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AN EXPLICIT FORMULA FOR FUNDAMENTAL SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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The theme of this note is the classical theorem of Ehrenpreis [1] and Malgrange [5, 6] on the existence of fundamental solutions for all linear partial differential operators with constant coefficients. The aim is to present an explicit formula for such fundamental solutions. The procedure is that of Malgrange, but conducted in such a manner that one remains within the frame of equalities and avoids the transition to estimations, which in the former proof required the Hahn-Banach theorem. We note that the well-known method of construction of fundamental solutions via the so-called Hörmander staircase is not explicit in the present sense, since it involves some kind of partition of unity (and sometimes a transformation of variables). It has been termed "explicit" (in quotation marks) in Hörmander [3, p. 67]; for more recent presentations we refer to Folland [2] and to the sophisticated version in Hörmander [4, pp. 189–191]. Furthermore, the recent elementary proof of Rosay [7] leads to the mere existence assertion too. We shall restrict ourselves to the statement and proof of our formula. It will of course be natural to use the formula in order to obtain further results on fundamental solutions.

As a rule the notation is that of Hörmander [4]. Let λ^n denote Lebesgue measure on $\mathbb{R}^n$ and $\tau^n := \tau \times \cdots \times \tau$ on T^n, where τ is arc length on $T := \{s \in \mathbb{C}: |s| = 1\}$ normalized to $\tau(T) = 1$. In addition, we define $[\![\cdot]\!]: \mathbb{C} \to \mathbb{C}$ to be $[\![z]\!] = \overline{z}/z$ for $z \neq 0$ and $= 0$ for $z = 0$. Thus $[\![\cdot]\!]$ is continuous outside of 0 and of modulus ≤ 1.

We consider a polynomial P in n variables with complex coefficients of precise degree $m \geq 0$

$$P(z) = \sum_{|\alpha| \leq m} c(\alpha) z^\alpha \quad \text{with } P_m(z) := \sum_{|\alpha| = m} c(\alpha) z^\alpha$$

and

$$S(P) := \sum_{|\alpha| = m} |c(\alpha)|^2 > 0$$

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and its differential operator

$$P(\partial) := \sum_{|\alpha|\le m} c(\alpha)\partial^\alpha.$$

We fix a natural number N with $2N > n$. For $\varepsilon > 0$ and an n-fold multi-index α we define $K_\alpha^\varepsilon\colon \mathbb{R}^n \to \mathbb{C}$ to be

$$K_\alpha^\varepsilon(x) = \int_{\mathbb{R}^n\times T^n} \frac{[\![(it+\varepsilon s)^\alpha P(it+\varepsilon s)]\!]}{(1+|it+\varepsilon s|^2)^N} P_m(s) e^{\langle it+\varepsilon s, x\rangle} d(\lambda^n\times\tau^n)(t,s) \quad \text{for } x\in\mathbb{R}^n.$$

For fixed $x\in\mathbb{R}^n$ the integrand is in $L^1(\lambda^n\times\tau^n)$, and for $|x|\le c<\infty$ it has a majorant in $L^1(\lambda^n\times\tau^n)$. Hence, K_α^ε is continuous on $\mathbb{R}^n$, and we have

$$|K_\alpha^\varepsilon(x)| \le \|P_m|T^n\|_{\sup} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|t|^2)^N}\, d\lambda^n(t)\right) \exp\left(\varepsilon\sum_{l=1}^n |x_l|\right) \quad \text{for } x\in\mathbb{R}^n.$$

Thus for $\varepsilon > 0$

$$E^\varepsilon := \frac{1}{(2\pi)^n S(P)\varepsilon^m} \sum_{|\alpha|\le N} \frac{N!}{(N-|\alpha|)!\alpha!} \partial^{2\alpha} K_\alpha^\varepsilon$$

is a distribution of finite order $\le 2N$ on $\mathbb{R}^n$.

