A DISC-HULL IN $\mathbb{C}^2$

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Abstract. We construct a compact set in $\mathbb{C}^2$ whose disc-hull is a proper dense subset of its polynomial hull.

Let $Z$ be a compact subset of $\mathbb{C}^n$. The polynomially convex hull of $Z$ is the set

$$
\hat{Z} = \left\{ p \in \mathbb{C}^n : |P(p)| \leq \max_{z \in Z} |P(z)|, \text{ for all polynomials } P \right\}.
$$

Ahern and Rudin [1] have studied a particular subset of $\hat{Z}$ which they call the disc-hull $D(Z)$ of $Z$. This is defined as follows. An $H^\infty$-disc is the image of a nonconstant $H^\infty$ map $\Phi : U \to \mathbb{C}^n$. Here $U$ is the open unit disc in $\mathbb{C}$. The boundary value function $\Phi^*$ exists almost everywhere on the unit circle $bU$. The $H^\infty$-disc is said to have its boundary lying in $Z$ if $\Phi^*(e^{i\theta}) \in Z$ for almost all $e^{i\theta} \in bU$. The disc-hull $D(Z)$ is defined as the union of $Z$ and all $H^\infty$-discs with boundary in $Z$. The maximum principle implies that $D(Z) \subseteq \hat{Z}$.

Ahern and Rudin determined the disc-hull for some interesting classes of three-spheres in $\mathbb{C}^3$. Their work led them to ask [1, §XII] whether $D(Z)$ is always compact for $Z$ compact in $\mathbb{C}^n$. The object of this note is to supply a negative answer. We shall construct a compact set $Z$ in $\mathbb{C}^2$ such that $D(Z)$ is a proper dense subset of $\hat{Z}$.

The construction of $Z$ will be based on Wermer's beautiful example [5] of a hull without analytic structure. The first such example was due to Stolzenberg [4]. Let $z$ and $w$ be the coordinate functions in $\mathbb{C}^2$. Then according to [5] there exists a sequence of polynomials $\{P_n\}_{n \geq 1}$ on $\mathbb{C}^2$ and a sequence of positive numbers $\{e_n\}_{n \geq 1}$ with the following properties. $P_n(z, w)$ is monic and of degree $2^n$ in $w$ and $\{|P_{n+1}| \leq e_{n+1}, |z| \leq \frac{1}{2}\} \subseteq \{|P_n| \leq e_n, |z| \leq \frac{1}{2}\}$ for $n = 1, 2, 3, \ldots$.

Set

$$
X = \bigcap_{n=1}^{\infty} \{|P_n| \leq e_n, |z| \leq \frac{1}{2}\}
$$

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and set $\Sigma_n = \{P_n = 0, \ |z| \leq \frac{1}{2}\}$. Then $X$ is compact and is contained in the closure of $\bigcup_{n=1}^{\infty} \Sigma_n$ and $X = Y$ where $Y = X \cap \{|z| = \frac{1}{2}\}$. The polynomial hull $X$ contains no analytic structure. This is Wermer’s example.

Now we can define $Z \subseteq \mathbb{C}^2$. Put $Y_n = \Sigma_n \cap \{|z| = \frac{1}{2}\}$. Then $\hat{Y}_n = \Sigma_n$. We define $Z = Y \cup \bigcup_{n=1}^{\infty} Y_n$. We shall establish the following assertions.

1. $Z$ is compact.
2. $\hat{Z} = \hat{Y} \cup \bigcup_{n=1}^{\infty} \hat{Y}_n = X \cup \bigcup_{n=1}^{\infty} \Sigma_n$.
3. $\mathcal{D}(Z) = Z \cup \bigcup_{n=1}^{\infty} \Sigma_n$.
4. $\mathcal{D}(Z)$ is a proper subset of $\hat{Z}$.

Then, by (2) and (3), $\mathcal{D}(Z)$ is dense in $\hat{Z}$ and, by (4), therefore is not compact.

