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THE STRUCTURE OF JOHNS RINGS

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ABSTRACT. In this paper we continue our study of *right Johns rings*, that is, right Noetherian rings in which every right ideal is an annihilator. Specifically we study *strongly right Johns rings*, or rings such that every $n \times n$ matrix ring R_n is right Johns. The main theorem (Theorem 1.1) characterizes them as the left FP-injective right Noetherian rings, a result that shows that not all Johns rings are strong. (This first was observed by Rutter for Artinian Johns rings; see Theorem 1.2.) Another characterization is that all finitely generated right R-modules are Noetherian and torsionless, that is, embedded in a product of copies of R. A corollary to this is that a strongly right Johns ring R is preserved by any group ring RG of a finite group (Theorem 2.1). A strongly right Johns ring is right FPF (Theorem 4.2).

INTRODUCTION

Rutter's theorem [R] characterizes the quasi-Frobenius ($= QF$) rings as right Artinian strongly right Johns. This raises the question: Are all strongly right Johns rings QF? We do not know, but we show that the only non-Artinian right Johns rings ever constructed (in [F-M]) are not strongly right Johns. Thus, it would appear that a counterexample to the conjecture that all strongly right Johns rings are QF would have to be of larger complexity than the [F-M] examples, which *interalia* required Cohn's and Resco's Theorems on existentially closed skew fields and V-domains. A number of conditions necessary and sufficient for a right Johns ring to be Artinian, hence for a strongly right Johns ring to be QF, were collected in [F-M]: R semilocal; or R has finite left Goldie dimension. We include these in Corollary 1.3 and add one: $J = \text{rad } R$ is finitely annihilated.

We now relate strong Johns rings to class rings that are also QF when they are Artinian: A ring R is right $(F)PF$ if all (finitely generated) faithful right

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R-modules are generators in the category of all right R-modules; equivalently, there exists an epimorphism $M^n \to R$, for some $n > 0$, hence an isomorphism $M^n \approx R \oplus X$ in mod-R. Right PF rings are rings R that are injective cogenerators in the category mod-R and have been characterized as the semilocal rings right self-injective rings with essential right socles. The Artinian FPF rings are the QF rings. In fact, as alluded to above, Artinian FPF rings are QF [F8, F9], and strongly right Johns rings are right FPF (Theorem 4.2).

1. Remarks on right annular rings and a theorem of Rutter

A ring R is right (*finitely*) *annular* if every (finitely generated) right ideal is a right annihilator (= right annulet). A classical theorem of ring theory states that a right Noetherian ring R is 2-sided annular iff R is quasi-Frobenuis (= QF). (See, e.g., [F, Chapter 24].) This shows that 2-sided Johns rings are QF. Nevertheless, there exist right annular rings R with just three right ideals, namely, R, $J = \operatorname{rad} R$, and 0, which are not left Noetherian, hence not QF.

We list properties of right annular rings that are either obvious or easy to prove.

(RA 1) R is right annular iff every cyclic right module is torsionless. See [F4] for this and the next.

(RA 2) Every matrix ring over R is right annular iff every finitely generated module is torsionless (= R is right FGT).

(RA 3) A left $\aleph_0$-injective ring is finitely right annular. (RA 3) is a theorem of Ikeda-Nakayama. (See, e.g., [F2, p. 189, 23.11].)

(RA 4) A ring R is left FP-injective iff every finitely presented right R-module is torsionless. ((RA 4) is a theorem of Stenstrom and Jain. See [Ja].)

A ring R is *right* $FG(T)F$ if every finitely generated (torsionless) module embeds in a free module. A right annulet I is said to be *finitely annihilated* (= FA) if $I = r(X)$ is the right annihilator of a finite subset X of R. A necessary and sufficient condition that $I = r(X)$ where X has n elements is that R/I embeds in R^n. Moreover, a sufficient condition for every right annulet to be FA is that R satisfies the acc on left annulets [F10].

(RA 5) A ring R is right $FGTF$ iff every right annulet in every matrix ring R_n is FA. (See, e.g., [F4] for (RA 5).)

Similarly,

(RA 6) A ring R is right FGF iff every matrix ring R_n is right annular and every right ideal is FA.

