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FIXED ALGEBRAS OF RESIDUALLY NILPOTENT LIE ALGEBRAS

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ABSTRACT. Let L_m be the free Lie algebra of rank $m > 1$ over a field K, and let J be an ideal of L_m such that $J \subset L_m''$ and the algebra L_m/J is residually nilpotent. Let $G \neq \langle 1 \rangle$ be a finite group of automorphisms of L_m/J and the order of G be invertible in K. We establish that the algebra of fixed points $(L_m/J)^G$ is not finitely generated. The class of algebras under consideration contains the free Lie algebra over an arbitrary field and the relatively free algebras in nonnilpotent varieties of Lie algebras over infinite fields of characteristic different from 2 and 3.

INTRODUCTION

By a result of Dyer and Scott [4, Theorem 2], for any finite group G of automorphisms of the free group F_m of finite rank, the subgroup of fixed points

$$F_m^G = \{f \in F_m | g(f) = f \text{ for all } g \in G\}$$

is finitely generated. For the free associative algebra $K\langle x_1, \ldots, x_m \rangle$ over a field K of characteristic 0 the situation is completely different. Dicks and Formanek [2, Theorem 5.3] and Kharchenko [6, Theorem 2] have established that, for a finite group G acting linearly on $\operatorname{span}\{x_1, \ldots, x_m\}$, the algebra of invariants $K\langle x_1, \ldots, x_m\rangle^G$ is finitely generated if and only if G acts by scalar multiplication on the generating set $\{x_1, \ldots, x_m\}$.

The starting point of this research was the problem for finite generation of the fixed algebra of the free Lie algebra L_m of rank $m > 1$ over an arbitrary field K. The only result known is due to Belov [1] who has proved that L_m^G is not finitely generated if $G \neq \langle 1 \rangle$ is a finite cyclic subgroup of $\operatorname{Aut} L_m$ and the order of G is invertible in K. Our main result is the following.

Let J be an ideal of L_m, $m > 1$, such that J is contained in $L_m'' = [[L_m, L_m], [L_m, L_m]]$, and let the algebra L_m/J be residually nilpotent, i.e.,

$$\bigcap_{n \geq 1} (L_m/J)^n = \{0\}, \quad \textit{where } (L_m/J)^n = \underbrace{[L_m/J, \ldots, L_m/J]}_{n}.$$

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For any finite subgroup $G \neq \langle 1 \rangle$ *of the automorphism group* $\mathrm{Aut}(L_m/J)$ *such that the order of* G *is invertible in* K, *the fixed algebra*

$$(L_m/J)^G = \{f \in L_m/J \mid g(f) = f \text{ for all } g \in G\}$$

is not finitely generated.

The following algebras belong to the class under consideration:

(i) The free Lie algebra L_m.

(ii) The relatively free algebras $F_m(\mathfrak{U})$ in nonnilpotent varieties $\mathfrak{U}$ of Lie algebras over infinite fields of characteristic different from 2 and 3.

Our approach is based on noncommutative invariant theory of finite linear groups and essentially depends on the Maschke theorem. First we solve the problem for the free metabelian Lie algebra and then, as a consequence, for any algebra from the class under consideration. To the best of our knowledge, the problem for finite generation of L_m^G when $\mathrm{char}\, K$ divides $|G|$ is still open.

1. Preliminaries

Let K be an arbitrary field, and let L_m be the free Lie algebra of rank $m > 1$ freely generated by $x_1, \dots, x_m$. We use left normed notation for the commutators in L_m:

$$[u_1, u_2] = u_1(\mathrm{ad}\, u_2), \qquad [u_1, \dots, u_{n-1}, u_n] = [[u_1, \dots, u_{n-1}], u_n].$$

Let $V_m = \mathrm{span}\{x_1, \dots, x_m\}$ be the vector space spanned on the free generators of L_m. The natural action of the general linear group $\mathrm{GL}_m = \mathrm{GL}(V_m)$ on V_m can be extended diagonally on L_m. Let U be an ideal of L_m which is closed under all endomorphisms of L_m; U is called a verbal ideal. The class $\mathfrak{U}$ of all Lie algebras satisfying the polynomial identities from U is called a variety of algebras, and the factor-algebra $F_m(\mathfrak{U}) = L_m/U$ is the relatively free algebra of rank m in $\mathfrak{U}$. We use the same symbols $x_1, \dots, x_m$ for the generators of $F_m(\mathfrak{U})$. Clearly $F_m(\mathfrak{U})$ inherits the GL_m-module structure of L_m.

