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LINEAR INDEPENDENCE OF ITERATES AND MEROMORPHIC SOLUTIONS OF FUNCTIONAL EQUATIONS

JENS PETER REUS CHRISTENSEN AND PAL FISCHER

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ABSTRACT. It is shown that except for trivial cases a sequence generated by iteration of meromorphic functions is always linearly independent. The nonexistence of meromorphic solutions and solutions having only isolated singularities of the Feigenbaum functional equation is proven.

1. INTRODUCTION

In order to solve the Feigenbaum conjecture concerning a class of one-parameter families of smooth unimodal transformations in the case when $\epsilon = 1$, Lanford considered the Feigenbaum functional equation [10, 11]. This equation has the form

$$f(f(\lambda x)) = -\lambda f(x), \qquad 0 < \lambda = -f(1) < 1, \tag{1}$$

where the solution f is an even and unimodal function on the interval $[-1, 1]$, with $f(0) = 1$, $f''(0) < 0$ and fairly smooth.

In [11] Lanford exhibited a computer-assisted proof of a solution of (1) which is analytic in a neighbourhood of the whole real line. It was shown in [6], as it was predicted in [11], that (1) does not have an entire solution. This proof in [6] was based upon a result of Polya concerning the growth of Nevanlinna's characteristics of composite functions. In [9], the same result was shown using various properties of the maximum modulus function. Professor R. M. Redheffer of UCLA was the first to ask the second-named author about the existence of meromorphic solutions of (1), and furthermore he was asking about the existence of solutions of (1) having only isolated singularities. The nonexistence of solutions of the above types of (1) will be presented here.

Equation (1) can be transformed into an eigenvalue problem. The following result was obtained in [6].

Theorem 1. *Let $\mathscr{E}$ be the set of all entire functions. Let f be an entire function which is not a polynomial of degree ≤ 1. Let $\mathscr{T}_f : \mathscr{E} \to \mathscr{E}$ be defined by*

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$\mathscr{T}_f(g) = g \circ f$ for some $f \in \mathscr{E}$. Then $\mathscr{T}_f$ has no eigenvalue $\neq \{0, 1\}$ and any eigenfunction with eigenvalue 1 is a constant.

If f is a function then $f^{\circ 1} = f$, $f^{\circ 2} = f \circ f, \dots, f^{\circ(n+1)} = f \circ f^{\circ n}, \dots$ for $n = 1, 2, \dots$ denote its successive iterates and $f^{\circ\circ}$ stands for the identity mapping.

Actually, the following stronger result was also established in [6].

Theorem 2. *Let g be a nonconstant entire function, and let f be an entire function which is not a polynomial of degree ≤ 1. Then $\{g \circ f^{\circ n} : n \geq 0\}$ is a linearly independent sequence of entire functions.*

Here, we extend the previous result to the case when the functions f and g are meromorphic.

2. Notation and preliminaries

The set of all meromorphic functions on $\mathbb{C}$ is designated by $\mathscr{M}$. The forward orbit of a point z_0 under f is denoted by $\mathscr{O}^+(z_0)$, and the closure of a set A is designated by $\overline{A}$. If $f = P/Q$, where P and Q are relatively prime polynomials, with $\deg P = n$ and with $\deg Q = m$, then the degree of f is defined to be $\max(m, n)$. If f and g are any two nonconstant rational functions, then the following relation holds (see [4]):

$$\deg(f \circ g) = \deg(f)\deg(g). \tag{2}$$

If f is either a rational function, or an entire function, the Julia set of f is denoted by $\mathscr{J}(f)$. If f is not a rational function of degree less than two, then $\mathscr{J}(f)$ has the following properties [1–5, 7–8]. It is a nonempty perfect set, which is completely invariant under f, and it is the closure of the repelling periodic points of f. The set $\{z_0 \in \mathscr{J}(f) : \overline{\mathscr{O}^+(z_0)} = \mathscr{J}(f)\}$ is dense in $\mathscr{J}(f)$ and it is a G_δ set.

The following lemma serves as an easy reference.

Lemma 1. *If $F \in \mathscr{M} \setminus \mathscr{E}$ and if G is a nonrational meromorphic function, then $G \circ F$ is not a meromorphic function.*

3. Meromorphic functions and the Feigenbaum equation

The topic of this section is to show that if $\lambda \neq 0$ then (1) has no nontrivial meromorphic solution. This result is prepared by the following.

Theorem 3. *Let $F \in \mathscr{M} \setminus \mathscr{E}$ be a solution of the functional equation*

$$F(z) = -\frac{1}{\lambda} F(F(\lambda z)), \tag{3}$$

where λ is a fixed nonzero complex number. Then $\lambda = i$ or $\lambda = -i$ and the corresponding solution F is a rational function of degree one.

