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A DISC-HULL IN $\mathbf{C}^2$

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ABSTRACT. We construct a compact set in $\mathbf{C}^2$ whose disc-hull is a proper dense subset of its polynomial hull.

Let Z be a compact subset of $\mathbf{C}^n$. The polynomially convex hull of Z is the set

$$\widehat{Z} = \left\{p \in \mathbf{C}^n : |P(p)| \leq \max_{z \in Z} |P(z)|, \text{ for all polynomials } P\right\}.$$

Ahern and Rudin [1] have studied a particular subset of $\widehat{Z}$ which they call the disc-hull $\mathscr{D}(Z)$ of Z. This is defined as follows. An H^∞-disc is the image of a nonconstant H^∞ map $\Phi: U \to \mathbf{C}^n$. Here U is the open unit disc in $\mathbf{C}$. The boundary value function Φ^* exists almost everywhere on the unit circle bU. The H^∞-disc is said to have its boundary lying in Z if $\Phi^*(e^{i\theta}) \in Z$ for almost all $e^{i\theta} \in bU$. The disc-hull $\mathscr{D}(Z)$ is defined as the union of Z and all H^∞-discs with boundary in Z. The maximum principle implies that $\mathscr{D}(Z) \subseteq \widehat{Z}$.

Ahern and Rudin determined the disc-hull for some interesting classes of three-spheres in $\mathbf{C}^3$. Their work led them to ask [1, §XI]) whether $\mathscr{D}(Z)$ is always compact for Z compact in $\mathbf{C}^n$. The object of this note is to supply a negative answer. We shall construct a compact set Z in $\mathbf{C}^2$ such that $\mathscr{D}(Z)$ is a proper dense subset of $\widehat{Z}$.

The construction of Z will be based on Wermer's beautiful example [5] of a hull without analytic structure. The first such example was due to Stolzenberg [4]. Let z and w be the coordinate functions in $\mathbf{C}^2$. Then according to [5] there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ on $\mathbf{C}^2$ and a sequence of positive numbers $\{\varepsilon_n\}_{n\geq 1}$ with the following properties. $P_n(z, w)$ is monic and of degree 2^n in w and $\{|P_{n+1}| \leq \varepsilon_{n+1},\ |z| \leq \frac{1}{2}\} \subseteq \{|P_n| \leq \varepsilon_n,\ |z| \leq \frac{1}{2}\}$ for $n = 1, 2, 3, \ldots$.

Set

$$X = \bigcap_{n=1}^{\infty} \{|P_n| \leq \varepsilon_n,\ |z| \leq \tfrac{1}{2}\}$$

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and set $\Sigma_n = \{P_n = 0,\ |z| \le \frac{1}{2}\}$. Then X is compact and is contained in the closure of $\bigcup_{n=1}^{\infty} \Sigma_n$ and $X = \widehat{Y}$ where $Y = X \cap \{|z| = \frac{1}{2}\}$. The polynomial hull X contains no analytic structure. This is Wermer's example.

Now we can define $Z \subseteq \mathbf{C}^2$. Put $Y_n = \Sigma_n \cap \{|z| = \frac{1}{2}\}$. Then $\widehat{Y}_n = \Sigma_n$. We define $Z = Y \cup \bigcup_{n=1}^{\infty} Y_n$. We shall establish the following assertions.

(1) Z is compact.
(2) $\widehat{Z} = \widehat{Y} \cup \bigcup_{n=1}^{\infty} \widehat{Y}_n = X \cup \bigcup_{n=1}^{\infty} \Sigma_n$.
(3) $\mathscr{D}(Z) = Z \cup \bigcup_{n=1}^{\infty} \Sigma_n$.
(4) $\mathscr{D}(Z)$ is a proper subset of $\widehat{Z}$.

Then, by (2) and (3), $\mathscr{D}(Z)$ is dense in $\widehat{Z}$ and, by (4), therefore is not compact.

To see (1) note that Z is bounded since Σ_n is contained in the compact set $\{|P_1| \le \varepsilon_1,\ |z| \le \frac{1}{2}\}$ for all n. Also Z is closed. In fact, if $p \in \overline{Z} \backslash Y$ then, as $Y_n \to Y$ by the choice of $\{\varepsilon_n\}$, there is a neighborhood of p which meets only a finite number of Y_n's and which is disjoint from Y. It follows that p is a point of one of the closed sets Y_n.

