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RATES OF GROWTH OF P.I. ALGEBRAS

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ABSTRACT. Let A be any p.i. algebra in characteristic zero. Then the GK-dimension of finitely generated subalgebras is linearly bounded in the number of generators.

Let A be any p.i. algebra in characteristic zero. For $a_1, \dots, a_k \in A$ we denote by $\langle a_1, \dots, a_k\rangle$ the subalgebra generated by these elements. By [1] the GK-dimension of $\langle a_1, \dots, a_k\rangle$ will be finite for any finite k. In this paper we will show how these dimensions depend on k. Namely,

Theorem. *For any p.i. algebra A there exists a linear function $f(k)$ such that, for all $a_1, \dots, a_k \in A$, $\operatorname{GKdim}\langle a_1, \dots, a_k\rangle \le f(k)$.*

The main tool in proving this paper will be the following theorem of Kemer's ([4, Corollary 1], also proven in [3, Corollary 8]):

Theorem. *Let $M_n(E)$ denote the $n \times n$ matrices over the infinite-dimensional Grassmann algebra E. Let A be any (characteristic zero!) p.i. algebra. Then for large n, A satisfies all of the identities of $M_n(E)$.*

Now let U be the universal p.i. algebra for $M_n(E)$ with canonical generators $x_1, x_2, \dots$, and let U_k be $\langle x_1, \dots, x_k\rangle$, the subalgebra generated by $x_1, \dots, x_k$. We showed in [2] that $\operatorname{GKdim} U_k = (k-1)n^2 + 1$. Without resorting to that work, it is not hard to show that $\operatorname{GKdim} U_k$ is bounded by a linear function in k. Here is a sketch suggested by the referee:

Let K be the algebra gotten by adjoining the commutative variables $t_{ij}^{(\alpha)}$ and the anticommuting variables $e_{ij}^{(\alpha)}$ to the field F, $i, y = 1, \dots, n$, $\alpha = 1, \dots, k$. For each α let X_α be the $n \times n$ matrix with (i, j)-entry $t_{ij}^{(\alpha)} + e_{ij}^{(\alpha)}$, for each (i, j). Then U_k is the subalgebra of $M_n(K)$ generated by $X_1, \dots, X_k$. Hence, $\operatorname{GKdim} U_k \le \operatorname{GKdim} M_n(K)$. It is then not hard to see that $\operatorname{GKdim} M_n(K) = kn^2$.

The proof of our theorem now follows. By Kemer's theorem A satisfies all of the identities of $M_n(E)$ for some n. Hence, $\langle a_1, \dots, a_k\rangle$ will be a homomorphic image of U_k, so $\operatorname{GKdim}\langle a_1, \dots, a_k\rangle \le \operatorname{GKdim} U_k$ which is linear in k.

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