Remark 1. We are saying that a meromorphic function F satisfies (3) if it satisfies (3) for all z which belongs to the domain of $F(z)$ and to the domain of $F(F(\lambda z))$.

Proof. Let $F \in \mathscr{M} \setminus \mathscr{E}$ be a solution of (3) for a given λ. Now, if F were a nonrational meromorphic function, then one could derive by Lemma 1, that

$D(F \circ F(\lambda z))$, where D is a nonzero complex number, cannot be a meromorphic solution of (3). Thus, F must be a rational function.

Equation (2) indicates that a rational function of degree larger than one cannot be a solution of (3) for any given nonzero λ. One can assume that F has the form

$$F(z) = \frac{Az + B}{z + C}, \quad \text{where} \quad AC - B \neq 0. \tag{4}$$

It can be verified by substitution that F can be a solution of (3) only when each of the quantities A, B, C, and $A + C$ are different from zero. Note that

$$F(F(\lambda z)) = \frac{(A^2 + B)\lambda z + B(A + C)}{(A + C)\lambda z + (B + C^2)}.$$

Observe that F can satisfy (3) only if each of the following hold:

$$\frac{B}{\lambda} = -B\lambda, \quad \frac{A^2 + B}{A + C} = -A\lambda, \quad \frac{B + C^2}{(A + C)\lambda} = C. \tag{5}$$

Thus, λ can be only i or $-i$. Conversely, let $\lambda = i$ or $\lambda = -i$, and let λ be fixed. Then, it is clear that if F has the form (4), if none of the quantities A, B, C, or $A + C$ is zero, and if, in addition, A, B, and C satisfy (5), then F satisfies (3) for the given λ.

Examples. Let

$$F_1(z) = \frac{-iz + i}{z + 1} \quad \text{and} \quad F_2(z) = (-i)\frac{z + 1}{z - 1}; \tag{6}$$

then F_1 is a solution of (3) for $\lambda = i$ and F_2 is a solution of (3) for $\lambda = -i$.

Now, we can prove the main result of this section.

Theorem 4. *The Feigenbaum functional equation* (1) *does not have a meromorphic solution f such that $f(0) = 1$ and $f(1)$ is real.*

Proof. Theorem 4 of [6] shows that (1) does not have an entire solution when $f(0) = 1$. The condition $f(1) = -\lambda$ implies that λ must be real. Then, Theorem 3 indicates that the proof is complete.

4. Problems from iteration theory

A generalization of Theorem 2 will be given first in this section.

Theorem 5. *Let f be an entire function which is not a polynomial of degree ≤ 1, and let g be a nonconstant meromorphic function. Then $\{g \circ f^{\circ n} : n \geq 0\}$ is a linearly independent sequence of functions on $\mathbb{C}$.*

Proof. The beginning of the proof is similar to the one of Theorem 2 as it is given in [6]. Thus, let $P(z) = \sum \lambda_p z^p$ be a polynomial with complex coefficients, and let $P(T)$ be a linear operator from the space of meromorphic functions into itself defined by

$$P(T)(g) = \sum \lambda_p (g \circ f^{\circ p}). \tag{7}$$

The mapping $P \to P(T)$ is linear and multiplicative. Suppose the theorem is false. Then there exist functions f and g and a nonconstant polynomial P

so that $P(T)(g) = 0$. Without loss of generality, one can assume that $\lambda_0 \neq 0$. Hence, $P(z) = \prod_{i=1}^{n}(z - \lambda_i)$, where the degree of P is n and $\lambda_i \neq 0$ for $1 \leq i \leq n$. Since the mapping $P \to P(T)$ is multiplicative, it suffices to show that the assumption

$$g \circ f = \lambda g + c, \qquad \lambda \neq 0, \ c = \text{const}, \tag{8}$$

leads to a contradiction.

From (8), by iteration, we obtain that

$$g \circ f^{\circ k} = \begin{cases} \lambda^k g + \dfrac{\lambda^k - 1}{\lambda - 1} c = \lambda^k \left(g + \dfrac{c}{\lambda - 1} \right) - \dfrac{c}{\lambda - 1} & \text{for } \lambda \neq 1, \\ g + kc & \text{for } \lambda = 1. \end{cases} \tag{9}$$

Equation (9) shows that if z_0 is a pole of g, then so is $f(z_0)$, and we can deduce by induction that $f^{\circ k}(z_0)$ is also a pole for all $k \in \mathbb{N}$. Let $w \in \mathcal{J}(f)$, such that $\overline{\mathcal{O}^+(w)} = \mathcal{J}(f)$. Then w cannot be a pole of g, since the poles of g cannot be dense in a perfect set. We separate five cases, and in all the cases the assumption that g is a nonconstant meromorphic function will lead to a contradiction.

