STABLE ABSOLUTELY UBIQUITOUS STRUCTURES

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Abstract. It is proved that absolutely ubiquitous structures with trivial algebraic closures are monadically stable.

Let $L$ be a finite language, and let $M$ be an $L$-structure with countably infinite domain. The age of $M$, denoted by $\mathcal{J}(M)$, is the set of all isomorphism types of finite substructures of $M$. A model $M$ is absolutely ubiquitous if $M$ is uniformly locally finite and every countable locally finite $L$-structure with age $\mathcal{J}(M)$ is isomorphic to $M$. It is known that an absolutely ubiquitous structure is countably categorical and model-complete. It is not known if an absolutely ubiquitous structure has stable or superstable theory. In [1-3] a classification was given for absolutely ubiquitous structures over relational languages. Such structures are finitely partitioned. This will be proved here in a slightly more general case using completely different techniques.

Notation. $B < M$ if $B$ is a substructure of $M$, 

$$\text{Str}(B)$$

is the substructure of $M$ generated by $B$.

1. Lemma of E. A. Palyutin

The following lemma is a simple generalization of an old result of E. A. Palyutin on categorical universal theories [4].

Lemma 1. A countable structure $M$ is absolutely ubiquitous iff

1. $M$ is uniformly locally finite; and

2. there is a function $s: \omega \to \omega$ such that for every $n$, for every substructure $B$ of $M$, and for every type $p(y, b_0, \ldots, b_{n-1})$ over $\{b_0, \ldots, b_{n-1}\} \subseteq B$, if $B$ contains all isomorphism types of $s(n)$-generated substructures of $M$ then there is $b \in B$ such that $M \models p(b, b_0, \ldots, b_{n-1})$.

Proof. Let $M$ be absolutely ubiquitous. We construct $s$ by induction. Suppose $s$ is defined for all $i < n$. Suppose $R_0, \ldots, R_i$ (and $T_0, \ldots, T_i$) are complete formulas which determine all complete $n$-types ($(n + 1)$-types) over $\emptyset$ ($\text{Th}(M)$ is $\omega$-categorical). By model-completeness we may suppose that $R_i$
are \( \forall \)-formulas and \( T_j \) are \( \exists \)-formulas. We define the set \( G_{k}^{i,j} \) as follows. Let

\[
\Phi_{i,j} = \forall x_0, \ldots, x_{n-1}(R_i(x_0, \ldots, x_{n-1}) \rightarrow \exists y(T_j(y, x_0, \ldots, x_{n-1})).
\]

If \( M \models \Phi_{i,j} \) let \( G_{k}^{i,j} \) be

\[
\text{Th}_v(M) \cup \{ \text{there are } z_1, \ldots, z_m \text{ which generate substructure embedding all types of } k \text{-generated substructures of } M \}
\]

\[
\cup \{ \neg \Phi_{i,j} \}.
\]

If \( M \models \neg \Phi_{i,j} \) let \( G_{k}^{i,j} \) be \( \forall x(x \neq x) \).

If for all \( k \) the set \( G_{k}^{i,j} \) is consistent then there is a countable model \( N \) of the theory \( \text{Th}_v(M) \) with \( \mathcal{F}(N) = \mathcal{F}(M) \) and \( N \models \neg \Phi_{i,j} \). It is a contradiction.

Let

\[
k(i, j) = \min(k: G_{k}^{i,j} \text{ is inconsistent}),
\]

\[
s(n) = \max(k(i, j): 0 \leq i \leq t, 0 \leq j \leq l).
\]

Now if \( B \) and \( p \) are as in (2), formula \( R_i \) determines \( \text{tp}((b_0, \ldots, b_{n-1})) \) and formula \( T_j \) determines type \( p(y, x_0, \ldots, x_{n-1}) \), then since \( G_{s(n)} \) is inconsistent we have

\[
B \models \forall x_0, \ldots, x_{n-1}(R_i(x_0, \ldots, x_{n-1}) \rightarrow \exists y T_j(y, x_0, \ldots, x_{n-1})).
\]

Since \( R_i \) is universal, there is \( b \in B \) such that \( B \models T_j(b, b_0, \ldots, b_{n-1}) \), and since \( T_j \) is \( \exists \)-formula, \( M \models T_j(b, b_0, \ldots, b_{n-1}) \).

If (1) and (2) hold in \( M \) then \( M \) is countably saturated. If for countable \( N \) we have \( \mathcal{F}(N) = \mathcal{F}(M) \) then \( N \) is isomorphic to a substructure of \( M \). By back-and-forth argument using (2) we can prove \( M \) is isomorphic to \( N \).

2. Structures with a finite number of term functions

We suppose here that there are terms \( d_1, \ldots, d_t \) such that for every term \( d \) the corresponding function under essential variables is one of the functions defined by \( d_1, \ldots, d_t \) in \( M \). For example, a uniformly locally finite unary algebra is such a structure.

**Proposition 1.** Let \( M \) be an absolutely ubiquitous structure with a finite number of term functions. Then the theory of \( M \) is stable.

