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EXACT COVERING SYSTEMS AND THE GAUSS-LEGENDRE MULTIPLICATION FORMULA FOR THE GAMMA FUNCTION

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ABSTRACT. The Gauss-Legendre multiplication formula for the gamma function is $(2\pi)^{(m-1)/2}m^{1/2-mz}\Gamma(mz) = \Gamma(z)\Gamma(z+\frac{1}{m})\cdots\Gamma(z+\frac{m-1}{m})$. Let $\{a_i \pmod{b_i} : 1 \le i \le m\}$ be an exact covering system with standardized offsets. Then

$$\Gamma(z) = \frac{\Gamma(z/b_1)}{{b_1}^{1-z/b_1}} \prod_{i=2}^{m} \frac{\Gamma((z+a_i)/b_i)}{b_i^{-z/b_i}\Gamma(a_i/b_i)}.$$

Conversely, if the above identity holds, then $\{a_i \pmod{b_i} : 1 \le i \le m\}$ is an exact covering system with standardized offsets. The Gauss-Legendre multiplication formula is a special case of this identity.

Let Z_{ab} be the arithmetic progression (AP) $\{x : x = a + nb, n \in Z\}$. Another notation for this arithmetic progression is $a(\text{mod } b)$. A finite collection of disjoint AP's $C = \{Z_{a_ib_i} : 1 \le i \le m\}$ is called an exact covering system (or exact cover) if each integer belongs to exactly one AP $Z_{a_ib_i}$. We usually assume that the offsets a_i have been standardized so that $0 \le a_i < b_i$. A consequence of the fact that C is an exact cover with standardized offsets is that there is one and only one offset that is zero, and we assume it is always a_1. Another property of exact covers is that $\sum_{i=1}^{m} 1/b_i = 1$. If the offsets are standardized, $\sum_{i=1}^{m} a_i/b_i = (m-1)/2$, by Theorem 1 of Fraenkel [1]. Let $M = \text{lcm}\{b_i\}$. If each of the integers $\{0, 1, \dots, M-1\}$ is covered by C, then all the integers are covered by C. The collection $Z = \{Z_{a_ib_i} : 1 \le i < \infty\}$ is an infinite exact cover if each integer belongs to exactly one AP. There are two classes of infinite exact covers. A saturated cover has $\sum_{i=1}^{\infty} 1/b_i = 1$. An unsaturated cover has $\sum_{i=1}^{\infty} 1/b_i < 1$.

In [2] I proved that

$$\sin(\pi z) = b_1 \sin\left(\frac{\pi z}{b_1}\right) \prod_{i=2}^{m} \frac{\sin \pi((a_i - z)/b_i)}{\sin(\pi a_i/b_i)}$$

if and only if $C = \{Z_{a_ib_i} : 1 \le i \le m\}$ is an exact cover. A special case of this

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is the well-known identity

$$\sin(mz) = 2^{m-1}\sin(z)\sin\left(z+\frac{\pi}{m}\right)\cdots\sin\left(z+\frac{(m-1)\pi}{m}\right).$$

In [1] Fraenkel proves that Raabe's identity for the Bernoulli polynomials,

$$B_n(mz) = m^{n-1}\left(B_n(z)+B_n\left(z+\frac{1}{m}\right)+\cdots+B_n\left(z+\frac{m-1}{m}\right)\right),$$

can be generalized to exact covers. (See also Beebee [3].) This last identity is an additive analogy to the Gauss-Legendre multiplication formula,

$$(2\pi)^{(m-1)/2}m^{1/2-mz}\Gamma(mz) = \Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right). \tag{1}$$

Evidently, these identities belong to a class which can be generalized to exact covers. For example, the referee has nominated the q-gamma function to this class (see [4]). The definition of this class and the common characteristics of the functions need to be determined.

