A NECESSARY CONDITION FOR AN ELLIPTIC ELEMENT TO BELONG TO A UNIFORM TREE LATTICE

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Abstract. Let $X$ be a universal cover of a finite connected graph, $G = \text{Aut}(X)$, and $\Gamma$ a group acting discretely and cocompactly on $X$, i.e., a uniform lattice on $X$. We give a necessary condition for an elliptic element of $G$ to belong to a uniform lattice or to the commensurability group. By using this condition, we construct some explicit examples.

Continuing the classic Bass-Serre theory on graphs of groups [S], Bass developed the covering theory for graphs of groups [B]. Using this, Bass and Kulkarni developed the uniform tree lattices theory in their joint paper [BK]. In that paper they obtained a lot of important results. It is fruitful to think of $(G, X, \Gamma)$ as a combinatorial analogue of $(\text{PSL}_2(\mathbb{R}), \text{upper half plane}, \text{fuchsian group})$.

Let $X$ be a 'uniform tree', i.e., the universal cover of a finite connected graph, $G = \text{Aut}(X)$, equipped with compact open topology. The subgroup $H < G$ is discrete iff every vertex stabilizer $H_x$ for $x \in VX$ is finite, where $VX$ is the set of all vertices of $X$. We call $\Gamma < G$ a uniform $X$-lattice if $\Gamma$ is discrete and the quotient graph $\Gamma \backslash X$ is finite (i.e., $VX$ has only finitely many $\Gamma$-orbits). Let $\Gamma_0, \Gamma_1$ be subgroups of $G$. $\Gamma_0$ and $\Gamma_1$ are said to be commensurable (denoted $\Gamma_0 \sim \Gamma_1$), if the index $[\Gamma_i : \Gamma_0 \cap \Gamma_1]$ is finite for $i = 0, 1$. The commensurator (or "virtual normalizer") of $\Gamma$ in $G$ is the group $C_G(\Gamma) = \{g \in G | g\Gamma g^{-1} \sim \Gamma\}$. It was shown in [BK] that, up to $G$-conjugacy, any two uniform lattices in $G$ are commensurable. Thus the commensurator $C_G(\Gamma)$ of a uniform lattice $\Gamma \leq G$ is, up to conjugacy, independent of $\Gamma$; we denote it by $C(X)$. It is proved in [L1] that $C(X)$ is dense in $G$, which was conjectured in [BK].

In this paper, we give a necessary condition for an elliptic element (i.e., one having fixed points) of $G$ to belong to a uniform lattice or to the commensurability group $C(X)$. By this condition, it is then easy to construct some automorphisms of $X$ which do not belong to a uniform lattice, nor do they belong to $C(X)$.

We address here the following questions:

Question. Let $X$ be a uniform tree, $G = \text{Aut}(X)$, $g \in G$. When is there a uniform $X$-lattice $\Gamma$: (a) such that $g \in \Gamma$; (b) such that $g \in C_G(\Gamma)$?
We begin by quoting

**Proposition 1** [BK, (4.2) Conjugacy Theorem]. If \( g \) is hyperbolic (i.e., acting on \( X \) freely), then \( g \) belongs to a uniform lattice.

So the case of main interest is when \( g \) is of finite order. The following notion, due to Gelfand, will be useful for our discussion.

**Definition.** Let \( G \) be a locally compact group. An element \( u \in G \) will be called \( G \)-unipotent if the closure of its \( G \)-conjugacy class \( C_G(u) \) contains 1, where \( C_G(u) = \{ gug^{-1} | g \in G \} \).

**Lemma 1.** Assume that \( \Gamma \setminus G \) is compact, in the sense that \( G = K \cdot \Gamma \) for some compact set \( K \subset G \). If \( \sigma \in G \) is \( G \)-unipotent then the closure of its \( \Gamma \)-conjugacy class \( C_\Gamma(\sigma) \) contains 1, where \( C_\Gamma(\sigma) = \{ \gamma \sigma \gamma^{-1} | \gamma \in \Gamma \} \).

