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THE ASSOCIATED GRADED RING AND THE INDEX OF A GORENSTEIN LOCAL RING

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ABSTRACT. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. It is shown that if the associated graded ring $G(R)$ of R is Cohen-Macaulay, then the index of R is equal to the generalized Loewy length of R.

INTRODUCTION

Let $(R, \mathfrak{m}, k)$ be a commutative noetherian Gorenstein local ring. Associated with a finitely generated R-module M, we have Auslander's δ-invariant $\delta_R(M)$ of M. It is defined to be the smallest integer n such that there is an epimorphism $X \oplus R^n \to M$ with X a maximal Cohen-Macaulay module with no free summands. Of particular interest is the δ-invariant of $R/\mathfrak{m}^n$. We know that $\delta_R(R/\mathfrak{m}^n) \leq 1$ and eventually equals to 1 [3]. The smallest n such that $\delta_R(R/\mathfrak{m}^n) = 1$ is called the index of R. One of the main questions is how the index is related to other invariants of R.

Let N be a module of finite length. The *Loewy length* of N, denoted by $ll(N)$, is the smallest integer n such that $\mathfrak{m}^n N = 0$. When R is 0-dimensional, the index of R is the same as the Loewy length of R. For R of positive dimension, the *generalized Loewy length* of R is the minimum of all integers $ll(R/(\mathbf{x}))$, where $\mathbf{x}$ is a system of parameters (sop) of R. In [4] we put forth the following

Conjecture. *Let R be a Gorenstein local ring. Then the index of R is equal to the generalized Loewy length of R.*

The conjecture was shown in the affirmative for hypersurface rings [4] and for homogeneous Gorenstein k-algebras [5]. (Here one has to extend the above concepts in an obvious way to homogeneous k-algebras.) The main result of this paper is to show that the conjecture holds for R if the associated graded ring $G(R)$ of R is Cohen-Macaulay. It is known that the associated graded rings of hypersurface rings and homogeneous Gorenstein k-algebras are Gorenstein. As a consequence, we obtain the earlier results. Moreover, much work has been done on the Cohen-Macaulayness of $G(R)$ of a Gorenstein local ring R; we thus obtain a larger class of Gorenstein rings for which the conjecture hold.

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We recall some basic facts on δ-invariant of a module in §1. Section 2 is devoted to proving the main result.

1. Some preliminaries

In this paper $(R, \mathfrak{m}, k)$ will always be a Gorenstein local ring, and all R-modules are assumed to be finitely generated. Since both the δ-invariant and the generalized Loewy length are invariants under faithfully flat local ring extensions, we may assume that k is an infinite field and R is complete, whenever necessary. Let $G(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$ be the associated graded ring. We denote the map $R \to G(R)$ which takes each element x of R to its initial form in $G(R)$ by "$-$". The following are some basic facts about the δ-invariant of a module over R.

Lemma 1.1 [1, 3]. *Let M and N be R-modules. Then:*

(a) *If N is an epimorphic image of M, then $\delta_R(M) \geq \delta_R(N)$.*
(b) *$\delta_R(M \oplus N) = \delta_R(M) + \delta_R(N)$.*

Lemma 1.2 [2]. *Let M be an R-module and $x \in \mathfrak{m}$ be regular on both R and M. Set $\overline{R} = R/xR$. Then $\delta_R(M) = \delta_{\overline{R}}(M/xM)$.*

Lemma 1.3 [1]. *For any integer $s \geq 1$, we have $\delta_R(\mathfrak{m}^s) = 0$.*

Lemma 1.4 [4]. *Let R be a 0-dimensional Gorenstein local ring and I an ideal of R. Then $\delta_R(R/I) \neq 0$ if and only if $I = (0)$. In particular,* $\mathrm{index}(R) = ll(R)$.

Let $x \in \mathfrak{m}$ be R-regular. Set $\overline{R} = R/xR$. An $\overline{R}$-module N is said to be weakly liftable to R if there exists an R-module L on which x is regular such that N is isomorphic to a direct summand of the module L/xL [2]. The following result is useful.

