RATES OF GROWTH OF P.I. ALGEBRAS

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Abstract. Let A be any p.i. algebra in characteristic zero. Then the GK-dimension of finitely generated subalgebras is linearly bounded in the number of generators.

Let A be any p.i. algebra in characteristic zero. For \( a_1, \ldots, a_k \in A \) we denote by \( \langle a_1, \ldots, a_k \rangle \) the subalgebra generated by these elements. By [1] the GK-dimension of \( \langle a_1, \ldots, a_k \rangle \) will be finite for any finite \( k \). In this paper we will show how these dimensions depend on \( k \). Namely,

**Theorem.** For any p.i. algebra \( A \) there exists a linear function \( f(k) \) such that, for all \( a_1, \ldots, a_k \in A \), \( \text{GKdim} \langle a_1, \ldots, a_k \rangle < f(k) \).

The main tool in proving this paper will be the following theorem of Kemer's ([4, Corollary 1], also proven in [3, Corollary 8]):

**Theorem.** Let \( M_n(E) \) denote the \( n \times n \) matrices over the infinite-dimensional Grassmann algebra \( E \). Let \( A \) be any (characteristic zero!) p.i. algebra. Then for large \( n \), \( A \) satisfies all of the identities of \( M_n(E) \).

Now let \( U \) be the universal p.i. algebra for \( M_n(E) \) with canonical generators \( x_1, x_2, \ldots \), and let \( U_k \) be \( \langle x_1, \ldots, x_k \rangle \), the subalgebra generated by \( x_1, \ldots, x_k \). We showed in [2] that \( \text{GKdim} U_k = (k - 1)n^2 + 1 \). Without resorting to that work, it is not hard to show that \( \text{GKdim} U_k \) is bounded by a linear function in \( k \). Here is a sketch suggested by the referee:

Let \( K \) be the algebra gotten by adjoining the commutative variables \( i_{ij}^{(\alpha)} \) and the anticommuting variables \( e_{ij}^{(\alpha)} \) to the field \( F \), \( i, y = 1, \ldots, n \), \( \alpha = 1, \ldots, k \). For each \( \alpha \) let \( X_\alpha \) be the \( n \times n \) matrix with \( (i, j) \)-entry \( i_{ij}^{(\alpha)} + e_{ij}^{(\alpha)} \), for each \( (i, j) \). Then \( U_k \) is the subalgebra of \( M_n(K) \) generated by \( X_1, \ldots, X_k \). Hence, \( \text{GKdim} U_k \leq \text{GKdim} M_n(K) \). It is then not hard to see that \( \text{GKdim} M_n(K) = kn^2 \).

The proof of our theorem now follows. By Kemer's theorem \( A \) satisfies all of the identities of \( M_n(E) \) for some \( n \). Hence, \( \langle a_1, \ldots, a_k \rangle \) will be a homomorphic image of \( U_k \), so \( \text{GKdim} \langle a_1, \ldots, a_k \rangle \leq \text{GKdim} U_k \) which is linear in \( k \).
REFERENCES


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