**Proof.** Say

\[
1 = \lim_{n} g_n \sigma g_n^{-1}, \quad g_n \in G, \ n = 1, 2, \ldots
\]

Write \( g_n = k_n \gamma_n \), where \( k_n \in K, \ \gamma_n \in \Gamma \).

Passing to a subsequence we can (compactness of \( K \)) assume that \( k_n \to k \) for some \( k \in K \). Then

\[
1 = \lim_{n} k_n \gamma_n \sigma \gamma_n^{-1} k_n^{-1} = \lim_{n} \gamma_n \sigma \gamma_n^{-1} k_n^{-1},
\]

and so

\[
1 = \lim_{n} \gamma_n \sigma \gamma_n^{-1}, \quad \gamma_n \in \Gamma. \quad \text{Q.E.D}
\]

**Proposition 2.** Let \( \Gamma \in \text{Lat}_u(X) \); then \( C_\Gamma(\Gamma) \), in particular \( \Gamma \), contains no \( G \)-unipotent element \( \neq 1 \).

**Proof.** Suppose that \( \sigma \in C_\Gamma(\Gamma) \) is \( G \)-unipotent.

Put \( \Gamma' = \Gamma \cap \sigma \Gamma \sigma^{-1} \), a subgroup of finite index in \( \Gamma \). Applying Lemma 1 to \( \Gamma' \), we have \( 1 = \lim_{\gamma_n} \gamma_n \sigma \gamma_n^{-1} \) with \( \gamma_n \in \Gamma' \). Hence,

\[
\sigma^{-1} = \lim_{\gamma_n} \sigma^{-1} \gamma_n \sigma \gamma_n^{-1}.
\]

But, for each \( n \), \( (\sigma^{-1} \gamma_n \sigma) \gamma_n^{-1} \in (\sigma^{-1} \Gamma' \sigma, \Gamma') \leq \Gamma \), and \( \Gamma \) is discrete. Hence, for \( n \gg 0 \),

\[
\sigma^{-1} = \sigma^{-1} \gamma_n \sigma \gamma_n^{-1};
\]

whence, \( \gamma_n \sigma \gamma_n^{-1} = 1 \), i.e., \( \sigma = 1 \). \quad \text{Q.E.D.}

Thus we get a necessary condition for an elliptic \( g \neq 1 \) to belong to a uniform tree lattice or \( C(X) \) that \( g \) is not a \( G \)-unipotent element.

**Lemma 2.** An element \( \sigma \in G \) is \( G \)-unipotent iff it is elliptic and its tree of fixed points contains a \( G \)-translate of any given finite subtree.

**Proof.** Assume that \( \sigma \in G \) is \( G \)-unipotent. Thus, by the definition, there is a sequence \( \{ g_n \in G, \ n = 1, 2, \ldots \} \) such that \( \lim_n g_n^{-1} \sigma g_n = 1 \). In other words, for any given finite subtree \( Y \) of \( X \) and for \( n \gg 0 \), we have \( g_n^{-1} \sigma g_n | Y = \text{id} | Y \), i.e., \( \sigma | (g_n Y) = \text{id} | (g_n Y) \). So, \( \sigma \) is elliptic and its tree of fixed points contains \( g_n Y \), where \( g_n \in G \) and \( Y \) is any given finite subtree of \( X \).

Conversely, suppose that \( \sigma \) is elliptic and its tree of fixed points contains a \( G \)-translate of any given finite subtree.
For \( a \in VX \), put \( B_a(n) = \{ x \in VX | d(a, x) \leq n \} \). Then \{\( B_a(n), \ n = 1, 2, \ldots \}\) is a sequence of finite subtrees of \( X \). For each \( B_a(n) \), by the assumption, there is \( g_n \in G \), such that

\[
\sigma(g_nB_a(n)) = \text{id}(g_nB_a(n)),
\]
i.e.,

\[
g_n^{-1} \sigma g_nB_a(n) = \text{id}B_a(n), \quad g_n \in G, \ n = 1, 2, \ldots.
\]

So, \( \lim_n g_n^{-1} \sigma g_n = 1, \ g_n \in G \). Q.E.D.