Theorem. *For $\varepsilon>0$ the distribution E^ε satisfies $P(\partial)E^\varepsilon = \delta$.*

The proof will be subdivided into five sections. The idea to integrate over the unit torus instead of the unit ball is from Rudin [8, pp. 192–197, 378]. Let $D(\mathbb{R}^n)$ consist of the functions $\mathbb{R}^n\to\mathbb{C}$ of class C^∞ with compact support.

(1) Application of the one-dimensional Cauchy formula. For $g\colon\mathbb{C}\to\mathbb{C}$ an entire function and $Q\colon Q(z)=\sum_{l=0}^m c_l z^l$ a polynomial, one has

$$g(0)\overline{c}_m = \int_T g(t)\overline{Q(t)}t^m\, d\tau(t).$$

This follows upon application of the Cauchy formula to the function

$$G\colon G(z) = g(z)\sum_{l=0}^m \overline{c}_l z^{m-l} \quad \text{for } z\in\mathbb{C},$$

since for $t\in T$ one has $G(t) = g(t)\overline{Q(t)}t^m$.

(2) Transition to the n-dimensional Cauchy formula. For $f\colon\mathbb{C}^n\to\mathbb{C}$ an entire function and P a polynomial as above, one has

$$f(z)S(P)\varepsilon^m = \int_{T^n} f(z+\varepsilon s)P(z+\varepsilon s)[\![P(z+\varepsilon s)]\!]P_m(s)\, d\tau^n(s)$$

for $z\in\mathbb{C}^n$ and $\varepsilon>0$. To see this one applies (1) for fixed $s\in T^n$ to the functions

$$g\colon g(t) = f(z+\varepsilon ts),$$
$$Q\colon Q(t) = P(z+\varepsilon ts) = P_m(s)\varepsilon^m t^m + \cdots \quad \text{for } t\in\mathbb{C}.$$

One obtains

$$f(z)\overline{P_m(s)}\varepsilon^m = \int_T f(z+\varepsilon ts)\overline{P(z+\varepsilon ts)}t^m\, d\tau(t),$$
$$f(z)|P_m(s)|^2\varepsilon^m = \int_T f(z+\varepsilon ts)\overline{P(z+\varepsilon ts)}P_m(ts)\, d\tau(t).$$

We integrate over T^n with respect to τ^n and note that

$$S(P) = \int_T |P_m(s)|^2 \, d\tau^n(s).$$

From the Fubini theorem we conclude

$$\begin{aligned} f(z)S(P)\varepsilon^m &= \int_T \left(\int_{T^n} f(z + \varepsilon ts)\overline{P(z + \varepsilon ts)} P_m(ts) \, d\tau^n(s) \right) d\tau(t) \\ &= \int_{T^n} f(z + \varepsilon s)\overline{P(z + \varepsilon s)} P_m(s) \, d\tau^n(s), \end{aligned}$$

where the second equation is obvious since the inner integral on its left side does not depend on $t \in T$. The assertion follows.

(3) The Fourier-Laplace transformation. For $u \in D(\mathbb{R}^n)$ one defines $\hat{u}: \mathbb{C}^n \to \mathbb{C}$ to be

$$\hat{u}(z) = \int_{\mathbb{R}^n} e^{\langle z, x \rangle} u(x) \, d\lambda^n(x) \quad \text{for } z \in \mathbb{C}^n.$$

Thus $\hat{u}$ is an entire function. One proves that

$$(-1)^{|\alpha|} (\partial^\alpha u)^\wedge(z) = z^\alpha \hat{u}(z) \quad \text{for } z \in \mathbb{C}^n,$$

and hence $(P(-\partial)u)^\wedge = P\hat{u}$. The usual Fourier transform of u is $t \mapsto \hat{u}(it)$ for $t \in \mathbb{R}^n$. Thus the inversion formula reads

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, t \rangle} \hat{u}(it) \, d\lambda^n(t) \quad \text{for } x \in \mathbb{R}^n.$$

(4) The well-known multinomial formula implies for $z \in \mathbb{C}^n$

$$(1 + |z|^2)^N = \sum_{|\alpha| \le N} \frac{N!}{(N - |\alpha|)! \alpha!} \overline{z}^\alpha z^\alpha = \sum_{|\alpha| \le N} \frac{N!}{(N - |\alpha|)! \alpha!} [\![z^\alpha]\!] z^{2\alpha}.$$

