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BOUNDED COMMON EXTENSIONS OF CHARGES

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ABSTRACT. Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of a set X and let $\mu\colon \mathscr{A} \to \mathbf{R}$ and $\nu\colon \mathscr{B} \to \mathbf{R}$ be consistent, bounded, finitely additive measures (i.e., charges). We give necessary and sufficient conditions for μ and ν to have a bounded common extension to $\mathscr{A} \vee \mathscr{B}$. Conditions on $\mathscr{A}$ and $\mathscr{B}$ are given under which any bounded consistent charges μ and ν have a bounded common extension.

0. INTRODUCTION

Let $\mathscr{A}$ be a field of subsets of a set X. We denote by $F(X, \mathscr{A}) = F(\mathscr{A})$ the linear space spanned by indicator functions I_A of sets $A \in \mathscr{A}$. The functions in $F(X, \mathscr{A})$ have finite range and are therefore bounded. If $f \in F(X, \mathscr{A})$, then $\|f\|$ is the supremum norm of f.

A *charge* is a finitely additive set function $\mu\colon \mathscr{A} \to \mathbf{R}$, i.e., $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ whenever $A_1, A_2 \in \mathscr{A}$ are disjoint. If μ is a charge, let $|\mu|$ denote the corresponding *total variation* and let $\|\mu\| = |\mu|(X)$ be the *total variation norm* of μ. A charge μ is *bounded* if $\|\mu\| < \infty$. A charge μ taking only the values 0 and 1 is a 0-1 *charge.*

Now $(F(X, \mathscr{A}), \|\cdot\|)$ is a normed linear space whose dual may be identified with $(\mathrm{ba}(X, \mathscr{A}), \|\cdot\|)$, the space of all bounded charges on $\mathscr{A}$. Each bounded charge μ induces a linear functional $f \to \int f\,d\mu$ on $F(\mathscr{A})$ whose norm is $\|\mu\|$. For these and other facts about charges, we refer the reader to [1].

Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of a set X and let μ, ν be charges on $\mathscr{A}, \mathscr{B}$, respectively. Say that μ and ν are *consistent* if $\mu(C) = \nu(C)$ for all $C \in \mathscr{A} \cap \mathscr{B}$. Let $\mathscr{A} \vee \mathscr{B}$ be the field generated by $\mathscr{A} \cup \mathscr{B}$.

0.1 **Lemma.** *Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of X and suppose that μ and ν are charges on $\mathscr{A}$ and $\mathscr{B}$, respectively. If μ and ν are consistent, then they have a common extension, i.e., there is a charge ρ on $\mathscr{A} \vee \mathscr{B}$ such that $\rho(C) = \mu(C)$ for $C \in \mathscr{A}$ and $\rho(C) = \nu(C)$ for $C \in \mathscr{B}$.*

Indication. This is a well-known result. See, e.g., Theorem 3.6.2 of [1].

When do two bounded consistent charges have a bounded common extension? By the lemma, some extension exists, but might not be bounded. This

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problem seems to have been first studied by Lipecki; some results (in a more general context) were obtained by Schmidt and Waldschaks [3]. A solution is provided by our Theorem 1.5 below. Earlier, Lipecki [2] gave some examples to show that the answer to the question is "not always". Lipecki's positive results, using global conditions on the fields $\mathscr{A}$, $\mathscr{B}$, follow from our work.

1. Chain conditions and bounded extensions

To explain our approach, we begin with the following definition and a few abbreviations. Call $\varnothing = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{N+1} = X$ a *chain* in $\mathscr{A} \cup \mathscr{B}$ (of *length* N) if all the C_i's are in $\mathscr{A} \cup \mathscr{B}$. The following elementary fact lays the groundwork for our main result. Let μ and ν be consistent charges on $\mathscr{A}$ and $\mathscr{B}$, respectively: we define η on $\mathscr{A} \cup \mathscr{B}$ by putting $\eta(C) = \mu(C)$ if $C \in \mathscr{A}$ and $\eta(C) = \nu(C)$ if $C \in \mathscr{B}$.