Let $\mathrm{char}\, K = 0$. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ (i.e., $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$) we denote by $W_m(\lambda)$ the irreducible GL_m-module corresponding to λ. The vector space L_m is multigraded in a natural way counting the entries of each variable x_i in the commutators from L_m. Every GL_m-submodule of L_m is a graded subspace, and its Hilbert (or Poincaré) series is

$$H(W, t_1, \dots, t_m, t) = \sum W^{\mathbf{n}} t_1^{n_1} \cdots t_m^{n_m} t^n,$$

where $\mathbf{n} = (n_1, \dots, n_m)$, $n_1 + \cdots + n_m = n$, and $W^{\mathbf{n}}$ is the multihomogeneous component of W of degree n_i in x_i. The Hilbert series of W is the character of the GL_m-module. If $g \in \mathrm{GL}_m$ has eigenvalues $\xi_1, \dots, \xi_m$, then

$$\chi_W(g, t) = H(W, \xi_1, \dots, \xi_m, t) = \sum \mathrm{tr}_{W^{(n)}}(g) t^n,$$

where $\mathrm{tr}_{W^{(n)}}(g)$ is the trace of g acting on the homogeneous component of degree n of W. Let G be a finite subgroup of GL_m. There is an analogue of the classical Molien formula for the invariants of a finite group on a GL_m-submodule of $K\langle x_1, \dots, x_m \rangle$ and hence on L_m.

Proposition 1.1 [5, Theorem 7]. *Let* $\mathrm{char}\, K = 0$, *let* W *be a* GL_m-*submodule of* L_m *with Hilbert series* $H(W, t_1, \dots, t_m, t)$, *and let* G *be a finite subgroup* *of* GL_m. *The graded vector space* $W^G = \{w \in W | g(w) = w$ *for all* $g \in G\}$ *has a Hilbert series*

$$H(W^G, t) = \frac{1}{|G|} \sum_{g \in G} \chi_W(g, t).$$

Proposition 1.2. *Let* $\operatorname{char} K = p \neq 0$, *and let the variety* $\mathfrak{U}$ *of Lie algebras over* K *be defined by multilinear polynomial identities.*

(i) *The relatively free algebra* $F_m(\mathfrak{U})$ *is a multigraded vector space, and its Hilbert series* $H(F_m(\mathfrak{U}), t_1, \dots, t_m, t)$ *is the* K*-character of the* GL_m*-module* $F_m(\mathfrak{U})$. *If* $g \in \mathrm{GL}_m$ *has eigenvalues* $\xi_1, \dots, \xi_m$, *then in* $K[[t]]$

$$\chi_{F_m(\mathfrak{U})}(g, t) = \operatorname{tr}_{F_m(\mathfrak{U})}(g, t) = H(F_m(\mathfrak{U}), \xi_1, \dots, \xi_m, t).$$

(ii) *If* G *is a finite subgroup of* GL_m *and* p *does not divide* $|G|$, *then*

$$H(F_m(\mathfrak{U})^G, t) \equiv \frac{1}{|G|} \sum_{g \in G} \chi_{F_m(\mathfrak{U})}(g, t) \pmod p,$$

where the congruence modulo p *means that the formal power series are equal as elements in* $(\mathbb{Z}/p\mathbb{Z})[[t]]$.

Proof. Let $\mathfrak{U}$ be defined by the multilinear polynomial identities $f_i(x_1, \dots, x_{n_i}) = 0$, $i = 1, 2, \dots$. Fixing a basis $y_1, \dots, y_m$ of V_m, the verbal ideal U of L_m corresponding to $\mathfrak{U}$ is spanned on

$$\{f_i(u_1, \dots, u_{n_i}) \operatorname{ad} w_1 \cdots \operatorname{ad} w_{q_i} | u_k, \ w_l \text{ commutators of } y_1, \dots, y_m\}.$$