Theorem 1. *If* $\Gamma(z) = g(z)\prod_{i=1}^{m}\Gamma((z+a_i)/b_i)$ *where* g *has no zeros at the nonpositive integers and is defined for all complex z, then the set of AP's* $C = \{Z_{a_ib_i} : 1 \le i \le m\}$ *is an exact cover with standardized offsets; and conversely if* C *is an exact cover with standardized offsets, then*

$$\Gamma(z) = \frac{\Gamma(z/b_1)}{{b_1}^{1-z/b_1}}\prod_{i=2}^{m}\frac{\Gamma((z+a_i)/b_i)}{b_i^{-z/b_i}\Gamma(a_i/b_i)}. \tag{2}$$

Proof. Suppose $\Gamma(z) = g(z)\prod_{i=1}^{m}\Gamma((z+a_i)/b_i)$. The gamma function has the nonpositive integers for its only poles, and these poles have order 1. Then $-a_i$ is a pole of the function on the right and hence of the function on the left. Thus $-a_i$ is a nonpositive integer, so a_i is a nonnegative integer. If n is a nonnegative integer, then $-a_i - nb_i$ is a pole on the right and hence on the left. Thus $a_i + nb_i$ is a nonnegative integer. Thus b_i is a positive integer. If n is a nonnegative integer, $-n$ is a pole of order 1 on the left, and hence $(-n+a_i)/b_i$ is a pole for precisely one i on the right and thus is a nonpositive integer: $(-n+a_i)/b_i = -m$. Hence $n = a_i + mb_i$ for each nonnegative n, so each nonnegative integer belongs to exactly one AP $Z_{a_ib_i}$. For finite m, this means C is an exact cover. (There are infinite saturated systems of disjoint AP's that cover the nonnegative integers but not the integers. See Example 3 below.) Zero is a pole of the left side, so $(0+a_i)/b_i$ is a pole on the right for some i and hence a_i/b_i is a nonpositive integer for some i. But $a_i/b_i \ge 0$, so $a_i = 0$ for some i. Suppose $a_i \ge b_i$. Then $b_i - a_i \le 0$. Hence $b_i - a_i$ is a pole on the left. But this implies it is a pole on the right. Hence $(b_i - a_i + a_j)/b_j = -n$, for some nonnegative n. If $j = i$, this implies 1 is a pole of the gamma function, so $j \ne i$. Thus $a_i - b_i = a_j + nb_j$, which is a contradiction, because $a_i - b_i$ cannot belong to two AP's in the exact cover. Hence $0 \le a_i < b_i$, so C has standardized offsets.

Now suppose C is an exact cover with $a_1 = 0$, $0 \le a_i < b_i$. The referee suggested the following derivation of (2). It is similar to the derivation of (1) in Rainville [5] or Marsden [6]. Let $(z)_n = z(z+1)\cdots(z+n-1)$ and $N =$

a multiple of all the moduli, b_i. Since C is an exact cover with standardized offsets,

$$\{0, 1, 2, \ldots, N-1\} = \bigcup_{i=1}^{m} \{a_i + nb_i : 0 \le n \le N/b_i - 1\}.$$

Thus

$$\begin{aligned}(z)_N &= \prod_{i=1}^{m} (z+a_i)(z+a_i+b_i)\cdots\left(z+a_i+\left(\frac{N}{b_i}-1\right)b_i\right)\\ &= \prod_{i=1}^{m} b_i^{N/b_i}\left(\frac{z+a_i}{b_i}\right)_{N/b_i}.\end{aligned}$$

By Theorem 9 of [5], $(z)_n = \Gamma(z+n)/\Gamma(z)$. Thus

$$\frac{\Gamma(z+N)}{\Gamma(z)} = \prod_{i=1}^{m} b_i^{N/b_i} \frac{\Gamma((z+a_i)/b_i + N/b_i)}{\Gamma((z+a_i)/b_i)}. \tag{3}$$