Now, it is easy to construct \( G \)-unipotent elements of finite order, which thus lie in no uniform lattice (or even the commensurator of one).

**Example 1.** Let \( X \) be the following virtually linear tree:

![Diagram of a virtually linear tree]

Clearly, \( X \) is a uniform tree. In fact, let \( g \in \text{Aut}(X) \) be defined by

\[
g(x_n) = x_{n+1}, \quad g(y_n) = y_{n+1}, \quad g(z_n) = z_{n+1}, \quad n = 0, \pm 1, \pm 2, \ldots,
\]

then \( \langle g \rangle \) is a uniform lattice of \( X \): \( \langle g \rangle \) is discrete and \( \langle g \rangle \backslash X \) is finite.

Define \( \sigma \in G = \text{Aut}(X) \), such that \( \sigma(x_0) = y_0, \ \sigma(y_0) = x_0 \), and \( \sigma \) acts on \( X - \{x_0, y_0\} \) trivially.

Clearly, the subtree of fixed points of \( \sigma \) contains a \( G \)-translate of any given finite subtree of \( X \). By Lemma 2, \( \sigma \) is a nontrivial \( G \)-unipotent. Hence, by Proposition 2, \( \sigma \) does not belong to any uniform lattice nor even to the commensurator of any uniform lattice.

**Example 2.** Let \( X \) be the Cayley tree \( \text{Cay}(F(x, y), \{x, y\}) \), where \( F(x, y) \) is a free group on a basis \( \{x, y\} \). Let \( \alpha \in \text{Aut}(F(x, y)) \), such that \( \alpha(x) = y, \ \alpha(y) = x \). Put

\[
P = \{ u \in F(x, y) | \text{reduced word of } u \text{ begins with } x \text{ or } y \}.
\]

Note that \( \alpha \) defines an automorphism of \( X \) and \( \alpha P = P \). Define \( \sigma \in \text{Aut}(X) \) by

\[
\sigma(u) = \begin{cases} 
\alpha(u) & \text{if } u \in P, \\
u & \text{if } u \notin P.
\end{cases}
\]
Since \( \sigma \) switches two branches of \( X \) and fixes the other two branches, the subtree of fixed points of \( \sigma \) contains \( G \)-translate of any given finite subtree of \( X \). By Lemma 2, \( \sigma \) is a nontrivial \( G \)-unipotent. Hence, \( \sigma \) lies in no uniform lattice nor even the commensurator of one.

On the other hand, we have

**Proposition 3.** Let \( \Gamma \leq G \) be a uniform lattice and \( F \leq C_G(\Gamma) \) a subgroup such that \( F \cdot \Gamma / \Gamma \) is finite. Then \( F \leq \Gamma' \) for some \( \Gamma' \sim \Gamma \).

**Proof.** We may assume that \( F \cdot \Gamma = S \cdot \Gamma \), where \( S \) is a finite subset of \( C_G(\Gamma) \). Put

\[
\Gamma_0 = \bigcap_{g \in F \cdot \Gamma} g \Gamma g^{-1} = \bigcap_{s \in S} s \Gamma s^{-1}.
\]

As \( s \in C_G(\Gamma) \), \( s \Gamma s^{-1} \sim \Gamma \) for each \( s \in S \). Since the intersection of two subgroups of finite index has finite index, it follows that a commensurability class of subgroups of \( G \) is stable under finite intersection. Thus the finite intersection \( \Gamma_0 \) is commensurable with \( \Gamma \). And, clearly, \( \Gamma_0 \) is normalized by \( F \), i.e., \( F \leq N_G(\Gamma_0) \). According to [BK, Corollary (6.4)], \( \Gamma_0 \setminus N_G(\Gamma_0) \) is finite, so \( N_G(\Gamma_0) \sim \Gamma_0 \sim \Gamma \). Thus the proposition is proved by taking \( \Gamma' = N_G(\Gamma_0) \).

**Remark.** Proposition 3 applies notably when \( F \leq C_G(\Gamma) \) is finite or when \( F = \langle g \rangle \) with \( g^n \in \Gamma \) for some \( n > 0 \).

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