Hence $F_m(\mathfrak{U})$ is a multigraded vector space with respect to any basis $\{y_1, \dots, y_m\}$. If $\widetilde{K}$ is an extension of the base field K, the $\widetilde{K}$-algebra $\widetilde{L}_m = \widetilde{K} \otimes_K L_m$ is the free Lie algebra generated by $\tilde{x}_i = 1 \otimes x_i$, $i = 1, \dots, m$, and $\widetilde{F}_m(\widetilde{\mathfrak{U}}) = \widetilde{K} \otimes_K F_m(\mathfrak{U})$ is the relatively free algebra of the variety $\widetilde{\mathfrak{U}}$ of $\widetilde{K}$-algebras defined by the polynomial identities $\tilde{f}_i = 1 \otimes f_i = 0$, $i = 1, 2, \dots$. Let $g \in \mathrm{GL}_m$. Considering an extension $\widetilde{K}$ of K we may assume that g acts as an upper triangular matrix on $\widetilde{V}_m = \widetilde{K} \otimes_K V_m$. The Hilbert series of $F_m(\mathfrak{U})$ is independent of the choice of the basis of V_m. Hence

$$\operatorname{tr}_{F_m(\mathfrak{U})}(g, t) = H(F_m(\mathfrak{U}), \xi_1, \dots, \xi_m, t).$$

(ii) For any finite-dimensional G-module W the linear operator

$$\pi : w \to \frac{1}{|G|} \sum_{g \in G} g(w), \qquad w \in W,$$

projects W onto W^G. Hence $\operatorname{tr}_W(\pi) \equiv \dim W^G \pmod p$, and this completes the proof because $F_m(\mathfrak{U})^G$ is a direct sum of its homogeneous components of degree $n = 1, 2, \dots$. □

We denote by $\mathfrak{A}^2$ the metabelian variety of Lie algebras defined by the polynomial identity $[[x_1, x_2], [x_3, x_4]] = 0$. Clearly $F_m(\mathfrak{A}^2) = L_m / L_m''$.

Proposition 1.3. *Over an arbitrary field,*

$$H(F_m(\mathfrak{A}^2), t_1, \dots, t_m, t) = 1 + (t_1 + \cdots + t_m)t + ((t_1 + \cdots + t_m)t - 1) \prod_{i=1}^{m} \frac{1}{1 - t_i t}.$$

Proof. The algebra $F_m(\mathfrak{A}^2)$ has a basis

$$x_i\,,\ [x_{i_1}\,,\, x_{i_2}\,,\,\dots\,,\, x_{i_n}]\,, \qquad i_1 > i_2 \leq \cdots \leq i_n\,,\ \ n = 2\,,\,3\,,\,\dots$$

(see, e.g., [8, §2, pp. 274–276 of the English translation] where bases of the free polynilpotent Lie algebras are found). Since the basis of $F_m(\mathfrak{A}^2)$ does not depend on the characteristic of the field, we may assume that $\operatorname{char} K = 0$. It is well known (see, e.g., [3, Proof of Lemma 2.5]) that as a GL_m-module

$$F_m(\mathfrak{A}^2) \cong W_m(1) + \sum_{n\geq 2} W_m(n-1\,,\,1).$$

The character of the irreducible GL_m-module $W_m(\lambda)$ is equal to the Schur function $S_\lambda(t_1\,,\,\dots\,,\,t_m)$. For $\lambda = (n)$ the Schur function $S_{(n)}(t_1\,,\,\dots\,,\,t_m)$ is equal to the nth complete symmetric function $h_n(t_1\,,\,\dots\,,\,t_m)$. By [7, Chapter 1, 5.15, p. 42], for $n \geq 1$

$$\begin{aligned} h_n(t_1,\dots,t_m)h_1(t_1,\dots,t_m) &= S_{(n)}(t_1,\dots,t_m)S_{(1)}(t_1,\dots,t_m) \\ &= S_{(n+1)}(t_1,\dots,t_m) + S_{(n,1)}(t_1,\dots,t_m) \\ &= h_{n+1}(t_1,\dots,t_m) + S_{(n,1)}(t_1,\dots,t_m), \end{aligned}$$

