{Im}(gf) \subset \operatorname{Ker}(\beta) = \operatorname{Im}(\rho_2)$ and gf can be lifted to a nontrivial morphism $h: \Omega^1_{\mathbf{P}^n}(1)^{a_1} \to \operatorname{Im} g(\rho_2)$. Finally, applying the functor $\operatorname{Hom}(\Omega^1_{\mathbf{P}^n}(1)^{a_1}, \cdot)$ to the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\rho_2) \hookrightarrow \Omega^2_{\mathbf{P}^n}(2)^{a_2} \xrightarrow[\rho_2]{} \operatorname{Im} g(\rho_2) = \operatorname{Ker}(\beta) \longrightarrow 0$$

and taking into account that $\operatorname{Ext}^1(\Omega^1_{\mathbf{P}^n}(1)^{a_1}, \operatorname{Ker}(\rho_2)) = 0$, we get that h and, hence, fg can be lifted to a nontrivial morphism $\Omega^1_{\mathbf{P}^n}(1)^{a_1} \to \Omega^2_{\mathbf{P}^n}(2)^{a_2}$. This is a contradiction because $\operatorname{Hom}(\Omega^1_{\mathbf{P}^n}(1), \Omega^2_{\mathbf{P}^n}(2)) = 0$.

As a corollary, we have the following well-known result:

Corollary 2.2. *Let E be a rank ρ vector bundle on $\mathbf{P}^n$ such that $H^iE(*) = 0$ for $0 < i < n$ with the only exception $h^1E(-1) = 1$. Then, $E \cong \Omega^1(1) \oplus$ (line-bundles).*

Proof. Set $t = \min\{l | H^0E(l) \neq 0\}$. If $t \leq 0$, then $H^iE(t-i-1) = 0$ for $0 \leq i < n$. Hence, by [AO, Theorem 2], $E \cong F_1 \oplus \mathscr{O}(-t)^{h^0F(t)}$ where F_1 is a locally free sheaf on $\mathbf{P}^n$ such that $H^iF_1(*) = 0$ for $0 < i < n$ with only exception $h^1F_1(-1) = 1$ and $\min\{l|H^0E(l) \neq 0\} < \min\{l|H^0F_1(l) \neq 0\}$. Repeating this process we may assume that $E \cong F \oplus$ (line-bundles) where F is a locally free sheaf on $\mathbf{P}^n$ such that $H^iF(*) = 0$ for $0 < i < n$ with only exception $h^1F(-1) = 1$ and $\min\{l|H^0F(l) \neq 0\} > 0$. Now, applying Theorem 2.1, we have $F \cong \Omega^1(1) \oplus$ (line-bundles), which gives the desired result.

Question 2.4. Given a vector bundle E on $\mathbf{P}^n$, are there sufficient conditions involving only a finite number of suitably chosen cohomology groups in order that the vector bundle E be the direct sum of $F^p_{n,r}$ and line bundles?

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