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STABLE ABSOLUTELY UBIQUITOUS STRUCTURES

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ABSTRACT. It is proved that absolutely ubiquitous structures with trivial algebraic closures are monadically stable.

Let L be a finite language, and let M be an L-structure with countably infinite domain. The age of M, denoted by $\mathcal{J}(M)$, is the set of all isomorphism types of finite substructures of M. A model M is absolutely ubiquitous if M is uniformly locally finite and every countable locally finite L-structure with age $\mathcal{J}(M)$ is isomorphic to M. It is known that an absolutely ubiquitous structure is countably categorical and model-complete. It is not known if an absolutely ubiquitous structure has stable or superstable theory. In [1–3] a classification was given for absolutely ubiquitous structures over relational languages. Such structures are finitely partitioned. This will be proved here in a slightly more general case using completely different techniques.

Notation. $B < M$ if B is a substructure of M,

$\mathrm{Str}(B)$ is the substructure of M generated by B.

1. LEMMA OF E. A. PALYUTIN

The following lemma is a simple generalization of an old result of E. A. Palyutin on categorical universal theories [4].

Lemma 1. *A countable structure M is absolutely ubiquitous iff*

(1) *M is uniformly locally finite; and*

(2) *there is a function $s\colon \omega \to \omega$ such that for every n, for every substructure B of M, and for every type $p(y, b_0, \dots, b_{n-1})$ over $\{b_0, \dots, b_{n-1}\} \subseteq B$, if B contains all isomorphism types of $s(n)$-generated substructures of M then there is $b \in B$ such that $M \models p(b, b_0, \dots, b_{n-1})$.*

Proof. Let M be absolutely ubiquitous. We construct s by induction. Suppose s is defined for all $i < n$. Suppose $R_0, \dots, R_t$ (and $T_0, \dots, T_l$) are complete formulas which determine all complete n-types ($(n+1)$-types) over $\varnothing$ ($\mathrm{Th}(M)$ is ω-categorical). By model-completeness we may suppose that R_i

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are $\forall$-formulas and T_j are $\exists$-formulas. We define the set $G_k^{i,j}$ as follows. Let

$$\Phi_{i,j} = \forall x_0, \dots, x_{n-1}(R_i(x_0, \dots, x_{n-1}) \to \exists y(T_j(y, x_0, \dots, x_{n-1})).$$

If $M \models \Phi_{i,j}$ let $G_k^{i,j}$ be

$$\mathrm{Th}_\forall(M) \cup \{\text{there are } z_1, \dots, z_m \text{ which generate substructure embedding all types of } k\text{-generated substructures of } M\} \cup \{\neg\Phi_{i,j}\}.$$

If $M \models \neg\Phi_{i,j}$ let $G_k^{i,j}$ be $\forall x(x \neq x)$,

If for all k the set $G_k^{i,j}$ is consistent then there is a countable model N of the theory $\mathrm{Th}_\forall(M)$ with $\mathscr{J}(N) = \mathscr{J}(M)$ and $N \models \neg\Phi_{i,j}$. It is a contradiction.

Let

$$k(i, j) = \min(k\colon G_k^{i,j} \text{ is inconsistent}),$$
$$s(n) = \max(k(i, j)\colon 0 \le i \le t,\ 0 \le j \le l).$$

Now if B and p are as in (2), formula R_i determines $\mathrm{tp}(\langle b_0, \dots, b_{n-1}\rangle)$ and formula T_j determines type $p(y, x_0, \dots, x_{n-1})$, then since $G_{s(n)}$ is inconsistent we have

$$B \models \forall x_0, \dots, x_{n-1}(R_i(x_0, \dots, x_{n-1}) \to \exists y T_j(y, x_0, \dots, x_{n-1})).$$

Since R_i is universal, there is $b \in B$ such that $B \models T_j(b, b_0, \dots, b_{n-1})$, and since T_j is $\exists$-formula, $M \models T_j(b, b_0, \dots, b_{n-1})$.