By Lemma 7 of [5], $\lim_{n\to\infty}((n-1)!n^z/\Gamma(z+n)) = 1$. Rearranging (3) and using this,

$$\frac{\Gamma(z)}{\prod_{i=1}^{m}\Gamma((z+a_i)/b_i)} = \lim_{N\to\infty} \frac{(N-1)!N^z}{\prod_{i=1}^{m} b_i^{N/b_i}(N/b_i-1)!(N/b_i)^{(z+a_i)/b_i}},$$

or

$$\frac{\Gamma(z)}{b_i^{z/b_i}\prod_{i=1}^{m}\Gamma((z+a_i)/b_i)} = \lim_{N\to\infty} \frac{(N-1)!}{N^{(m-1)/2}\prod_{i=1}^{m} b_i^{(N-a_i)/b_i}} = \text{const}, \tag{4}$$

because $\sum 1/b_i = 1$ and $\sum a_i/b_i = (m-1)/2$. Taking $\lim_{z\to 0}$ on the left, we see that

$$\text{const} = \left(b_1 \prod_{i=2}^{m} \Gamma\left(\frac{a_i}{b_i}\right)\right)^{-1}.$$

Substituting this value of the constant in (4) yields (2). Neither this or any other proof that I have found applies to infinite exact covers.

Example 1. The Gauss-Legendre multiplication formula is a special case of (2).

Proof. It is easy to see that $C_m = \{Z_{i-1,m} : 1 \le i \le m\}$ is an exact cover. In [2] I proved

Lemma. *If* $C = \{Z_{a_i b_i} : 1 \le i \le m\}$ *is an exact cover then*

$$\prod_{i=2}^{m} \sin \pi \frac{a_i}{b_i} = \frac{b_1}{2^{m-1}}.$$

When we apply this lemma to the exact cover C_m, we get

$$\prod_{i=2}^{m} \sin \pi \left(\frac{i-1}{m}\right) = \frac{m}{2^{m-1}}.$$

But $\Gamma(z)\Gamma(1-z) = \pi/(\sin \pi z)$. Hence

$$\prod_{i=2}^{m} \Gamma\left(\frac{i-1}{m}\right)\Gamma\left(1-\frac{i-1}{m}\right) = \prod_{i=2}^{m} \frac{\pi}{\sin \pi((i-1)/m)} = \frac{(2\pi)^{m-1}}{m}.$$

But $\prod_{i=2}^{m}\Gamma\left(\frac{i-1}{m}\right)=\prod_{i=2}^{m}\Gamma\left(1-\frac{i-1}{m}\right)$, so

$$\prod_{i=2}^{m}\Gamma\left(\frac{i-1}{m}\right)=\frac{(2\pi)^{(m-1)/2}}{\sqrt{m}}. \tag{5}$$

Now substitute mz for z, $i-1$ for a_i, and m for b_i in (2), and use (5):

$$\Gamma(mz)=\frac{\Gamma(z)\Gamma(z+1/m)\cdots\Gamma(z+(m-1)/m)}{(2\pi)^{(m-1)/2}m^{1/2-mz}}.$$

Rearrangement of this gives the Gauss-Legendre multiplication formula (1).

Example 2. As an example of (2), consider the exact cover

$$C=\{Z_{04}, Z_{24}, Z_{16}, Z_{36}, Z_{56}\}.$$

Substituting in (2) yields

$$\Gamma(z)=\frac{\Gamma\left(\frac{z}{4}\right)\Gamma\left(\frac{z+2}{4}\right)\Gamma\left(\frac{z+1}{6}\right)\Gamma\left(\frac{z+3}{6}\right)\Gamma\left(\frac{z+5}{6}\right)}{4^{1-z/4}4^{-z/4}6^{-z/6}6^{-z/6}6^{-z/6}\Gamma\left(\frac{2}{4}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{3}{6}\right)\Gamma\left(\frac{5}{6}\right)}.$$

For $z=10$, this equation is

$$362,880=\frac{(1.329)(2)(.941)(1.082)(1.329)}{4^{-1.5}4^{-2.5}6^{-1.667}6^{-1.667}6^{-1.667}(1.772)(5.566)(1.772)(1.129)}.$$