We call a ring with the latter property *right FA-annular.* Thus (RA 6) states that R_n is right FA-annular for all $n \leq 1$ iff R is right FGF.

We restate (RA 6) as:

(RA 7) The f.a.e:

(RA 7a) R is right FGF,

(RA 7b) R_n is right FA-annular for all $n \geq 1$,

(RA 7c) R is right $FGTF$ and R_n is right annular for all $n \geq 1$.

It is known that every QF ring R has the characterizing property: (*right* GF) every right module embeds in a free module.

Furthermore, by the symmetric properties of QF rings, we have:

(RA 8) Every QF ring R is right and left GF.
By results stated in [F3]:
(RA 9) R is QF iff left and right FGF.
also:
(RA 10) Let R be right FGF ring. Then f.a.e.c.'s
(FGF 1) R is QF,
(FGF 2) R is right Noetherian,
(FGF 3) R is semilocal with essential right socle,
(FGF 4) R has finite essential right socle,
(FGF 5) R is right self-injective,
(FGF 6) R is left FGF,
(FGF 7) R is right and left FA-annular.

For a proof, see [F3]. The proof of (FGF 2) in [F3] uses John's lemma.

1.1. **Theorem.** *For a ring R, the f.a.e.:*

(1) *R is strongly right Johns.*
(2) *R is left FP-injective and right Noetherian.*
(3) *R is right FGT and right Noetherian.*
(4) *Every finitely generated right R-module is Noetherian torsionless.*

Proof. $(1) \Leftrightarrow (3)$, $(2) \Leftrightarrow (4)$, and $(3) \Leftrightarrow (4)$ are obvious by (RA4) and (RA2), for right Noetherian R.

We now prove what is essentially:

1.2. **Rutter's Theorem.** *For a right Artinian ring, the f.a.e:*

(1) *R is QF.*
(2) *R_n is ring annular for $n > 1$.*
(3) *R is strong right Johns.*

Proof. $(2) \Leftrightarrow (3)$ is trivial since R_n is right Artinian, hence right Noetherian, and QF is a Morita invariant property, i.e., is inherited by R_n.

$(2) \Rightarrow (1)$. We show this follows from (RA 10). Since R_n is right Artinian, it satisfies

$$(\perp \text{acc}) = \text{the acc on left annulets.}$$

But by a theorem of [F10], $\perp$acc implies that R_n is right FA. Thus, by (RA 7), R is right FGF, hence QF by (FG 2) of (RA 10). Conversely, $(1) \Rightarrow (2)$ is obvious, since (1) implies that R_n is QF and hence annular by (RA 10).

1.3. **Corollary.** *Let R be strongly right Johns. The f.a.e.:*

(1) *R is QF.*
(2) *R is semilocal.*
(3) *R has finite left Goldie dimension.*
(4) *R is left Noetherian.*
(5) *R_n is right FA for all n.*
(6) *$J = \text{rad}\, R$ is right FA, i.e., $J = r(X)$, for a finite set X.*

Proof. By Rutter's theorem, Theorem 1.2, to prove QF it suffices to prove R is right Artinian. That $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow R$ right Artinian holds in right Johns rings (see [F-M]); and obviously $(1) \Rightarrow (4)$, and $(4) \Rightarrow (5) \Rightarrow (6)$. Also,

(5) $\Rightarrow R$ is right FGF by (RA 7), and then R is QF by (RA 10), especially (FGF 2). Finally, (6) implies that R/J embeds in R^n where $n = |X|$. Now R has finite essential socle by John's results in [J], so R/J has the same, but being semiprime, this implies that R/J is semisimple, so (2) holds.

2. Finite group rings over Johns rings

Let G be a finite group. We raise the question: if R is right Johns, is the group ring RG?

Using Theorem 1.1, we show the answer is affirmative assuming that R is strong right Johns.

2.1. **Theorem.** *If R is a strong right John ring, then so is the group ring RG of a finite group G.*

Proof. First, R right Noetherian implies RG right Noetherian. Because of the natural isomorphism of functors

$$(*) \qquad \operatorname{Hom}_{RG}(\ ,RG) \approx \operatorname{Hom}_R(\ ,R),$$

it is easy to see that R left FP-injective implies that RG is left FP-injective when R is right Noetherian, so RG is strong right John by Theorem 1.1.