If (1) and (2) hold in M then M is countably saturated. If for countable N we have $\mathscr{J}(N) = \mathscr{J}(M)$ then N is isomorphic to a substructure of M. By back-and-forth argument using (2) we can prove M is isomorphic to N.

2. Structures with a finite number of term functions

We suppose here that there are terms $d_1, \dots, d_t$ such that for every term d the corresponding function under essential variables is one of the functions defined by $d_1, \dots, d_t$ in M. For example, a uniformly locally finite unary algebra is such a structure.

Proposition 1. *Let M be an absolutely ubiquitous structure with a finite number of term functions. Then the theory of M is stable.*

Proof. It is sufficient to prove that $\mathrm{Th}(M)$ has not the independence property because by [5] $\mathrm{Th}(M)$ has not the strict order property. For a contradiction suppose that there are a formula $F(\mathbf{x}, \mathbf{y})$ and sequences $\langle \mathbf{a}_i\colon i \in \omega\rangle$ and $\langle \mathbf{c}_\sigma\colon \sigma \in 2^\omega\rangle$ such that $\models F(\mathbf{a}_i, \mathbf{c}_\sigma)$ iff $\sigma(i) = 0$. For ease of notation suppose that $\mathbf{a}_i$ and $\mathbf{c}_\sigma$ are singletons. Assume terms $d_1, \dots, d_t$ depend on at most m variables. Let B be a finite substructure of M containing parameters of F and all types of $s(2m+2)$-generated substructures of M. We may suppose that $\mathbf{a}_1, \dots, \mathbf{a}_{2m+2}$ is a sequence of order indiscernibles over B, and by Lemma 1 suppose

$$\{\mathbf{c}_\sigma\colon \sigma \in 2^{2m+2}\} \subseteq \mathrm{Str}(B \cup \{\mathbf{a}_1, \dots, \mathbf{a}_{2m+2}\})$$

(of course, we neglect parameters). Define $\sigma \in 2^{2m+2}$ by $\sigma(i) = 0$ iff i is odd. We know that for some $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_k}$ $(k \le m)$ and $\mathbf{b} \subseteq B$ and term d_i, $\mathbf{c}_\sigma = d_i(\mathbf{b}, \mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_k})$. There are $\mathbf{a}_{2n-1}$ and $\mathbf{a}_{2n}$ $(n \le m+1)$ not among

$\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_k}$. By indiscernibility, $\models F(\mathbf{a}_{2n-1}, \mathbf{c}_\sigma)$ iff $\models F(\mathbf{a}_{2n}, \mathbf{c}_\sigma)$, which is a contradiction.

3. Structures with trivial closure

A structure M has trivial closure if $\operatorname{acl}(X \cup Y) = \operatorname{acl}(X) \cup \operatorname{acl}(Y)$ holds in M. For example, absolutely ubiquitous structures with unary terms only have trivial closure. Indeed, by Lemma 1 there is some type of finite substructures of M such that for every B of this type $\operatorname{acl}(X \cup Y)$ is in the substructure generated by $X \cup Y \cup B$. Since every term is unary, we have $\operatorname{acl}(X \cup Y) \subseteq \operatorname{acl}(X) \cup \operatorname{acl}(Y) \cup B$. We can choose B with $B \cap \operatorname{acl}(X \cup Y) \subseteq \operatorname{acl}(\varnothing)$. So, $\operatorname{acl}(X \cup Y) = \operatorname{acl}(X) \cup \operatorname{acl}(Y)$.