1.1 **Lemma.** *Let μ on $\mathscr{A}$ and ν on $\mathscr{B}$ be consistent bounded charges. If ρ on $\mathscr{A} \vee \mathscr{B}$ is a common extension of μ and ν, then for any finite chain $\varnothing = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{N+1} = X$ in $\mathscr{A} \cup \mathscr{B}$,*

$$(*) \qquad \sum_{i=0}^{N} |\eta(C_{i+1}) - \eta(C_i)| \le \|\rho\|,$$

where η is defined as above.

From the lemma, it follows that if μ and ν have a bounded common extension, then the supremum of the left-hand side of $(*)$, taken over all possible finite chains, must be finite. Our main result establishes the converse statement (Theorem 1.5).

Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of X and let μ on $\mathscr{A}$ and ν on $\mathscr{B}$ be consistent bounded charges. We define

$$I = I(\mu, \nu) = \inf\{\|\rho\| : \rho \text{ a common extension of } \mu \text{ and } \nu \text{ to } \mathscr{A} \vee \mathscr{B}\},$$

$$S = S(\mu, \nu) = \sup\left\{\int f\,d\mu + \int g\,d\nu : f \in F(\mathscr{A}),\ g \in F(\mathscr{B}),\ \|f+g\| \le 1\right\},$$

$$SC = SC(\mu, \nu) = \sup\left\{\sum_{i=0}^{N} |\eta(C_{i+1}) - \eta(C_i)| : \varnothing = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{N+1} = X \text{ chain in } \mathscr{A} \cup \mathscr{B},\ N \ge 0\right\}.$$

1.2 **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of a set X and suppose that μ and ν are consistent charges on $\mathscr{A}$ and $\mathscr{B}$, respectively. Then $S(\mu, \nu) = I(\mu, \nu)$.*

The infimum defining $I = I(\mu, \nu)$ is attained at some choice of ρ. If $\mathscr{A}$ and $\mathscr{B}$ are finite, then the supremum defining $S = S(\mu, \nu)$ is attained for some f and g.

Proof. Given $f \in F(\mathscr{A})$, $g \in F(\mathscr{B})$, $\|f+g\| \le 1$, and some common extension ρ of μ and ν, we have

$$\int f\,d\mu + \int g\,d\nu = \int (f+g)\,d\rho \le \int \|f+g\|\,d|\rho| \le \|\rho\|,$$

so that $S \leq I$. If $S = \infty$, there is nothing to prove. If $S < \infty$, consider the linear subspace M of $F(\mathscr{A} \vee \mathscr{B})$ defined by

$$M = \{f + g : f \in F(\mathscr{A}),\ g \in F(\mathscr{B})\}.$$

Let $L: M \to \mathbf{R}$ be the linear functional defined by

$$L(f+g) = \int f\,d\mu + \int g\,d\nu.$$

The consistency of μ and ν ensures that L is well defined. In fact, L is a bounded linear functional on M with norm $\|L\| = S < \infty$. The Hahn-Banach theorem implies that L may be extended to a linear functional L_0: $F(\mathscr{A} \vee \mathscr{B}) \to \mathbf{R}$ with $\|L_0\| = \|L\|$. Then $\rho(C) = L_0(I_C)$ defines a charge ρ on $\mathscr{A} \vee \mathscr{B}$ with $\|\rho\| = \|L_0\| = S$, so that $S = I$, and the infimum is attained at ρ.

If $\mathscr{A}$ and $\mathscr{B}$ are finite, then $F(\mathscr{A} \vee \mathscr{B})$ is a finite-dimensional vector space, and the last statement of the theorem becomes elementary. Q.E.D.