Thus we obtain from (3) for $v \in D(\mathbb{R}^n)$

$$\hat{v}(z) = \frac{1}{(1 + |z|^2)^N} \sum_{|\alpha| \le N} \frac{N!}{(N - |\alpha|)! \alpha!} [\![z^\alpha]\!] (\partial^{2\alpha} v)^\wedge(z) \quad \text{for } z \in \mathbb{C}^n.$$

(5) Finale. For $v \in D(\mathbb{R}^n)$ we obtain from the Fubini theorem and (4)

$$\begin{aligned} (2\pi)^n S(P)\varepsilon^m \langle E^\varepsilon, v \rangle &= \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \le N} \frac{N!}{(N - |\alpha|)! \alpha!} K_\alpha^\varepsilon(x) (\partial^{2\alpha} v)(x) \right) d\lambda^n(x) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \times T^n} \left(\sum_{|\alpha| \le N} \frac{N!}{(N - |\alpha|)! \alpha!} \frac{[\![(it + \varepsilon s)^\alpha P(it + \varepsilon s)]\!]}{(1 + |it + \varepsilon s|^2)^N} \right. \\ &\qquad \left. \times P_m(s) e^{\langle it + \varepsilon s, x \rangle} (\partial^{2\alpha} v)(x) \right) d(\lambda^n \times \tau^n)(t, s) \, d\lambda^n(x) \\ &= \int_{\mathbb{R}^n \times T^n} \left(\sum_{|\alpha| \le N} \frac{N!}{(N - |\alpha|)! \alpha!} \frac{[\![(it + \varepsilon s)^\alpha P(it + \varepsilon s)]\!]}{(1 + |it + \varepsilon s|^2)^N} \right. \\ &\qquad \left. \times P_m(s) (\partial^{2\alpha} v)^\wedge(it + \varepsilon s) \right) d(\lambda^n \times \tau^n)(t, s) \\ &= \int_{\mathbb{R}^n \times T^n} \hat{v}(it + \varepsilon s) [\![P(it + \varepsilon s)]\!] P_m(s) \, d(\lambda^n \times \tau^n)(t, s). \end{aligned}$$

If now $v = P(-\partial)u$ for some $u \in D(\mathbb{R}^n)$, then $\hat{v} = P\hat{u}$ from (3), and in view of (2), (3) the above is

$$= \int_{\mathbb{R}^n} \int_{T^n} \hat{u}(it + \varepsilon s) P(it + \varepsilon s) [\![P(it + \varepsilon s)]\!] P_m(s)\, d\tau^n(s)\, d\lambda^n(t)$$

$$= \int_{\mathbb{R}^n} S(P)\varepsilon^m \hat{u}(it)\, d\lambda^n(t) = (2\pi)^n S(P)\varepsilon^m u(0).$$

Thus we have $u(0) = \langle E^\varepsilon, v\rangle = \langle E^\varepsilon, P(-\partial)u\rangle = \langle P(\partial)E^\varepsilon, u\rangle$ for $u \in D(\mathbb{R}^n)$; that is, $P(\partial)E^\varepsilon = \delta$. The proof is complete.

Remarks (added December 26, 1992). (1) I owe to Gerhard May the remark that the simplest expression for the fundamental solution E^ε obtained above appears to be

$$\langle E^\varepsilon, u\rangle = \frac{1}{(2\pi)^n S(P)\varepsilon^m} \int_{\mathbb{R}^n \times T^n} \hat{u}(it + \varepsilon s) [\![P(it + \varepsilon s)]\!] P_m(s)\, d(\lambda^n \times \tau^n)(t, s),$$

that is, in terms of the Fourier-Laplace transform $\hat{u}$ of $u \in D(\mathbb{R}^n)$. This formula is contained in the above proof. It can be viewed to be not less explicit.

(2) The author also wants to express his thanks to Jürgen Voigt and to the referee for valuable remarks and hints to the literature.

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