As corollary to the proof we have:

2.2. **Corollary.** *Let G be a finite group. If R is right FGT, then RG is right FGT.*

Proof. This follows from the natural isomorphism $(*)$. To wit, if M is a finitely generated RG-module, then M is a finitely generated canonical R-module, and by $(*)$, the canonical map

$$M \to R^{\operatorname{Hom}_{RG}(M,RG)}$$

is an embedding iff the same is true of the canonical map

$$M \to R^{\operatorname{Hom}_R(M,R)}.$$

Thus, R right FGT implies that RG is right FGT.

3. Structure of strongly Johns rings

A module M is *finitely embedded* (= f.e.) provided M has finite essential socle. It is known that M is f.e. iff M has the finite intersection property (= f.i.p.), namely, if $\{S_i\}_{i\in I}$ is any family of submodules and if $\bigcap_{i\in I} S_i = 0$, then there is a finite subset $A \subseteq I$ with $\bigcap_{a\in A} S_a = 0$. (See [F2, p. 69] for background and references.) We need the following elementary property of f.e. modules.

3.1.A. **Proposition.** *If M is a f.e. right R-module and if M is contained in a product $N = \prod_{i\in I} N_i$ of right R-modules, then there is a finite subset $A \subseteq I$ such that $M \hookrightarrow \bigoplus_{a\in A} N_a$.*

Proof. Let $\rho_i : N \to N_i$ be the canonical projections from the direct product, and let $\bar{\rho}_i : M \to N_i$ be the induced map for all $i \in I$. Since $\bigcap_{i\in i} \ker \rho_i = 0$, and since M is f.e., for some finite subset A of I we have $\bigcap_{a\in A} \ker \bar{\rho}_a = 0$, and then the direct sum of the maps $\{\bar{\rho}_a\}_{a\in A}$ is an embedding $M \hookrightarrow \bigoplus_{a\in A} N_a$.

3.1.B. **Corollary.** *Every finitely embedded torsionless right R-module M embeds in free module R^n of finite rank n.*

Proof. Since M is torsionless and then embeds in a direct product of copies of R, the corollary follows from Proposition 3.1.A.

3.2. **Theorem.** *If R is strongly right Johns, then the injective hull $E = E(R_R)$ of R is a flat right R-module.*

Proof. The proof requires a theorem of Rutter [R] to the effect that an injective module is flat iff every finitely generated submodule embeds in a flat module. Actually we can prove more: every finitely generated submodule M of E embeds in a free R-module. For R is finitely embedded [J], hence so is E whence M. By Theorem 1.1, each finitely generated right R-module is torsionless, hence by Corollary 3.1.B, M embeds in R^n. This proves what was needed.

3.3. **Lemma.** *If A is a semiprime right Goldie ring with right quotient ring $Q = Q^r_{\text{cl}}(A)$, then any torsionfree divisible right A-module F is injective and a canonical Q-module.*

Proof. Let $G = E(F_A)$. Then G is also torsionfree, for if $0 \neq g \in G$ and if $a \in A$ is such that $ga = 0$, we can find a regular element $d \in A$ such that $0 \neq gd \in F$. (This follows because the conductor $(g\colon F)$ of g to F is an essential right ideal and every essential right ideal in a semiprime right Goldie ring contains a regular element.)

Next, since F is torsionfree over A, F is a canonical Q-module, so we may apply $d^{-1}a$ to gd to get

$$(gd)(d^{-1}a) = ga = 0.$$

But F is torsionfree, contradicting $gd \neq 0$. Thus, $d^{-1}a = 0$, so $a = 0$, hence G is torsionfree. But, since F is divisible, $gd = xd$ for $x \in F$, so

$$(g - x)d = 0,$$

whence $g - x = 0$, that is, $g = x \in F$. Therefore, $F = G$ is injective.

We shall need the next proposition which was originally in [F-M] but excised in keeping with instructions from the referee and the corresponding editor.