1.3 **Corollary.** *In order that consistent bounded charges μ and ν have a bounded common extension, it is necessary and sufficient that $S(\mu, \nu) < \infty$.*

The following technical lemma will be used in the proof of our main theorem.

1.4 **Lemma.** *Let $\mathscr{A}$ be a field of subsets of X and let μ be a bounded charge on $\mathscr{A}$. If $\int f\,d\mu = \|\mu\|$ for $f \in F(X, \mathscr{A})$ with $\|f\| \leq 1$, then*

(i) $\mu \geq 0$ *on subsets of* $\{x : f(x) = 1\}$;
(ii) $\mu \leq 0$ *on subsets of* $\{x : f(x) = -1\}$;
(iii) $|\mu|(\{x : -1 < f(x) < 1\}) = 0$.

Indication. The proof is easy, using the fact that if $\sum a_i b_i = \sum |a_i|$ for numbers $a_i \neq 0$ and $|b_i| \leq 1$, then $b_i = 1$ if $a_i > 0$ and $b_i = -1$ if $a_i < 0$.

1.5 **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of X and suppose that μ and ν are consistent charges on $\mathscr{A}$ and $\mathscr{B}$, respectively. Then $SC(\mu, \nu) = I(\mu, \nu)$.*

Proof. That $SC \leq I$ follows from Lemma 1.1.

In order to prove the reverse inequality, we use Theorem 1.2. Suppose that $f_0 \in F(\mathscr{A})$, $g_0 \in F(\mathscr{B})$, such that $\|f_0 + g_0\| \leq 1$ are given. We shall demonstrate that

$$SC \geq \int f_0\,d\mu + \int g_0\,d\nu \qquad (**)$$

from which fact follows $SC \geq S = I$ as desired.

Let $\mathscr{A}_0$ (respectively $\mathscr{B}_0$) be the smallest field for which f_0 (respectively g_0) is measurable. The $\mathscr{A}_0 \subseteq \mathscr{A}$ and $\mathscr{B}_0 \subseteq \mathscr{B}$ are finite. In order to prove $(**)$, we may assume that f_0 and g_0 have been chosen so that

$$\int f_0\,d\mu + \int g_0\,d\nu$$

is the supremum of $\int f\,d\mu + \int fg\,d\nu$ over all choices of $f \in F(\mathscr{A}_0)$ and $g \in F(\mathscr{B}_0)$ with $\|f + g\| \leq 1$. (We use the final sentence of Theorem 1.2.) Applying Theorem 1.2 to μ_0 and ν_0, the restrictions of μ and ν to $\mathscr{A}_0$ and

$\mathscr{B}_0$, respectively, we find some common extension ρ of μ_0 and ν_0 to $\mathscr{A}_0 \vee \mathscr{B}_0$ such that

$$\int f_0 \, d\mu + \int g_0 \, d\nu = \|\rho\|.$$

Hence

$$\int (f_0 + g_0) \, d\rho = \|\rho\|,$$

so that, by Lemma 1.4, $f_0 + g_0 = \pm 1$ ($|\rho|$-a.e.). By the same Lemma, $\rho \geq 0$ for subsets of $\{x\colon f_0(x) + g_0(x) = 1\}$ and $\rho \leq 0$ for subsets of $\{x\colon f_0(x) + g_0(x) = -1\}$.

Now f_0 and g_0 may be replaced with $f_0 + c$ and $g_0 - c$ for any constant c, with no effect on the norm or integral of their sum. Thus, without loss of generality, we may assume that $f_0 \geq 0$, $g_0 \leq 0$. Let N be an even integer such that $N \geq \max\{\|f_0\| \cdot \|g_0\|\}$ and define

$$C_i = \begin{cases} \{x \in X\colon g_0(x) \leq -N + i - 1\} & \text{if } i \text{ is odd}, \\ \{x \in X\colon f_0(x) \geq N - i + 1\} & \text{if } i \text{ is even}. \end{cases}$$

Then $\varnothing = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{N+1} = X$ is a chain of sets in $\mathscr{A} \cup \mathscr{B}$. If i is odd, then $f_0 + g_0 > -1$ on $C_{i+1} - C_i$, so that $f_0 + g_0 = 1$ $|\rho|$-a.e. on $C_{i+1} - C_i$. Likewise, if i is even, then $f_0 + g_0 = -1$ $|\rho|$-a.e. on $C_{i+1} - C_i$.