In the first case $|\lambda| > 1$. Let $w \in \mathcal{J}(f)$, such that $\overline{\mathcal{O}^+(w)} = \mathcal{J}(f)$, and let n_k be an increasing sequence of positive integers so that $f^{\circ n_k}(w) \to w$. Along this sequence the left side of (9) tends to $g(w)$, while the right side can have a finite limit only when $g(w) = -c/(\lambda - 1)$. Thus g must be constant on a dense subset of a perfect set, and hence g must also be a constant function.

In the second case $|\lambda| < 1$. By choosing w and the sequence n_k the same way as in the case $|\lambda| > 1$, one can deduce that $g = -c/(\lambda - 1)$ on a dense subset of a perfect set, and therefore g must be a constant function.

In the third case $\lambda = e^{i\theta}$, $\lambda \neq 1$, where θ is a nonzero real number, which is commensurable with 2π. By choosing w and the sequence n_k as in the previous cases and observing that w is not a periodic or an eventually periodic point of f, we see that (9) implies that

$$g(w_{n_k}) = \lambda^{n_k} \left(g(w) + \frac{c}{\lambda - 1} \right) - \frac{c}{\lambda - 1}. \tag{10}$$

The sequence $\{g(w_{n_k})\}$ has infinitely many different elements, but the right side of (10) can take only finitely many different values.

In the fourth case $\lambda = e^{i\theta}$, where the nonzero real number θ is not commensurable with 2π. We show that, if $z_0 \in \mathcal{J}(f)$, then z_0 cannot be a pole of g. As already observed, if $w \in \mathcal{J}(f)$, and if $\overline{\mathcal{O}^+(w)} = \mathcal{J}(f)$, then w cannot be a pole of g. Let $w \in \mathcal{J}(f)$ so that the forward orbit of f starting at w is dense in $\mathcal{J}(f)$. The right side of (9) is bounded along the points of $\mathcal{O}^+(w)$. If z_0 were a pole of g, then there would exist an infinite subsequence of $\mathcal{O}^+(w)$ along which the left side of (9) would be unbounded. This contradiction shows that g cannot have a pole at any point of $\mathcal{J}(f)$.

Let w_0 be a periodic point of $\mathcal{J}(f)$, and let m be its smallest positive period. The substitution of w_0 into (9) yields the relation

$$0 = (\lambda^m - 1) \left(g(w_0) + \frac{c}{\lambda - 1} \right).$$

Since θ is noncommensurable with 2π, it follows that $g(w_0) = -c/(\lambda - 1)$. Thus, g assumes the same value at each periodic point of $\mathscr{J}(f)$; because $\mathscr{J}(f)$ is the closure of the repelling periodic points of f, g must be constant on $\mathscr{J}(f)$. Hence, g must be a constant function.

In the last case $\lambda = 1$. The point w and the sequence n_k are chosen in the same way as in the first case. One derives from (9) that $g \circ f^{\circ n_k}(w) - g(w) = n_k c$ for all n_k. Clearly, this is possible only when $c = 0$. Assume $c = 0$; then (9) implies that

$$g(w_n) = g(w), \quad \text{where} \quad w_n = f^{\circ n}(w). \tag{11}$$

The sequence $\{w_n\}$ is dense in $\mathscr{J}(f)$; hence, (11) indicates that g must be a constant function.

Now, a generalization of the previous theorem will be given.

Theorem 6. *Let f be a meromorphic function which is not a rational function of degree ≤ 1, and let g be a nonconstant meromorphic function. Then $\{g \circ f^{\circ n} : n \geq 0\}$ is a linearly independent sequence of functions on $\mathbb{C}$.*

Proof. Suppose the theorem is false. By arguing essentially the same way as in the proof of the previous theorem, the existence of nonconstant meromorphic functions f and g satisfying (8) can be obtained, where f is not a rational function of degree ≤ 1. Without loss of generality let $f \in \mathscr{M} \setminus \mathscr{E}$. Hence f has a pole at a finite complex number z_0. We deduce that f and g must be rational functions.

Indeed, if z_0 is also a pole of g, then, as $z \to z_0$, we have by (8) that $|\lambda g(z) + c| \to \infty$. Letting $w \to \infty$ we get $|g(w)| \to \infty$. Therefore, g must be a rational function (see [12]). If z_0 is not a pole of g, then $\lambda g(z) + c \to \lambda g(z_0) + c$, as $z \to z_0$. It follows from (8) that $g(w) \to \lambda g(z_0) + c$ as $w \to \infty$. Hence, g must be a rational function. If f were a nonrational meromorphic function, then there would exist sequences of finite complex numbers $\{u_k\}$ and $\{w_k\}$ so that $|u_k| \to \infty$, $|w_k| \to \infty$ as $k \to \infty$ and $f(u_k) \to z_1$ while $f(w_k) \to z_2$ where z_1 and z_2 are regular points of g with the property that $g(z_1) \neq g(z_2)$. The existence of such sequences shows that f and g cannot satisfy (8).