By [1, 6] absolutely ubiquitous structures in a relational language are monadically stable. By the description of countably categorical monadically stable structures [6], every absolutely ubiquitous monadically stable structure has trivial closure. It is not hard to find an absolutely ubiquitous structure with a finite number of term functions which does not have trivial closure. So, the situation of Proposition 1 is quite different than that of [1–3]. But in the case of trivial closure we have the following

Proposition 2. *Absolutely ubiquitous structures with trivial closure are monadically stable.*

Proof. Let us prove that the relation $x \in \operatorname{acl}(y)$ on nonalgebraic elements is an equivalence relation. If $a \in \operatorname{acl}(b)$ and $b \notin \operatorname{acl}(a)$ then there is some type of finite substructures of our model M such that for every B of this type some realization of $\operatorname{tp}(b|a)$ is in the substructure generated by $B \cup \{a\}$. By triviality of closure it is in $\operatorname{acl}(B)$ and so $a \in \operatorname{acl}(B)$. But we can choose $\operatorname{acl}(B)$ so that $\operatorname{acl}(B) \cap \{a\} \subseteq \operatorname{acl}(\varnothing)$. We have now $a \in \operatorname{acl}(\varnothing)$.

Let us prove that Shelah's R-rank of M is one. Let $\mathbf{a}_i$, $i \in \omega$, be an infinite sequence, and suppose the set of nonalgebraic complete formulas $F(x, \mathbf{a}_i)$ is n-inconsistent. Choose a finite $B < M$ containing all types of $s(\ln(\mathbf{a}_i))$-generated substructures of M. By Lemma 1 $\operatorname{Str}(B \cup \{\mathbf{a}_i : i \in \omega\})$ contains realizations of all formulas $F(x, \mathbf{a}_i)$. Of course, the corresponding realization of formula $F(x, \mathbf{a}_i)$ is in $\operatorname{Str}(B \cup \mathbf{a}_i)$ and by triviality of closure it is in $\operatorname{acl}(B)$. This contradicts n-inconsistency of $\{F(x, \mathbf{a}_i) : i \in \omega\}$.

Since ω-categorical superstable theory is ω-stable, the Morley rank of M is one. There is $E_0 \in FE(\varnothing)$ with strongly minimal classes. Let $G = \{\operatorname{acl}(a) - \operatorname{acl}(\varnothing) : a \text{ is nonalgebraic}\}$. For $X_1, X_2 \in G$ let $\langle X_1, X_2 \rangle \in E_1$ iff X_1, X_2 meet the same E_0-classes. Hence E_1 is a finite equivalence relation.

If a and b are in the same E_0-class and $\mathbf{c}$ is a finite sequence disjoint to $(\operatorname{acl}(a) \cup \operatorname{acl}(b)) - \operatorname{acl}(\varnothing)$ then $\operatorname{tp}(a|\mathbf{c}) = \operatorname{tp}(b|\mathbf{c})$. Indeed otherwise for some formula $H(x, \mathbf{c})$, $M \models H(a, \mathbf{c}) \& \neg H(b, \mathbf{c})$. For the corresponding strongly minimal type $p(x, \mathbf{d})$ (consisting of the E_0-class with a and b) we have that, for example, $p(x, \mathbf{d}) \& H(x, \mathbf{c})$ is algebraic. Since acl is trivial and $x \in \operatorname{acl}(y)$ is an equivalence relation, $a \in \operatorname{acl}(\varnothing)$. This is a contradiction.

We now have a 0-definable equivalence relation E on M with finite classes such that M/E is finitely partitioned: E is $\operatorname{acl}(x) = \operatorname{acl}(y)$, E_1-classes on $M/E - \operatorname{acl}(\varnothing)/E$ are infinite, E_1 is invariant under $\operatorname{Aut}(M)$, and for every such E_1-class Q the pointwise stabilizer of $M/E - Q$ in $\operatorname{Aut}(M)$ induces

$\operatorname{Sym} Q)$. By [6] M is monadically stable (M is cellular in the notation of Schmerl [2]).

Proposition 2 yields an easy proof of the main result of [1]. It is enough to note that in a relational language $\operatorname{acl}(a) = \operatorname{acl}(\varnothing) \cup \{a\}$. But this is an easy corollary of Lemma 1.

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