Define functions $f_1 \in F(\mathscr{A})$ and $g_1 \in F(\mathscr{B})$ by putting

$$\begin{aligned} f_1 &= N - 2n - 1 \quad \text{for } x \in C_{2n+2} - C_{2n}, \\ g_1 &= -N + 2n \quad \text{for } x \in C_{2n+1} - C_{2n-1} \end{aligned}$$

for $n = 0, 1, \ldots, N/2$, noting that $C_{-1} = \varnothing$ and $C_{N+2} = X$. For i odd, $f_1 + g_1 = 1$ on $C_{i+1} - C_i$. For i even, $f_1 + g_1 = -1$ on $C_{i+1} - C_i$. Thus

$$\begin{aligned} SC &\geq \sum_{i=0}^{N} |\eta(C_{i+1}) - \eta(C_i)| = \int (f_1 + g_1) \, d\rho \\ &= \int (f_0 + g_0) \, d\rho = \int f_0 \, d\mu + \int g_0 \, d\nu. \quad \text{Q.E.D.} \end{aligned}$$

1.6 **Corollary.** *In order that consistent bounded charges μ, ν have a bounded common extension, it is necessary and sufficient that $SC(\mu, \nu) < \infty$.*

Inspection of the proof of Theorem 1.5 yields the following useful sharpening of the result.

1.7 **Corollary.** *In the supremum used to define $SC(\mu, \nu)$, it suffices to restrict attention to chains of the form $\varnothing = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{N+1} = X$, where $C_i \in \mathscr{A}$ if i is even, and $C_i \in \mathscr{B}$ if i is odd.*

2. Global conditions of fields

Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of X. Then $\mathscr{A}$ and $\mathscr{B}$ are *independent* if $A \cap B \neq \varnothing$ whenever $A \in \mathscr{A}$ and $B \in \mathscr{B}$ are nonvoid sets. In particular, this implies that $\mathscr{A} \cap \mathscr{B} = \{\varnothing, X\}$. The following result is probably known (reference unknown), but now follows from the theory of the previous section.

2.1 **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be independent fields of subsets of X and suppose that μ and ν are consistent charges on $\mathscr{A}$ and $\mathscr{B}$. (Consistency means only that $\mu(X) = \nu(X)$.) Then μ and ν have a common extension ρ on $\mathscr{A} \vee \mathscr{B}$ such that $\|\rho\| = \max\{\|\mu\|, \|\nu\|\}$.*

Proof. We apply Theorem 1.5 and Corollary 1.7. Independence essentially limits the length of chains as in Corollary 1.7. It suffices to consider chains of the form $\varnothing \subseteq A \subseteq X$ for $A \in \mathscr{A}$ or $\varnothing \subseteq B \subseteq X$ for $B \in \mathscr{B}$. The supremum $SC(\mu, \nu)$ is thus taken over quantities of the form $|\mu(A)| + |\mu(X - A)|$ or $|\nu(B)| + |\nu(X - B)|$. The result follows. Q.E.D.

Fields $\mathscr{A}$ and $\mathscr{B}$ over X are *weakly independent*, a notion due to Lipecki [2], if whenever $X = A_1 \cup \cdots \cup A_n$ and $X = B_1 \cup \cdots \cup B_m$ are partitions of X into nonempty sets $A_i \in \mathscr{A}$ and $B_i \in \mathscr{B}$, then there is some k and some l such that $A_k \cap B_i \neq \varnothing$ (each i) and $A_i \cap B_l \neq \varnothing$ (each i). The following is a slight improvement on a result of Lipecki [2, Proposition 1].