$$\begin{aligned} &H(F_m(\mathfrak{A}^2), t_1,\dots,t_m,t) \\ &\quad = S_{(1)}(t_1,\dots,t_m)t + \sum_{n\geq 2} S_{(n-1,1)}(t_1,\dots,t_m)t^n \\ &\quad = h_1(t_1,\dots,t_m)t + \sum_{n\geq 2}(h_{n-1}(t_1,\dots,t_m)h_1(t_1,\dots,t_m) - h_n(t_1,\dots,t_m))t^n \\ &\quad = h_1(t_1,\dots,t_m)t\left(1 + \sum_{n\geq 1} h_n(t_1,\dots,t_m)t^n\right) - \sum_{n\geq 2} h_n(t_1,\dots,t_m)t^n \\ &\quad = (h_1(t_1,\dots,t_m)t - 1)\sum_{n\geq 0} h_n(t_1,\dots,t_m)t^n + 1 + h_1(t_1,\dots,t_m)t, \end{aligned}$$

and the assertion follows because

$$h_1(t_1,\dots,t_m) = t_1 + \cdots + t_m\,, \qquad \sum_{n\geq 0} h_n(t_1,\dots,t_m)t^n = \prod_{i=1}^m \frac{1}{1-t_it}. \quad \square$$

Proposition 1.4. *If G is a finite subgroup of* $\operatorname{Aut} F_m(\mathfrak{A}^2)$, $m > 1$, *and the order of G is invertible in K, then the fixed algebra $F_m(\mathfrak{A}^2)^G$ is infinite dimensional.*

Proof. First, let G be a subgroup of GL_m. If $F_m(\mathfrak{A}^2)^G$ is finite dimensional, then its Hilbert series $H(F_m(\mathfrak{A}^2)^G, t)$ is a polynomial $f(t)$. By Propositions 1.1 and 1.2

$$H(F_m(\mathfrak{A}^2)^G, t) = \frac{1}{|G|}\sum_{g\in G} \chi_{F_m(\mathfrak{A}^2)}(g\,,\,t),$$

where the equality is replaced by a congruence modulo p if $\operatorname{char} K = p \neq 0$. Let $g \in G$ have characteristic roots $\xi_1\,,\,\dots\,,\,\xi_m$. Then

$$\xi_1 + \cdots + \xi_m = \operatorname{tr} g\,, \qquad \prod_{i=1}^m (1 - \xi_i t) = \det(1 - tg),$$

$$H(F_m(\mathfrak{A}^2)^G, t) = 1 + \frac{1}{|G|}\sum_{g\in G}\left((\operatorname{tr} g)t + \frac{(\operatorname{tr} g)t - 1}{\det(1-tg)}\right) = f(t),$$

$$\sum_{g\in G}\frac{(\operatorname{tr} g)t - 1}{\det(1-tg)} = |G|(f(t)-1) - \sum_{g\in G}(\operatorname{tr} g)t = f_1(t),$$

where $f_1(t)$ is a polynomial and $\det^{-1}(1-tg)$ is a short notation for the formal series $\prod_{i=1}^m(\sum_{n\geq 0}\xi_i^n t^n)$ if $\operatorname{char} K \neq 0$. Replacing t by 0 we obtain that $f_1(0) = -|G| \neq 0$. Obviously $\det(1-tg)$ is a polynomial of degree m. We compare the degrees of both the sides of the equality

$$\left(\prod_{g\in G}\det(1-tg)\right)\sum_{g\in G}\frac{(\operatorname{tr} g)t - 1}{\det(1-tg)} = \left(\prod_{g\in G}\det(1-tg)\right) f_1(t)$$

and obtain that the right-hand side is of degree $\geq |G|m$ while the left-hand side is of degree $\leq 1 + (|G|-1)m$. Since $m > 1$, this is impossible; i.e., $H(F_m(\mathfrak{A}^2)^G, t)$ is not a polynomial. In the general case, let $G \subset \operatorname{Aut} F_m(\mathfrak{A}^2)$ and let $\dim F_m(\mathfrak{A}^2) < \infty$. Hence the elements of $F_m(\mathfrak{A}^2)^G$ are of degree bounded by an integer n_0. The map $\rho\colon G \to \mathrm{GL}_m$ defined by

$$(\rho(g))(v) \equiv g(v) \pmod{F'_m(\mathfrak{A}^2)}, \qquad v \in V_m,\ g \in G,$$

is a group homomorphism. We consider the algebra of invariants $F_m(\mathfrak{A}^2)^{\rho(G)}$ of the linear group $\rho(G)$ acting on $F_m(\mathfrak{A}^2)$. Since $F_m(\mathfrak{A}^2)^{\rho(G)}$ is not finite dimensional, there exists a homogeneous $f(x_1, \dots, x_m) \in F_m(\mathfrak{A}^2)^{\rho(G)}$ of degree n greater than n_0. Obviously