Lemma 1.5 [2]. *For an $\overline{R}$-module N the following are equivalent.*

(a) *N is weakly liftable to R.*
(b) *$\Omega_R(N)/x\Omega_R(N) \cong N \oplus \Omega_{\overline{R}}(N)$, where $\Omega_R(N)$ (resp. $\Omega_{\overline{R}}(N)$) is the first syzygy of N over R (resp. $\overline{R}$).*

The weak liftability and the δ-invariant of a module are closely related. For further details see [2].

2. The main result

Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. The index of R is defined to be the smallest integer $n > 0$ such that $\delta(R/\mathfrak{m}^n) \neq 0$. The generalized Loewy length of R is the minimum of all integers $ll(R/(\mathbf{x}))$, where $\mathbf{x}$ is a sop of R. Our main result in this paper is the following

Theorem 2.1. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. Suppose the associated graded ring $G(R)$ of R is Cohen-Macaulay. Then the following two numbers are the same:*

(a) *the index of R, and*
(b) *the generalized Loewy length of R.*

The proof of our main result is based on the following two lemmas which allow us to use reduction argument.

Lemma 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and s an integer. Suppose that $x \in \mathfrak{m}\backslash\mathfrak{m}^2$ is R-regular and the induced map $\overline{x}\colon \mathfrak{m}^{i-1}/\mathfrak{m}^i \to \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is injective for $1 \le i \le s$. Then the (R/xR)-module $R/(\mathfrak{m}^s, x)$ is weakly liftable to R.*

Proof. By Lemma 1.5, to show that the (R/xR)-module $R/(\mathfrak{m}^s, x)$ is weakly liftable to R, it suffices to show that the monomorphism

$$xR/x(\mathfrak{m}^s, x) \hookrightarrow (\mathfrak{m}^s, x)/x(\mathfrak{m}^s, x)$$

is split. Let $I = xR \cap \mathfrak{m}^s$ and $W = I + \mathfrak{m}^{s+1}/\mathfrak{m}^{s+1}$. Since $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ is a vector space over k, we have a direct summand decomposition $\mathfrak{m}^s/\mathfrak{m}^{s+1} = W \oplus V$ as vector spaces. Let $e_1, \dots, e_n$ be a basis of V, and let $y_1, \dots, y_n$ be a set of preimages of e_i in $\mathfrak{m}^s$. Now let B be the submodule of $(\mathfrak{m}^s, x)/x(\mathfrak{m}^s, x)$ generated by the images of y_i, $i = 1, \dots, n$. We claim that

$$(\mathfrak{m}^s, x)/x(\mathfrak{m}^s, x) = xR/x(\mathfrak{m}^s, x) \oplus B.$$

It is easily seen that the submodules $A = xR/x(\mathfrak{m}^s, x)$ and B generate $(\mathfrak{m}^s, x)/x(\mathfrak{m}^s, x)$. Suppose $z \in A \cap B$. Then $z = xb = \sum a_i y_i$, where $b, a_i \in R$. This implies that $xb \in \mathfrak{m}^s$. Modulo $\mathfrak{m}^{s+1}$, we get $\overline{xb} = \sum \overline{a_i y_i} = 0$ in $\mathfrak{m}^s/\mathfrak{m}^{s+1}$. Therefore, we have $xb \in \mathfrak{m}^{s+1}$. By our assumption this means that $b \in \mathfrak{m}^s$. Hence, we get $z = xb = 0$ in $(\mathfrak{m}^s, x)/x(\mathfrak{m}^s, x)$. Since x is R-regular, we have $R/(\mathfrak{m}^s, x) \cong xR/x(\mathfrak{m}^s, x)$, and the proof is complete.