Example 3. $C_n=\{Z_{02}; Z_{14}; Z_{3,8}; \dots; Z_{2^{n-1}-1,2^n}; Z_{2^n-1,2^n}\}$ is an exact cover, the Grey cover, with $n+1$ AP's. Substitute C_n into (2).

$$\Gamma(z)=\frac{\Gamma(z/2)}{2^{1-z/2}}\frac{\Gamma((2^n-1+z)/2^n)}{(2^n)^{-z/2^n}\Gamma((2^n-1)/2^n)}\prod_{i=2}^{n}\frac{\Gamma((2^{i-1}-1+z)/2^i)}{(2^i)^{-z/2^i}\Gamma((2^{i-1}-1)/2^i)}.$$

Taking $\lim n\to\infty$ on both sides we see that the product converges and

$$\Gamma(z)=\frac{\Gamma(z/2)}{2^{1-z/2}}\frac{\Gamma(1)}{2^0\Gamma(1)}\prod_{i=2}^{\infty}\frac{\Gamma(1/2+(z-1)/2^i)}{2^{-iz/2^i}\Gamma(1/2-1/2^i)}.$$

But the set of disjoint AP's $C=\{Z_{02}; Z_{14}; Z_{3,8}; \dots; Z_{2^{n-1}-1,2^n}; \dots\}$ is not an infinite saturated exact cover, even though $\sum_{i=1}^{\infty}1/b_i=1$, because it does not cover -1. Thus for $m=\infty$ (2) can hold, but C is not an exact cover. The proof of Theorem 1 made use of $\sum_{i=1}^{m}1/b_i=1$, so I speculate that if C is an unsaturated infinite exact cover, (2) does not hold. But if C is a saturated infinite exact cover, then (2) does hold. Limited numerical experiments support this conjecture.

Example 4. Define

$$\Gamma_{ab}(z)=\begin{cases}\dfrac{\Gamma(z/b)}{b^{1-z/b}} & \text{if } a=0,\\[2ex] \dfrac{\Gamma((z+a)/b)}{b^{-z/b}\Gamma(a/b)} & \text{if } a\neq 0.\end{cases}$$

With this notation, (2) is

$$\Gamma(z)=\prod_{i=1}^{m}\Gamma_{ab}(z). \tag{6}$$

The function $\Gamma_{ab}(z)$ has about the same relation to the AP Z_{ab} that $\Gamma(z)$ has to the integers.

(a) $\Gamma(z)$ has its poles at the nonpositive integers. $\Gamma_{ab}(z)$ has its poles at the nonpositive integers in Z_{ab}.

(b) For $n \geq 1$, $\Gamma(1+z) = z\Gamma(z)$, and $\Gamma(n) = n-1!$.

For the functions $\Gamma_{ab}(z)$, we have

$$\begin{aligned}
\Gamma_{0b}(b) &= 1 \quad \text{and} \quad \Gamma_{0b}(nb) = (n-1)b \cdot (n-2)b \cdots (2b)(b), \\
\Gamma_{ab}(b) &= a \quad \text{and} \quad \Gamma_{ab}(nb) = (a+(n-1)b) \cdot (a+(n-2)b) \\
&\qquad \cdots (a+2b)(a+b)(a).
\end{aligned}$$

Let C be an exact cover, and let $z = M = \operatorname{lcm}\{b_i\}$. Then (6) is the self-evident formula

$$\begin{aligned}
M - 1! = &\left(\left(\frac{M}{b_1} - 1\right) b_1\right) \left(\left(\frac{M}{b_1} - 2\right) b_1\right) \cdots 2b_1 b_1 \\
&\cdot \prod_{i=2}^{m} \left(a_i + \left(\frac{M}{b_i} - 1\right) b_i\right) \left(a_i + \left(\frac{M}{b_i} - 2\right) b_i\right) \cdots (a_i + b_i)\, a_i.
\end{aligned}$$

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