**Proof.** It is sufficient to prove that \( \text{Th}(M) \) has not the independence property because by [5] \( \text{Th}(M) \) has not the strict order property. For a contradiction suppose that there are a formula \( F(x, y) \) and sequences \( (a_i: i \in \omega) \) and \( (c_\sigma: \sigma \in 2^\omega) \) such that \( B \models F(a_i, c_\sigma) \) iff \( \sigma(i) = 0 \). For ease of notation suppose that \( a_i \) and \( c_\sigma \) are singletons. Assume terms \( d_1, \ldots, d_t \) depend on at most \( m \) variables. Let \( B \) be a finite substructure of \( M \) containing parameters of \( F \) and all types of \( s(2m+2) \)-generated substructures of \( M \). We may suppose that \( a_1, \ldots, a_{2m+2} \) is a sequence of order indiscernibles over \( B \), and by Lemma 1 suppose

\[
\{ c_\sigma: \sigma \in 2^{2m+2} \} \subseteq \text{Str}(B \cup \{ a_1, \ldots, a_{2m+2} \})
\]

(of course, we neglect parameters). Define \( \sigma \in 2^{2m+2} \) by \( \sigma(i) = 0 \) iff \( i \) is odd. We know that for some \( a_{j_1}, \ldots, a_{j_k} (k \leq m) \) and \( b \subseteq B \) and term \( d_i, c_\sigma = d_i(b, a_{j_1}, \ldots, a_{j_k}) \). There are \( a_{2n-1} \) and \( a_{2n} (n \leq m + 1) \) not among
3. Structures with trivial closure

A structure $M$ has trivial closure if $\text{acl}(X \cup Y) = \text{acl}(X) \cup \text{acl}(Y)$ holds in $M$. For example, absolutely ubiquitous structures with unary terms only have trivial closure. Indeed, by Lemma 1 there is some type of finite substructures of $M$ such that for every $B$ of this type $\text{acl}(X \cup Y)$ is in the substructure generated by $X \cup Y \cup B$. Since every term is unary, we have $\text{acl}(X \cup Y) \subseteq \text{acl}(X) \cup \text{acl}(Y) \cup B$. We can choose $B$ with $B \cap \text{acl}(X \cup Y) \subseteq \text{acl}(\emptyset)$. So, $\text{acl}(X \cup Y) = \text{acl}(X) \cup \text{acl}(Y)$.

By [1, 6] absolutely ubiquitous structures in a relational language are monadically stable. By the description of countably categorical monadically stable structures [6], every absolutely ubiquitous monadically stable structure has trivial closure. It is not hard to find an absolutely ubiquitous structure with a finite number of term functions which does not have trivial closure. So, the situation of Proposition 1 is quite different than that of [1-3]. But in the case of trivial closure we have the following

**Proposition 2.** Absolutely ubiquitous structures with trivial closure are monadically stable.

**Proof.** Let us prove that the relation $x \in \text{acl}(y)$ on nonalgebraic elements is an equivalence relation. If $a \in \text{acl}(b)$ and $b \notin \text{acl}(a)$ then there is some type of finite substructures of our model $M$ such that for every $B$ of this type some realization of $\text{tp}(b|a)$ is in the substructure generated by $B \cup \{a\}$. By triviality of closure it is in $\text{acl}(B)$ and so $a \in \text{acl}(B)$. But we can choose $\text{acl}(B)$ so that $\text{acl}(B) \cap \{a\} \subseteq \text{acl}(\emptyset)$. We have now $a \in \text{acl}(\emptyset)$.

Let us prove that Shelah's $R$-rank of $M$ is one. Let $a_i, i \in \omega$, be an infinite sequence, and suppose the set of nonalgebraic complete formulas $F(x, a_i)$ is $n$-inconsistent. Choose a finite $B < M$ containing all types of $s(\text{ln}(a_i))$-generated substructures of $M$. By Lemma 1 $\text{Str}(B \cup \{a_i : i \in \omega\})$ contains realizations of all formulas $F(x, a_i)$. Of course, the corresponding realization of formula $F(x, a_i)$ is in $\text{Str}(B \cup a_i)$ and by triviality of closure it is in $\text{acl}(B)$. This contradicts $n$-inconsistency of $\{F(x, a_i) : i \in \omega\}$.

Since $\omega$-categorical superstable theory is $\omega$-stable, the Morley rank of $M$ is one. There is $E_0 \in FE(\emptyset)$ with strongly minimal classes. Let $G = \{\text{acl}(a) - \text{acl}(\emptyset) : a$ is nonalgebraic$\}$. For $X_1, X_2 \in G$ let $(X_1, X_2) \in E_1$ iff $X_1, X_2$ meet the same $E_0$-classes. Hence $E_1$ is a finite equivalence relation.

If $a$ and $b$ are in the same $E_0$-class and $c$ is a finite sequence disjoint to $\text{acl}(a) \cup \text{acl}(b) - \text{acl}(\emptyset)$ then $\text{tp}(a|c) = \text{tp}(b|c)$. Indeed otherwise for some formula $H(x, c)$, $M \models H(a, c) & \neg H(b, c)$. For the corresponding strongly minimal type $p(x, d)$ (consisting of the $E_0$-class with $a$ and $b$) we have that, for example, $p(x, d) \& H(x, c)$ is algebraic. Since $\text{acl}$ is trivial and $x \in \text{acl}(y)$ is an equivalence relation, $a \in \text{acl}(\emptyset)$. This is a contradiction.

We now have a $0$-definable equivalence relation $E$ on $M$ with finite classes such that $M/E$ is finitely partitioned: $E$ is $\text{acl}(x) = \text{acl}(y)$, $E_1$-classes on $M/E - \text{acl}(\emptyset)/E$ are infinite, $E_1$ is invariant under $\text{Aut}(M)$, and for every such $E_1$-class $Q$ the pointwise stabilizer of $M/E - Q$ in $\text{Aut}(M)$ induces
Sym \( Q \). By [6] \( M \) is monadically stable (\( M \) is cellular in the notation of Schmerl [2]).

Proposition 2 yields an easy proof of the main result of [1]. It is enough to note that in a relational language \( acl(a) = acl(\varnothing) \cup \{a\} \). But this is an easy corollary of Lemma 1.

REFERENCES