$$(\rho(g))(f(x_1, \dots, x_m)) \equiv g(f(x_1, \dots, x_m)) \pmod{F_m^{n+1}(\mathfrak{A}^2)},$$

$$h = \frac{1}{|G|}\sum_{g\in G} g(f) \in F_m(\mathfrak{A}^2)^G, \qquad h \equiv \frac{1}{|G|}\sum_{g\in G}(\rho(g))(f) = f \pmod{F_m^{n+1}(\mathfrak{A}^2)},$$

$h \neq 0$, and $\deg h \geq n$. Hence $\dim F_m(\mathfrak{A}^2)^G = \infty$. □

2. The main results

Lemma 2.1. *Let R be a residually nilpotent Lie algebra generated by $r_1, \dots, r_m$, and let $r_1, \dots, r_m$ be linearly independent modulo R'. If G is a finite subgroup of $\operatorname{Aut} R$ of order invertible in K, then the canonical homomorphism $\rho\colon G \to \mathrm{GL}(R/R')$ is injective.*

Proof. Every automorphism ϕ of R is determined by $\phi(r_i)$, $i = 1, \dots, m$. The ideal R^n is closed under all endomorphisms of R; the map $\rho_n\colon G \to \operatorname{Aut}(R/R^{n+1})$ defined by

$$(\rho_n(g))(r_i) \equiv g(r_i) \pmod{R^{n+1}}, \qquad i = 1, \dots, m,$$

is a group homomorphism; $\rho = \rho_1$; and $G \supset \operatorname{Ker}\rho_1 \supset \operatorname{Ker}\rho_2 \supset \cdots$. The algebra R is residually nilpotent, i.e., $\bigcap_{n\geq 1} R^n = \{0\}$, and $\bigcap_{n\geq 1}\operatorname{Ker}\rho_n = \langle 1\rangle$. If $g \in \operatorname{Ker}\rho_n \backslash \operatorname{Ker}\rho_{n+1}$, $n \geq 1$, then $g(r_i) = r_i + f_i$, where $f_i = f_i(r_1, \dots, r_m) \in R^{n+1}$, $i = 1, \dots, m$, and f_{i_0} is not contained in R^{n+2} for some i_0. It is easy to see that

$$g^k(r_{i_0}) \equiv r_{i_0} + kf_{i_0}(r_1, \dots, r_m) \pmod{R^{n+2}}.$$

If $\operatorname{char} K = 0$, then $g^k \neq 1$ for all $k \geq 1$, which is impossible because $|G| < \infty$. If $\operatorname{char} K = p \neq 0$, then $g^k(r_{i_0}) \equiv r_{i_0} \pmod{R^{n+2}}$ if and only if p divides k. Hence $|G|$ is not invertible in K, which is a contradiction, and $\operatorname{Ker} \rho_n = \langle 1 \rangle$ for all $n \geq 1$. □

Lemma 2.2. *Every finite subset of $F_m(\mathfrak{A}^2)$ which is linearly independent modulo $F'_m(\mathfrak{A}^2)$ generates a free metabelian subalgebra of $F_m(\mathfrak{A}^2)$.*

Proof. The following arguments hold for any variety $\mathfrak{U}$ defined by homogeneous polynomial identities. Let $w_1, \dots, w_q \in F_m(\mathfrak{A}^2)$ be linearly independent modulo $F'_m(\mathfrak{A}^2)$. Applying an invertible linear transformation to the free generators of $F_m(\mathfrak{A}^2)$ we may assume that $w_i \equiv x_i \pmod{F'_m(\mathfrak{A}^2)}$, $i = 1, \dots, q$. Let $0 \neq f(x_1, \dots, x_r) \in F^k_m(\mathfrak{A}^2) \backslash F^{k+1}_m(\mathfrak{A}^2)$. Since

$$f(w_1, \dots, w_q) \equiv f(x_1, \dots, x_q) \not\equiv 0 \pmod{F^{k+1}_m(\mathfrak{A}^2)},$$

we obtain that all the relations between $w_1, \dots, w_q$ follow from the metabelian identity, i.e., $w_1, \dots, w_q$ generate a free metabelian algebra. □