Equation (2) implies that when f and g are nonconstant rational functions, where the degree of f is at least two, then they cannot satisfy (8).

Remark 2. With the aid of an example, we can show that the assumption about f in the previous theorem is essential. Indeed, let

$$f(z) = \frac{1}{z} \quad \text{and} \quad g(z) = \frac{z^n + 1}{z^n - 1},$$

where n is a fixed positive integer. Then $g \circ f = -g$.

5. On the Feigenbaum equation

In this section it will be shown that (1) does not have a solution having only isolated singularities.

Let $\mathscr{N}$ be the set of all complex-valued functions f such that f is analytic everywhere but at most countable many points of $\mathbb{C}$, with the property that the sequence of the points of singularities does not have an accumulation point in the finite complex plane.

Theorem 7. *A function $F \in \mathcal{N} \setminus \mathcal{M}$ cannot be a solution of the functional equation* (3), *where λ is a fixed nonzero complex number.*

Proof. Assume, to obtain a contradiction, that $F \in \mathcal{N} \setminus \mathcal{M}$ is a solution of (3) with a given nonzero complex number λ. Hence a finite complex number λz_0 is an essential singularity of F. As a first step, it will be shown that the point z_0 is also an essential singularity of F. Indeed, consider a sufficiently small neighbourhood of the point λz_0, which does not contain any other singularities of F. If z_0 were a regular point of F, then, as $z \to z_0$, the left side of (3) would tend to $F(z_0)$, but the right side of (3) clearly does not have a limit. A similar argument shows that z_0 cannot be a pole of F.

As a second step, it will be shown that F cannot have more than one essential singularity in the finite complex plane. Indeed, assume that z_1 and z_2 are essential singularities of F in the finite complex plane. Consider a small deleted neighbourhood of z_1. By Picard's theorem, there are infinitely many points u in this deleted neighbourhood so that $F(u) = z_1$ or $F(u) = z_2$. Note that the point u/λ is an essential singularity of F. Therefore, the point z_1/λ is not an isolated singularity of F. Since if u is an essential singularity of F so is u/λ, it follows that the proof is complete for all the cases, except when $\lambda = 1$.

Assume that $\lambda = 1$. Let $z = F(w)$ for some $w \in \mathbb{C}$. Then by (3) it follows that

$$F(z) = F(F(w)) = -F(w) = -z.$$

Hence for each number z which belongs to the domain and to the range of F, we have $F(z) = -z$. It is easily checked that such a function cannot have an isolated essential singularity.

The previous result together with Theorem 4 yield the following.

Theorem 8. *The Feigenbaum functional equation* (1) *does not have a solution f in the linear space of functions $\mathcal{N}$ such that $f(0) = 1$ and $f(1)$ is real.*

References

1. I. N. Baker, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.
2. ——, *Limit functions and sets of non-normality in iteration theory*, Ann. Acad. Sci. Fenn. Ser. A I Math. **467** (1970), 1–11.
3. ——, *The iteration of polynomials and transcendental entire functions*, J. Austral. Math. Soc. Ser. A **30** (1981), 483–495.
4. A. F. Beardon, *Iteration of rational functions*, Springer-Verlag, New York, 1991.
5. P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), 85–139.
6. J. P R. Christensen and P. Fischer, *Linear independence of iterates and entire solutions of functional equations*, Proc. Amer. Math. Soc. **103** (1988), 1120–1124.
7. R. L. Devaney, *An introduction to chaotic dynamical systems*, Addison-Wesley, New York, 1987.
8. P. Fatou, *Sur l'iteration des fonction transcendantes entiers*, Acta Math. **47** (1926), 337–370.
9. L. B. Gonzalez, *Linear independence of iterates of entire functions*, Proc. Amer. Math. Soc. **112** (1991), 1033–1036.
10. O. E. Lanford III, *Smooth transformations of intervals, Seminaire Bourbaki*, Lecture Notes in Math., vol. 901, Springer-Verlag, New York , 1981, pp. 36–54.

11. ——, *A computer-assisted proof of the Feigenbaum conjectures*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 427–434.

12. R. Nevanlinna and V. Paatero, *Introduction to complex analysis*, Addison-Wesley, Reading, MA, 1969.

MATEMATISK INSTITUT, KØBENHAVENS UNIVERSITET, UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN, DENMARK

E-mail address, J. P. R. Christensen: `reus@math.ku.dk`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GUELPH, GUELPH, ONTARIO, CANADA N1G 2W1

E-mail address, P. Fischer: `pfischer@msnet.mathstat.uoguelph.ca`