2.2 **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be weakly independent fields of subsets of a set X and suppose that μ and ν are consistent charges on $\mathscr{A}$ and $\mathscr{B}$ (this means only that $\mu(X) = \nu(X)$). Then there is a common extension ρ of μ and ν such that $\|\rho\| \leq \|\mu\| + |\mu(X)| + \|\nu\|$.*

Proof. Apply Theorem 1.5 and Corollary 1.7. Weak independence limits the length of the chains as in Corollary 1.7. They are either of the form $\varnothing \subseteq A \subseteq B \subseteq X$ or $\varnothing \subseteq B \subseteq A \subseteq X$ for $A \in \mathscr{A}$ and $B \in \mathscr{B}$. The supremum $SC(\mu, \nu)$ is thus taken over quantities

$$|\mu(A)| + |\nu(B) - \mu(A)| + |\nu(X) - \nu(B)|$$

or

$$|\nu(B)| + |\mu(A) - \nu(B)| + |\mu(X) - \mu(A)|.$$

Both of these are bounded by

$$|\mu(A)| + |\mu(X) - \mu(A)| + |\mu(X)| + |\nu(B)| + |\nu(X) - \nu(B)|,$$

which quantity does not exceed $\|\mu\| + |\mu(X)| + \|\nu\|$. Q.E.D.

Let us call a chain (finite or infinite) a *real chain* if each difference $C_{i+1} - C_i$ is nonempty, and each of its sets C_i (except for the first and last) is in exactly one of $\mathscr{A}$, $\mathscr{B}$, and C_{i+1} is in $\mathscr{B}$ (respectively $\mathscr{A}$) whenever C_i is in $\mathscr{A}$ (respectively $\mathscr{B}$). With reference to the preceding theorems, we note that $\mathscr{A}$ and $\mathscr{B}$ are independent if and only if every real chain in $\mathscr{A} \cup \mathscr{B}$ is of length ≤ 1. If $\mathscr{A}$ and $\mathscr{B}$ are weakly independent, then every real chain in $\mathscr{A} \cup \mathscr{B}$ is of length ≤ 2. This latter property is slightly weaker than weak independence, as is shown by the following example. This bound on the length of real chains (≤ 2) is, however, all that was needed to prove Theorem 2.2.

2.3 **Example.** Set $X = \{1, 2, 3, 4, 5, 6\}$, let $\mathscr{A}$ be generated by the partition $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, and let $\mathscr{B}$ be generated by the partition $\{1, 5\}$, $\{2, 3\}$, $\{4, 6\}$. Then each real chain in $\mathscr{A} \cup \mathscr{B}$ is of length ≤ 2, so that the conclusion of Theorem 2.2 applies, but $\mathscr{A}$ and $\mathscr{B}$ are not weakly independent.

2.4 **Theorem.** *Let $\mathscr{A}$ and $\mathscr{B}$ be fields of subsets of a set X such that $\mathscr{A} \cap \mathscr{B} = \{\varnothing, X\}$. Consider the following statements:*

(i) *All real chains in* $\mathscr{A} \cup \mathscr{B}$ *are of bounded length.*

(ii) *Any two bounded, consistent charges on* $\mathscr{A}$ *and* $\mathscr{B}$ *have a bounded common extension.*

(iii) *Any two consistent* 0-1 *charges on* $\mathscr{A}$ *and* $\mathscr{B}$ *have a bounded common extension.*

(iv) *All real chains in* $\mathscr{A} \cup \mathscr{B}$ *are of finite length.*

Then these implications obtain: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *Also,* (iii) *does not imply* (ii).