To see (1) note that $Z$ is bounded since $\Sigma_n$ is contained in the compact set $\{|P_n| \leq \epsilon_n, \ |z| \leq \frac{1}{2}\}$ for all $n$. Also $Z$ is closed. In fact, if $p \in \overline{Z} \setminus Y$ then, as $Y_n \to Y$ by the choice of $\{\epsilon_n\}$, there is a neighborhood of $p$ which meets only a finite number of $Y_n$’s and which is disjoint from $Y$. It follows that $p$ is a point of one of the closed sets $Y_n$.

Clearly the right-hand side of (2) is contained in $\hat{Z}$. We must show the reverse containment. Suppose that $p \in \hat{Z}$ and $p \notin X = \hat{Y}$. It follows from the definition of $X$ that there exists $n_0 > 0$ such that $p \notin \{|P_{n_0}| \leq \epsilon_{n_0}, \ |z| = \frac{1}{2}\}$.

Set $Q = Y \cup \bigcup_{n \geq n_0} Y_n$. Then $p \notin \hat{Q}$ and $Q \subseteq Z$. Let $\sigma$ be a Jensen measure on $Z$ representing evaluation at $p$ for polynomials [3]. Since $p \notin \hat{Q}$, $\sigma(Z \setminus Q) > 0$. Put $P = P_1 \cdot P_2 \cdots P_{n_0-1}$. By the definition of Jensen measure

$$-\infty \leq \log |P(p)| \leq \int \log |P| \, d\sigma.$$ 

Since $Z \setminus Q \subseteq \bigcup_{n=1}^{n_0-1} Y_n$ and $P \equiv 0$ on this last set, the integral $= -\infty$ as $\sigma(Z \setminus Q) > 0$. Hence $P(p) = 0$. This implies that $p \in \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_{n_0-1}$. This gives (2).

Half of (3) is clear: Let $\Phi_n: X \to \Sigma_n \setminus Y_n$ be the universal covering map of an analytic component of $\Sigma_n \setminus Y_n$. In fact, $\Sigma_n \setminus Y_n$ is irreducible, but we do not need to verify this. Then $\Phi_n$ is an $H^\infty$-disc with boundary in $Y_n$; cf. [2, p. 337]. It follows that $\Sigma_n \subseteq \mathcal{D}(Z)$ for all $n$.

For the opposite inclusion, let $\Phi: U \to \mathbb{C}^2$ be an $H^\infty$-disc with boundary in $Z$. Then $\Phi^*(e^{i\theta}) \in Z$ a.e. It cannot happen that $\Phi^*(e^{i\theta}) \in Y$ a.e., for then $\Phi(U) \subseteq \mathcal{D}(Y) \subseteq \hat{Y} = X$, in contradiction to Wermer’s example—the nonexistence of analytic structure in $X$. Hence, there exists $n_0$ such that the set $E_0 = \{e^{i\theta}: \Phi^*(e^{i\theta}) \in Y_{n_0}\}$ has positive measure (and is measurable). Set $F = P_{n_0} \circ \Phi$, an $H^\infty$ function on $U$. Then $F^* = 0$ on $E_0$ implies $F \equiv 0$. Hence $\Phi(U) \subseteq \Sigma_{n_0}$. We conclude that $\mathcal{D}(Z) \subseteq \bigcup_{n=1}^{\infty} \Sigma_n \cup Z$.

To establish (4) we argue by contradiction and suppose that $\mathcal{D}(Z) = \hat{Z}$. Set $C_n = X \cap \Sigma_n \cap \{|z| < \frac{1}{2}\}$. By our supposition, $\bigcup C_n = X \cap \{|z| < \frac{1}{2}\}$. By the Baire category theorem some $C_{n_0}$ contain a neighborhood of some point $q \in X$. Let $\Delta$ be a smoothly bounded parametric disc in $\Sigma_{n_0}$ which contains a neighborhood of $q$ in $X$. By the local maximum modulus principle, $q \in X \cap \Delta \subseteq (X \cap b\Delta)^-$. Since every proper subset of $b\Delta$ is polynomially convex and since $q \notin b\Delta$, it follows that $X \cap b\Delta = b\Delta$. But then $X \supseteq b\Delta \supseteq \Delta$, a contradiction to the fundamental property of $X$—it contains no analytic structure.
REFERENCES


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