Using the same argument, we have

Lemma 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring and s an integer. Suppose that $x \in \mathfrak{m}\backslash\mathfrak{m}^s$ is R-regular and the induced map $\overline{x}\colon \mathfrak{m}^{i-1}/\mathfrak{m}^i \to \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is injective for $1 \le i \le s$. Then the R-module $R/\mathfrak{m}^s$ is isomorphic to a direct summand of the R-module $(\mathfrak{m}^s, x)/x\mathfrak{m}^s$. In particular, $R/\mathfrak{m}^s$ is an epimorphic image of $(\mathfrak{m}^s, x)$.*

Proof. Let e_i, y_i, $i = 1, \dots, n$, be as in Lemma 2.2. Let D be the submodule of $(\mathfrak{m}^s, x)/x\mathfrak{m}^s$ generated by the images of y_i, $i = 1, \dots, n$. We claim that there is a direct summand decomposition

$$(\mathfrak{m}^s, x)/x\mathfrak{m}^s = xR/x\mathfrak{m}^s \oplus D.$$

It is easy to check that the modules $C = xR/x\mathfrak{m}^s$ is a submodule of $(\mathfrak{m}^s, x)/x\mathfrak{m}^s$ and that C and D generate module $(\mathfrak{m}^s, x)/x\mathfrak{m}^s$. Now let $z \in C \cap D$. Then we have $z = xb = \sum a_i y_i$ with $b, a_i \in R$. This implies that $xb \in \mathfrak{m}^s$. Modulo $\mathfrak{m}^{s+1}$, we get $\overline{xb} = \sum \overline{a_i y_i} = 0$ in $\mathfrak{m}^s/\mathfrak{m}^{s+1}$. This shows that $xb \in \mathfrak{m}^{s+1}$. By our assumption we have $b \in \mathfrak{m}^s$. Therefore, $z = xb = 0$ in $(\mathfrak{m}^s, x)/x\mathfrak{m}^s$. Since x is R-regular, we have that $R/\mathfrak{m}^s \cong xR/x\mathfrak{m}^s$, and this completes the proof.

Now we are ready to prove our main result.

Proof of Theorem 2.1. It is known that the index of R is always bounded by the generalized Loewy length of R [4]. Thus it is sufficient to show that $\delta_R(R/\mathfrak{m}^s) \neq 0$ implies that there exists a sop $\mathbf{x}$ of R such that $\mathfrak{m}^s \subset (\mathbf{x})$.

Now suppose that $\delta_R(R/\mathfrak{m}^s) = 1$. We may assume that k is an infinite field. Since $G(R)$ is Cohen-Macaulay, there exists an R-sequence $\mathbf{x} = x_1, \dots, x_d$, where $d = \dim R$, such that $x_i \in \mathfrak{m}\backslash\mathfrak{m}^2$ and $\overline{x_1}, \dots, \overline{x_d}$ is a regular sequence of $G(R)$. Now we proceed by induction on d. Suppose $d = 1$. By Lemma 2.3, we know that $R/\mathfrak{m}^s$ is an epimorphic image of the module $(\mathfrak{m}^s, x_1)$. Therefore,

$\delta_R((\mathfrak{m}^s, x_1)) \neq 0$ by Lemma 1.1. Lemmas 2.2 and 1.5 show that there is an R-module decomposition

$$(\mathfrak{m}^s, x_1)/x_1(\mathfrak{m}^s, x_1) \cong R/(\mathfrak{m}^s, x_1) \oplus (\mathfrak{m}^s, x_1)/x_1R.$$

By Lemma 1.3 we know that $\delta_{\overline{R}}((\mathfrak{m}^s, x_1)/x_1R) = 0$, where $\overline{R} = R/x_1R$. Therefore, we have $\delta_{\overline{R}}(R/(\mathfrak{m}^s, x_1)) = \delta_{\overline{R}}((\mathfrak{m}^s, x_1)/x_1(\mathfrak{m}^s, x_1)) = \delta_R((\mathfrak{m}^s, x_1)) \neq 0$ by Lemmas 1.2 and 1.1. Since $\dim \overline{R} = 0$, Lemma 1.4 implies that $\delta_{\overline{R}}(R/(\mathfrak{m}^s, x_1)) \neq 0$ if and only if $\mathfrak{m}^s \subset (x_1)$.