Clearly the right-hand side of (2) is contained in $\widehat{Z}$. We must show the reverse containment. Suppose that $p \in \widehat{Z}$ and $p \notin X = \widehat{Y}$. It follows from the definition of X that there exists $n_0 > 0$ such that $p \notin \{|P_{n_0}| \le \varepsilon_{n_0},\ |z| = \frac{1}{2}\}^{\wedge}$. Set $Q = Y \cup \bigcup_{n \ge n_0} Y_n$. Then $p \notin \widehat{Q}$ and $Q \subseteq Z$ is compact as in (1). Let σ be a Jensen measure on Z representing evaluation at p for polynomials [3]. Since $p \notin \widehat{Q}$, $\sigma(Z\backslash Q) > 0$. Put $P = P_1 \cdot P_2 \cdots P_{n_0 - 1}$. By the definition of Jensen measure

$$-\infty \le \log|P(p)| \le \int \log|P|\, d\sigma.$$

Since $Z\backslash Q \subseteq \bigcup_{n=1}^{n_0-1} Y_n$ and $P \equiv 0$ on this last set, the integral $= -\infty$ as $\sigma(Z\backslash Q) > 0$. Hence $P(p) = 0$. This implies that $p \in \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_{n_0-1}$. This gives (2).

Half of (3) is clear: Let $\Phi_n\colon U \to \Sigma_n \backslash Y_n$ be the universal covering map of an analytic component of $\Sigma_n\backslash Y_n$. In fact, $\Sigma_n \backslash Y_n$ is irreducible, but we do not need to verify this. Then Φ_n is an H^∞-disc with boundary in Y_n; cf. [2, p. 337]. It follows that $\Sigma_n \subseteq \mathscr{D}(Z)$ for all n.

For the opposite inclusion, let $\Phi\colon U \to \mathbf{C}^2$ be an H^∞-disc with boundary in Z. Then $\Phi^*(e^{i\theta}) \in Z$ a.e. It cannot happen that $\Phi^*(e^{i\theta}) \in Y$ a.e., for then $\Phi(U) \subseteq \mathscr{D}(Y) \subseteq \widehat{Y} = X$, in contradiction to Wermer's example—the nonexistence of analytic structure in X. Hence, there exists n_0 such that the set $E_0 = \{e^{i\theta}\colon \Phi^*(e^{i\theta}) \in Y_{n_0}\}$ has positive measure (and is measurable). Set $F = P_{n_0} \circ \Phi$, an H^∞ function on U. Then $F^* = 0$ on E_0 implies $F \equiv 0$. Hence $\Phi(U) \subseteq \Sigma_{n_0}$. We conclude that $\mathscr{D}(Z) \subseteq \bigcup_{n=1}^{\infty} \Sigma_n \cup Z$.

To establish (4) we argue by contradiction and suppose that $\mathscr{D}(Z) = \widehat{Z}$. Set $C_n = X \cap \Sigma_n \cap \{|z| < \frac{1}{2}\}$. By our supposition, $\bigcup C_n = X \cap \{|z| < \frac{1}{2}\}$. By the Baire category theorem some C_{n_0} contain a neighborhood of some point $q \in X$. Let Δ be a smoothly bounded parametric disc in Σ_{n_0} which contains a neighborhood of q in X. By the local maximum modulus principle, $q \in X \cap \Delta \subseteq (X \cap b\Delta)^{\wedge}$. Since every proper subset of $b\Delta$ is polynomially convex and since $q \notin b\Delta$, it follows that $X \cap b\Delta = b\Delta$. But then $X \supseteq \widehat{b\Delta} \supseteq \Delta$, a contradiction to the fundamental property of X—it contains no analytic structure.

References

1. P. Ahern and W. Rudin, *Hulls of 3-spheres in* $\mathbf{C}^3$, Proceedings of the Madison Symposium in Complex Analysis, Contemp. Math., vol. 137, Amer. Math. Soc., Providence, RI, 1992, 1–29.
2. H. Alexander, B. A. Taylor, and J. L. Ullman, *Areas of projections of analytic sets*, Invent. Math. **16** (1972), 335–341.
3. E. Bishop, *Holomorphic completions, analytic continuation and the interpolation of semi-norms*, Ann. of Math. (2) **78** (1963), 468–500.
4. G. Stolzenberg, *A hull with no analytic structure*, J. Math. Mech. **12** (1963), 103–112.
5. J. Wermer, *Polynomial convex hulls and analyticity*, Ark. Mat. **20** (1982), 129–135.

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