3.4. **Proposition.** *Let R be right Johns, and let $\bar{R} = R/J$, where J is the Jacobson radical of R. Then*:

(i) *If $a \in R$, then $r_R(a) = 0$ if and only if $r_{\bar{R}}(\bar{a}) = 0$.*
(ii) *If $\bar{R}$ is a domain, then $S = \text{soc}\, R$ is a minimal right ideal of R and the unique simple right R module.*

Proof. (1) Assume $r_R(a) = 0$, and let $ab \in J$ for some $b \in R$. Then, by Lemma 2.2 of [F-M], $a(b \cdot \text{soc} R) = 0$. Hence $b \cdot \text{soc}\, R = 0$, so $b \in J = l(\text{soc}\, R)$ by Lemma 2.2 of [F-M] again. Thus, $r_{\bar{R}}(\bar{a}) = 0$.

Conversely, let $a \in R$ be such that $r_{\bar{R}}(\bar{a}) = 0$. Then, since $\bar{R}$ is semiprime right Noetherian, necessarily $l_{\bar{R}}(\bar{a}) = 0$, hence

$$J = \{x \in R | xa \in J\} = (x : J).$$

Since $J = l_R(\text{soc}\, R)$, it follows that $J = l_R(a \cdot \text{soc}\, R)$, so we see that $\text{soc}\, R = a \cdot \text{soc}\, R$ since every right ideal of R is a right annihilator.

Since a therefore induces an epimorphism of the Noetherian module $\operatorname{soc} R$, $r_R(a) \cap (\operatorname{soc} R) = 0$. Since $\operatorname{soc} R$ is an essential right ideal of R by a lemma of Johns [J, Lemma 2], $r_R(a) = 0$, proving (i).

(ii) Let V be a right ideal of R properly contained in S. Then, by Lemma 2 of [J], $l_R(V) \supset l_R(S) = J$. By (i) above, any $a \in l_R(V)\backslash J$ satisfies $r_R(a) = 0$, a contradiction unless $V = 0$. Thus, S is a minimal right ideal. Since every right ideal of R is a right annihilator, every simple right R-module W of R embeds in R, hence coincides with S since S is essential. Thus, S is the unique simple R-module.

3.5. **Theorem.** *If R is right Johns, $S = \operatorname{soc} R_R$, $Y = \operatorname{ann}_E S$, and $\bar{E} = E/Y$, then $\bar{E}$ is an injective right A-module.*

Proof. Since J is the left annihilator in R of $S = \operatorname{soc} R_R$ (see proof of Proposition 3.4), $Y \supseteq EJ$, so Y is a canonical A-module. Now E (and any injective module) is divisible by every right regular element a of R. By (i) of Proposition 3.4, every (right) regular element $\bar{a}$ of $A = \bar{R}$ lift to a right regular element $a \in R$. Thus, $\bar{E}$ is divisible (by regular elements of A). Since $|S| < \infty$ and $aS \approx S$, it follows that $aS = S$ for any $a \in R$ above.

Now if $\bar{x}\bar{a} = 0$ for $\bar{x} \in \bar{E}$ and regular $\bar{a} \in A$, then $xa \in Y$, hence

$$0 = (xa)S = x(aS) = xS,$$

so $x \in Y$, i.e., $\bar{x} = 0$. This proves that $\bar{E}$ is torsionfree divisible over A, hence that E is injective by Lemma 3.3.

3.6. **Theorem.** *If R is strongly right Johns and $A = \bar{R} = R/J$ is a domain, then*

$$Q = Q_{\mathrm{cl}}^r(A) = Q_{\mathrm{cl}}^\ell(A) \overset{\mathrm{can}}{\approx} \operatorname{End} V_R \overset{\cdot}{=} \operatorname{End} V_A$$

where V is the unique simple R-module.

Proof. Since R is right Johns and $A = R/J$ is a domain, $V = \operatorname{soc} R$ is simple and the unique simple right R-module by Proposition 3.4. By Schur's lemma, $K = \operatorname{End} V_R = \operatorname{End} V_A$ is a field. If $\dim_K V = 1$, then $A = K$ and the result is trivial. Otherwise, we may suppose for every $0 \neq v \in V$ that $Kv \cap Kdv = 0$ and hence that

$$Av \cap Adv = 0.$$

Then, by a property of left $\aleph_0$-injective rings (which follows since R is left FP-injective by Theorem 1.1)