Theorem 2.3. *Let $R = F_m(\mathfrak{A}^2) = L_m/L''_m$ be the free metabelian Lie algebra of rank $m > 1$. If $G \neq \langle 1 \rangle$ is a finite subgroup of* $\operatorname{Aut} R$ *of order invertible in the base field K, then the fixed algebra $F_m(\mathfrak{A}^2)^G$ is not finitely generated.*

Proof. The ideal $F'_m(\mathfrak{A}^2)$ is a G-module and by the Maschke theorem there exist G-submodules W and Z of $F_m(\mathfrak{A}^2)$ such that

$$F_m(\mathfrak{A}^2) = W \oplus Z \oplus F'_m(\mathfrak{A}^2), \quad (W \oplus Z)^G = Z, \quad \dim(W \oplus Z) = m.$$

We fix bases $\{w_1, \dots, w_q\}$ of W and $\{z_{q+1}, \dots, z_m\}$ of Z. We may assume that

$$w_i \equiv x_i, \quad z_j \equiv x_j \pmod{F'_m(\mathfrak{A}^2)}, \qquad i = 1, \dots, q, \ j = q+1, \dots, m.$$

If $q = 0$, then G acts on $F_m(\mathfrak{A}^2)$ trivially modulo $F'_m(\mathfrak{A}^2)$ and, by Lemma 2.1, $G = \langle 1 \rangle$. Hence $q > 0$. Let the algebra $F_m(\mathfrak{A}^2)^G$ be finitely generated. As a vector space $F_m(\mathfrak{A}^2)^G$ is a direct sum of Z and $F'_m(\mathfrak{A}^2)^G$, and hence $F_m(\mathfrak{A}^2)^G$ is generated as an algebra by $z_{q+1}, \dots, z_m$ and some $f_1, \dots, f_r \in F'_m(\mathfrak{A}^2)^G$. Since the f_i's are in $F'_m(\mathfrak{A}^2)$, every element of $F'_m(\mathfrak{A}^2)^G$ is of the form

$$f = \sum (\alpha_{ik} f_i + \beta_{jk}[z_{j_1}, z_{j_2}]) \operatorname{ad}^{k_{q+1}} z_{q+1} \cdots \operatorname{ad}^{k_m} z_m.$$

The operators $\operatorname{ad} z_j$ and $\operatorname{ad} x_j$ are equal on $F'_m(\mathfrak{A}^2)$ and

$$f = \sum (\alpha_{ik} f_i + \beta_{jk}[z_{j_1}, z_{j_2}]) \operatorname{ad}^{k_{q+1}} x_{q+1} \cdots \operatorname{ad}^{k_m} x_m,$$

i.e., the elements of $F_m(\mathfrak{A}^2)^G$ are of bounded degree in $x_1, \dots, x_q$.

Case 1. $q > 1$. The group G acts on the algebra $\langle W \rangle$ generated by $w_1, \dots, w_q$ as a group of automorphisms, and the vector space W has no fixed points. By Lemma 2.2, $\langle W \rangle \cong F_q(\mathfrak{A}^2)$, and by Proposition 1.4 $\langle W \rangle^G$ is infinite dimensional. Since for $f \in \langle W \rangle^G$

$$\begin{aligned} f &= \sum \alpha_{ik}[w_{i_1}, w_{i_2}] \operatorname{ad}^{k_1} w_1 \cdots \operatorname{ad}^{k_q} w_q \\ &= \sum \alpha_{ik}[w_{i_1}, w_{i_2}] \operatorname{ad}^{k_1} x_1 \cdots \operatorname{ad}^{k_q} x_q \in F_m(\mathfrak{A}^2)^G, \end{aligned}$$

the degrees in $x_1, \dots, x_q$ of the elements from $F_m(\mathfrak{A}^2)^G$ are not bounded, which is a contradiction.