Assume that $d > 1$. Set $\overline{R} = R/x_1R$. Then the above argument shows that $\delta_R(R/\mathfrak{m}^s) = 1$ implies that $\delta_{\overline{R}}(R/(\mathfrak{m}^s, x_1)) \neq 0$. Let $\mathfrak{m}_1 = \mathfrak{m}/x_1R$. Then $\mathfrak{m}_1$ is the maximal ideal of $\overline{R}$ and $R/(\mathfrak{m}^s, x_1) \cong \overline{R}/\mathfrak{m}_1^s$. Also we have $G(R/x_1R) \cong G(R)/\overline{x_1}G(R)$. Therefore, by inductive hypothesis we get that $\delta_R(R/\mathfrak{m}^s) = 1$ implies that $\delta_{R_d}(R/(\mathfrak{m}^s, x_1, \dots, x_d)) = 1$, where R_d denotes $R/(\mathbf{x})$. Hence, we get $\mathfrak{m}^s \subset (\mathbf{x})$ since $\dim R_d = 0$. This completes the proof.

Remark. There are examples of Gorenstein local rings whose associated graded ring are not Cohen-Macaulay, but the conjecture holds. I know no example where the conjecture fails.

Now we give some applications.

Corollary 2.4 [4]. *The conjecture holds for hypersurface rings.*

Proof. Let $R = S/(f)$ where S is a complete regular local ring. Since $G(S)$ is a polynomial ring over a field and $G(R) = G(S)/\overline{f}G(S)$, we have that $G(R)$ is Cohen-Macaulay.

Let k be a field. A graded k-algebra $R = \bigoplus_{i\geq 0} R_i$ is called *homogeneous* if $R_0 = k$ and $R = k[R_1]$. The homogeneous k-algebra has a unique graded maximum ideal, namely, $\mathfrak{m} = \bigoplus_{i\geq 1} R_i$. All definitions we made and the results we obtained so far can be transferred accordingly to homogeneous Gorenstein algebras. We then have

Corollary 2.5 [5]. *The conjecture holds for homogeneous Gorenstein k-algebras.*

Proof. It follows from the fact that $G(R) = R$.

A quotient ring $R = S/I$ of a regular local ring S is called a *strict complete intersection* if the associated graded ring $G(R)$ is a complete intersection. Strict complete intersections are complete intersections. Thus we have

Corollary 2.6. *The conjecture holds for strict complete intersections.*

We denote by $e(R)$ the multiplicity of R and $\mu(\mathfrak{m})$ the minimal number of generators of $\mathfrak{m}$. It is true that $\mu(\mathfrak{m}) \leq e(R) + \dim(R) - 1$. When the equality holds, R is said to have minimal multiplicity. We have the following results.

Corollary 2.7. *Let R be a Gorenstein local ring of multiplicity at most* 4, *or with $\mu(\mathfrak{m}) = e(R) + \dim(R) - 2$. Then the conjecture holds for R.*

Proof. Sally showed in [6] that $G(R)$ is Gorenstein under the assumption.

Recall that, for a Gorenstein local ring R, $\operatorname{index}(R) = 1$ if and only if R is a regular local ring [3]. In the case where $\operatorname{index}(R) = 2$ we get

Corollary 2.8. *Suppose $G(R)$ is Cohen-Macaulay. Then* $\operatorname{index}(R) = 2$ *if and only if R has minimal multiplicity.*

Proof. Here we use the result in [6] that R has minimal multiplicity if and only if there exists a sop $\mathbf{x}$ of R such that $\mathfrak{m}^2 = (\mathbf{x})\mathfrak{m}$. Our result shows that, under the given condition, this is equivalent to $\delta_R(R/\mathfrak{m}^2) = 1$.

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