$$r_R(v) = r_R(v) + r_R(dv) = R$$

(see, e.g., the theorem of Ikeda-Nakayama in [F2, p. 139]). Thus, $v = 0$, a contradiction. This similarly implies that, for every $0 \neq v \in V$,

$$Ad_0v \cap Abv \neq 0.$$

Write $ad_0v = bv$ for some $a \in A$. Then $ad_0 = b$ (since ${}_KV$ is a vector space), hence

$$d_0 = a^{-1}b \in Q_{c\ell}^r(A),$$

i.e., $K = Q_{\mathrm{cl}}^r(A)$. Using the same argument with $d_0 = cd$, for $0 \neq c$, $d \in A$, we get $a^{-1}b = cd^{-1}$, hence that

$$K = Q_{\mathrm{cl}}^\ell(A) = Q_{\mathrm{cl}}^r(A).$$

For a bimodule V over a ring A, we let (A, V) denote the trivial, or split-null extension; namely, the ring consisting of all matrices $\begin{pmatrix} a & \nabla \\ 0 & a \end{pmatrix}$ with $a \in A$, $\nabla \in V$, and the usual matrix multiplication.

3.7. *Remark.* The right Johns ring $R = (A, V)$ constructed in [F-M] is not strongly right Johns.

For in this example, $A = D \otimes_C C(X)$, for an existentially closed field D with center C, and $V = D$ is an A-bimodule, so that $D \approx \operatorname{End} V_A$ has center C, whereas the center of $Q_{\text{cl}}(A) = C(X)$. Thus, since C is algebraically closed, then $C \not\approx C(X)$, hence $Q_{\text{cl}}(A) \neq \operatorname{End} V_A$ as required by Theorem 3.6 in strongly right Johns ring.

3.8. **Theorem.** *Let R be strongly right Johns, $A = R/J$, where $J = \operatorname{rad} R$, and $S = \operatorname{soc} R$. Then for $E = E(R_R)$, we have:*

(i) $Y = \operatorname{ann}_E S = EJ$,

and $\bar{E} = E/EJ$ is torsionfree over A, and an injective right Q-module, where $Q = Q^r_{\text{cl}}(A)$. Moreover,

(ii) *Q_A is flat, and*

(iii) $Q = Q^\ell_{\text{cl}}(A)$.

Proof. Since A is semiprime right Noetherian (hence Goldie), then $Q = Q^r_{\text{cl}}(A)$ exists by, e.g., [F1, Chapter 9] and is semisimple (theorem of A. W. Goldie). Moreover, any torsionfree divisible right module is canonically a right Q-module (see Lemma 3.3).

Let $y \in Y = \operatorname{ann}_E S$. By Corollary 3.1.B,

$$yR \approx R/\operatorname{ann}_R y$$

embeds in a free right R-module of finite rank, hence

$$\operatorname{ann}_R y = r_R(X)$$

is the right annihilator in R of a finite subset X of R. By the double annihilator condition (d.a.c.) for (quasi) injective modules (e.g., [F2, Chapter 19, Theorem 19.10, p. 66])

$$\operatorname{ann}_E \operatorname{ann}_R M = M$$

for any finitely generated left submodule of the left module E over $\Lambda = \operatorname{End} E_R$. Since

$$\operatorname{ann}_R \Lambda X = \operatorname{ann}_R y = \operatorname{ann}_R \Lambda y,$$

it follows that $y \in \Lambda X$. But,

$$r_R(X) = \operatorname{ann}_R y \supseteq \operatorname{ann}_R Y \supseteq S,$$

hence,

$$X \subseteq l_R r_R(X) \subseteq l_R(S) = J.$$

(The right equality is [F-M, Lemma 2.2].) So,

$$y \in \Lambda X \subseteq \Lambda(1)J = EJ,$$

proving (i).

(ii) and (iii). By a theorem of Levy [L] and Goodearl [G], (ii) $\Rightarrow$ (iii), so we proceed to prove (ii). By Theorem 3.2, E_R is flat, and hence E/EI is a flat (R/I)-module for any ideal I of R, so E/EJ is a flat A-module.

Since $\bar{E} = E/EJ$ is a torsionfree divisible, in fact, injective, right A-module by Theorem 3.5, and canonically a module over Q, it follows that the least generator Q_0 of mod-Q is a direct summand of $\bar{E}$, hence $(Q_0)_A$ whence Q_A is flat.