Case 2. $q = 1$. Every element $g \in G$ acts on w_1 by scalar multiplication, i.e., $g(w_1) = \xi_g w_1$ and $\xi_g^n = 1$, $n = |G|$. Hence

$$g(z_m \mathrm{ad}^{nk} w_1) = \xi_g^{nk} z_m \mathrm{ad}^{nk} w_1 = z_m \mathrm{ad}^{nk} w_1, \qquad g \in G, \ k = 1, 2, \dots,$$

and $z_m \mathrm{ad}^{nk} w_1 \in F_m(\mathfrak{A}^2)^G$. Again $z_m \mathrm{ad}^{nk} w_1 = [z_m, w_1] \mathrm{ad}^{nk-1} x_1$ and there exists no upper bound for the degree in x_1 of the elements from $F_m(\mathfrak{A}^2)^G$. $\square$

Theorem 2.4. *Let L_m be the free Lie algebra of* rank $m > 1$ *over a field K, and let J be an ideal of L_m such that $J \subset L_m''$ and the algebra L_m/J is residually nilpotent. If $G \neq \langle 1 \rangle$ is a finite group of automorphisms of L_m/J of order invertible in K, then the algebra of fixed points $(L_m/J)^G$ is not finitely generated.*

Proof. Every automorphism of $R = L_m/J$ induces an automorphism on R/R''. By Lemma 2.1 the group G acts faithfully on R/R' and hence on R/R''. Now $J \subset L_m''$ implies that $R/R'' \cong L_m/L_m'' \cong F_m(\mathfrak{A}^2)$. The map $R^G \to (R/R'')^G$ is surjective because $|G|$ is invertible in K. If R^G is finitely generated, the same holds for $(R/R'')^G$, and this is in contradiction with Theorem 2.3. $\square$

Corollary 2.5. *If $G \neq \langle 1 \rangle$ is a finite subgroup of* $\mathrm{Aut}\, L_m$, *$m > 1$, and $|G|$ is invertible in K, then L_m^G is not finitely generated.*

Theorem 2.6. *Let the field K be infinite,* $\mathrm{char}\, K \neq 2, 3$, *and let $\mathfrak{U}$ be a variety of Lie algebras over K. For a finite subgroup $G \neq \langle 1 \rangle$ of* $\mathrm{Aut}\, F_m(\mathfrak{U})$, *$m > 1$, the fixed algebra $F_m(\mathfrak{U})^G$ is finitely generated if and only if $\mathfrak{U}$ is nilpotent.*

Proof. Let $U \subset L_m$ be the verbal ideal corresponding to $\mathfrak{U}$. Since the base field is infinite, the algebra $F_m(\mathfrak{U})$ is multihomogeneous and hence residually nilpotent. It is well known that either $\mathfrak{U}$ satisfies the Engel identity $x_2(\mathrm{ad}^n x_1) = 0$ for some $n > 0$ or $U \subset L_m''$. If the variety $\mathfrak{U}$ satisfies $x_2(\mathrm{ad}^n x_1) = 0$, by the theorem of Zel′manov [9], $\mathfrak{U}$ is nilpotent. Hence $F_m(\mathfrak{U})$ is finite dimensional and $F_m(\mathfrak{U})^G$ is finitely generated. If $U \subset L_m''$ we apply Theorem 2.3. $\square$

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References

1. A. I. Belov, *Periodic automorphisms of free Lie algebras and their fixed points*, preprint.
2. W. Dicks and E. Formanek, *Poincaré series and a problem of S. Montgomery*, Linear and Multilinear Algebra **12** (1982), 21–30.
3. V. Drensky and A. Kasparian, *Polynomial identities of eighth degree for* 3×3 *matrices*, Ann. Univ. Sofia Fac. Math. Méc. **77** (1983), 175–195.
4. J. L. Dyer and G. P. Scott, *Periodic automorphisms of free groups*, Comm. Algebra **3** (1975), 195–201.
5. E. Formanek, *Noncommutative invariant theory*, Group Actions on Rings, Contemp. Math., vol. 43, Amer. Math. Soc., Providence, RI, 1985, pp. 87–119.

6. V. K. Kharchenko, *Noncommutative invariants of finite groups and Noetherian varieties*, J. Pure Appl. Algebra **31** (1984), 83–90.

7. I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, Clarendon, Oxford, 1979.

8. A. L. Shmel'kin, *Free polynilpotent groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 91–122; English transl. Amer. Math. Soc. Transl. Ser. 2, vol. 55, Amer. Math. Soc., Providence, RI, 1966, pp. 270–304.

9. E. I. Zelmanov, *Weakened Burnside problem*, Sibirsk. Math. Zh. **30** (1989), 68–74; English transl. Siberian Math. J. **30** (1989), 885–891.

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