Proof. (i) $\Rightarrow$ (ii): Let K be an upper bound for the length of real chains in $\mathscr{A} \cup \mathscr{B}$. Let μ and ν be bounded, consistent charges on $\mathscr{A}$ and $\mathscr{B}$ and let $\eta(C) = \mu(C)$ for $C \in \mathscr{A}$ and $\eta(C) = \nu(C)$ for $C \in \mathscr{B}$. Then for each real chain $\varnothing = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{N+1} = X$ we have

$$(***) \qquad \sum_{i=0}^{N} |\eta(C_{i+1}) - \eta(C_i)| \leq N(\|\mu\| + \|\nu\|).$$

Corollary 1.7 for the calculation of $SC(\mu, \nu)$ allows one to look only at real chains. It follows from $(***)$ that $SC(\mu, \nu) \leq K(\|\mu\| + \|\nu\|) < \infty$, so that μ and ν have a bounded common extension.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (iv): This was observed by Lipecki [2, Example 1]. If $\varnothing = C_0 \subseteq C_1 \subseteq \cdots$ is an infinite real chain with $C_i \in \mathscr{A}$ for i odd and $C_i \in \mathscr{B}$ for i even, take a 0-1 charge μ on $\mathscr{A}$ such that $\mu(C_1) = 1$ and a 0-1 charge ν on $\mathscr{B}$ such that $\nu(C_i) = 0$ for i even. Then μ and ν are consistent, and $SC(\mu, \nu) = \infty$, so that there is no bounded common extension.

To prove that (iii) does not imply (ii), we offer the following.

2.5 **Example.** Set $X = \{1, 2, 3, 4, \ldots\}$. Let $\mathscr{A}$ be the field generated by the sets $\{2k+1, 2k+2\}$, $k = 0, 1, 2, \ldots$. Let $\mathscr{B}$ be the field generated by the singleton sets $\{2^{l+1} - 1\}$ for $l = 0, 1, \ldots$ and the sets $\{2^{l+1} + 2k - 2, 2^{l+1} + 2k - 1\}$ for $l = 1, 2, \ldots$ and $1 \leq k \leq 2^l - 1$. Then $\mathscr{A} \cap \mathscr{B} = \{\varnothing, X\}$, all real chains in $\mathscr{A} \cup \mathscr{B}$ are of finite length, but are not of bounded length. Any two 0-1 charges on $\mathscr{A}$ and $\mathscr{B}$ have a bounded common extension.

Define charges μ and ν on $\mathscr{A}$ and $\mathscr{B}$ as follows. Put $\mu(A) = 0$ for $A \in \mathscr{A}$ finite and $\mu(A) = 1$ for $A \in \mathscr{A}$ cofinite. Put $\nu(\{2^{l+1} - 1\}) = 2^{-l-1}$ and $\nu(\{k, k+1\}) = 0$ if $\{k, k+1\} \in \mathscr{B}$. Also set $\nu(X) = 1$. Then μ and ν are bounded, consistent charges with no bounded common extension.

We shall conclude with an example to show that the hypothesis that $\mathscr{A} \cap \mathscr{B} = \{\varnothing, X\}$ is needed in Theorem 2.4. To wit, if this is not assumed, then even if there are infinite real chains in $\mathscr{A} \cup \mathscr{B}$, it is possible that any two consistent 0-1 charges have a bounded common extension.

2.6 **Example.** Set $X = \{1, 2, 3, \ldots\}$. Let $\mathscr{A}$ be the field generated by the collection $\{1\}, \{2, 3\}, \{4\}, \{5, 6\}, \{7\}, \{8, 9\}, \ldots$ and let $\mathscr{B}$ be the field generated by $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}, \{7, 8\}, \{9\}, \ldots$. Then $\mathscr{A} \cap \mathscr{B} \neq \{\varnothing, X\}$, there are infinite real chains in $\mathscr{A} \cup \mathscr{B}$, but any two bounded consistent charges on $\mathscr{A}$ and $\mathscr{B}$ have a bounded common extension.

2.7 **Question.** With reference to Theorem 2.4, can one prove the implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii)?

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