3.9. **Corollary.** *If R is strongly right Johns, then $S = \operatorname{soc} R_R$ is an injective right and left A-module, where $A = R/J$.*

Proof. By Theorem 2.3 of [F-M], A is a right V-ring, i.e., every simple right A-module is injective. Since R, whence A, is right Noetherian, every semisimple right A-module is injective, thus S_A is injective. By Lemma 3.3 if $\bar{a} \in A = R/J$, where $a \in R$, then $r_R(\bar{a}) = 0$ implies $r_R(a) = 0$, hence ${}_AS$ is torsionfree. Also, the fact that S_A has finite length and that $S \approx aS$ implies that $aS = S$, so ${}_AS$ is divisible. By Theorem 3.8 (iii), A is left Ore, hence left Goldie, so by Proposition 3.4, then ${}_AS$ is injective and canonically an injective left Q-module, where $Q = Q^r_{\text{cl}}(A) = Q^\ell_{\text{cl}}(A)$.

3.10. **Theorem.** *Let R be strongly right Johns, with $J^2 = 0$, $S = \operatorname{soc} R_R$, and suppose that*

$$Q \overset{\text{can}}{\approx} \operatorname{End} S_R$$

(e.g., if $A = R/J$ is a domain: see Theorem 3.6). *Then, $EJ = J$ and $E/J \approx Q$.*

Proof. By Theorem 3.8, $\bar{E} = E/EJ$ contains a copy of Q. For any ideal I, $\operatorname{ann}_E I$ is an injective (R/I)-module; in particular,

$$F_1 = \operatorname{ann}_E J$$

is an injective A-module. Since E is an essential extension of S_R, then F_1 is essential over S_A, hence $F_1 = S$ by injectivity of S (see Corollary 3.9).

Let $R \subseteq E_1 \subseteq E$ be such that

$$(1) \qquad E_1/J \approx Q.$$

Now E_1 exists since

$$(2) \qquad S = J \subseteq EJ \subseteq \operatorname{ann}_E J = S$$

(note $S = J$ by loc. cit.). Thus,

$$(3) \qquad EJ = J \quad \text{and} \quad \bar{E} = E/J,$$

so (1) exists since $\bar{E}$ contains a copy of Q.

Let $y \in E$. Then, by (2)

$$(4) \qquad yS \subseteq S = \operatorname{ann}_E J.$$

So y induces $\bar{y} \in K = \operatorname{End} S_R = \operatorname{End} S_A$. But $Q \subseteq K$ canonically, and by the assumption $Q = K$, we obtain $q \in Q \subseteq E$ such that $\bar{y} = \bar{q}$, i.e.,

$$s = y - q \in \operatorname{ann}_E S = \operatorname{ann}_E J \underset{\bullet}{=} S,$$

so $y = q + s \in E_1$. This proves that $E = E_1$, hence $\bar{E} \approx Q$.

3.11. **Corollary.** *If R in the theorem is the split-null extension $R = (A, W)$ (as in* [F-M]), *where $A = R/J$ and $W = \operatorname{Soc} R_R$, then:*

$$(1) \qquad E = (Q, W).$$

Moreover, in this case

(2) $$\operatorname{End} E_R \approx (Q, \operatorname{Hom}(Q_A, W_A)).$$

((1) *and* (2) *are trivial extensions.*)

Proof. (1) Straightforward application of Theorem 3.10. (2) follows by an easy calculation.

4. Strongly Johns rings are FPF

A ring R is right $(F)PF$ if every (finitely generated) faithful right R-module M is a generator of mod-R, the category of all right R-modules. This happens iff there is a (finite) direct sum of copies M and an epimorphism $M^n \longrightarrow R$, hence an isomorphism

$$M^n \approx R \oplus X \quad (\text{in mod-}R).$$

The relation between FPF, PF, and QF rings is studied in [F5–F9].

4.1. **Proposition.** *If $J = \operatorname{rad} R$ is nilpotent and M is a faithful torsionless right R-module, then the trace ideal $T(M) \not\subseteq J$.*

Proof. Suppose $J^n = 0$ and $J^{n-1} \neq 0$. Since M is faithful, $MJ^{n-1} \neq 0$, and since M is torsionless, there exist $f \in M^* = \operatorname{Hom}_R(M, R)$ such that

$$f(MJ^{n-1}) \neq 0.$$

Then

$$T(M)J^{n-1} = T(MJ^{n-1}) = \sum_{f \in M^*} f(MJ^{n-1}) \neq 0,$$

proving that $T(M) \not\subseteq J$.

4.2. **Theorem.** *A strongly right Johns ring R is right FPF.*

Proof. By Theorem 1.1(4) and [F-M, Lemma 2.2], if M is any finitely generated faithful right R-module, then $T = T(M) \not\subseteq J$ by Proposition 4.1. Since $A = R/J$ is a right Noetherian right V-ring, by [F-M], then A is a finite product of simple rings. (See [F1, Chapter 7, Theorem 7.36A, p. 357].) Suppose $TP \subseteq J$ for an ideal P of R containing J. Now $J = l_R(S)$, where $S = \operatorname{soc} R_R$ by a result of John stated in [F-M], so $TPS = 0$, and then, as in the proof of Proposition 4.1, $PS = 0$, so $P \subseteq J = l(S)$, whence $P = J$. This proves that $\bar{T} = T/J$ is a faithful module over A. Since A is finite a product of simple ideals, $\bar{T}$ must be a finite product of a collection of these , so $\bar{T}$ is faithful only if $\bar{T} = A$. This proves that $T = R$, so R is right FPF.

4.3. **Corollary.** *If, in Theorem 4.2, $A = R/J$ is a (right & left) PID and $J^2 = 0$, then every finitely generated right R-module M is (up to isomorphism) a unique direct sum of indecomposable cyclic modules. In fact, there exist unique nonnegative integers m and n such that*

$$M \approx R^{(n)} \oplus A^{(m)} \oplus M_0$$

where M_0 is a semisimple (hence injective) module of finite length. Moreover, every projective right R-module is free.

Proof. The proof of Proposition 4.1 shows that for every finitely generated faithful right module M there exists a map $f : M \to R$ such that $f(M) \not\subseteq J$.

Using the fact that A is a PID and Proposition 3.4, it follows that $f(M) = aR$ for some $a \in R\backslash J$ and that $r_R(a) = 0$, hence M maps epically onto R. Thus, by an induction argument, for *any* finitely generated R-module M,

$$M \approx R^{(n)} \oplus Y$$

where Y is unfaithful over R, and possibly $n = 0$ or $Y = 0$. In this case, since $J = \operatorname{soc} R$ is simple by (ii) of 3.4 (and its proof, using $J^2 = 0$), then Y is an A-module, and by the known theory of modules over PID's

$$Y \approx A^{(m)} \oplus M_0$$

where M_0 is a torsion hence Artinian right A-module, and a direct sum cyclic modules. However, since A is a right V-domain [F-M], then M_0 is actually semisimple of unique finite length. Moreover, m and n are unique since A is a PID, and R is right Noetherian hence has invariant basis number for free R-modules. This proves that M is a unique direct sum of cyclic R-modules.

Finally, if M is a finitely generated projective module, then M is free:

$$M \approx R^n$$

since R has no nontrivial idempotents. ($A = R/J$ has none, and idempotents lift.)

Addendum

(Added May 1993) Using the method of proof of Proposition 4.1 one can prove

Proposition 4.1*. *Let R be a ring such that every ideal $I \neq R$ has nonzero annihilator $I^{\perp}$. Then every torsionless faithful right R-module M generates* mod-R.

Corollary 4.2*. *If R is a right cogenerator ring, then R is right PF iff $I^{\perp} \neq 0$ for every ideal $I \neq R$.*

Proof. If R is right PF, then by a theorem of Kato, every maximal left ideal L has $L^{\perp} \neq 0$ (see [F8, Corollary 11]). Thus, if I is an ideal $\neq R$, then I is contained in a maximal left ideal L and hence $I^{\perp} \supset L^{\perp} \neq 0$.

Conversely, if R is a right cogenerator ring, then every right R-module M is torsionless, hence faithful modules generate mod-R by Proposition 4.1*, so R is right PF.

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