PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 1, May 1994

CONSTANT CODIMENSION FIXED SETS OF COMMUTING INVOLUTIONS

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(Communicated by Thomas Goodwillie)

ABSTRACT. The ideals $\mathcal{J}^r_{*,k}$ of cobordism classes in $\mathcal{N}_*$ containing a representative admitting a $(\mathbf{Z}_2)^k$-action with fixed point set of codimension r are determined for $2^k > r$.

1. INTRODUCTION

Following [P] let $\mathcal{J}^r_{*,k}$ denote the ideal in the unoriented cobordism ring $\mathcal{N}_*$ of classes containing a representative that admits a $(\mathbf{Z}_2)^k$-action (that is, an action generated by k commuting involutions) with fixed point set of constant dimension $n-r$. That article contains references for the cases $k=1$, $r \leq 8$ and complete calculations for the cases $k \geq 1$, $r \leq 2$.

Let $\chi : \mathcal{N}_* \to \mathbf{Z}_2$ denote the mod 2 Euler characteristic. It is well known [C-F, 27.2] that if an involution on M has fixed point set F then $\chi([M]) = \chi([F])$. Along with obvious dimension considerations this enables us to conclude that

$$\mathcal{J}^r_{*,k} \subseteq \begin{cases} \bigoplus_{n=r}^{\infty} \mathcal{N}_n, & r \text{ even}, \\ \bigoplus_{n=r}^{\infty} \mathcal{N}_n \cap \operatorname{Ker} \chi, & r \text{ odd}. \end{cases}$$

A result of tom Dieck implies that for equality to hold it is necessary that $2^k > r$. The present work shows that the condition is sufficient for $r > 1$.

This work is part of my doctoral dissertation at the University of Virginia under the direction of Professor R. E. Stong.

2. BACKGROUND

Throughout this work $\mathbf{Z}_2$ denotes the integers mod 2 and $\equiv$ denotes congruence mod 2. Binomial coefficients are $\binom{m}{n} = m!/n!(m-n)!$.

Manifolds are to be smooth, compact, of constant dimension, and without boundary but not necessarily connected. For a particular example $\phi : (\mathbf{Z}_2)^k \times M^n \to M^n$ of a smooth action on an n-dimensional manifold $T_1, T_2, \ldots, T_k$

Received by the editors August 5, 1992; portions of this paper were presented in a session of the joint meetings in San Antonio, TX, January 16, 1993.

1991 *Mathematics Subject Classification*. Primary 57S17; Secondary 57R85.

Key words and phrases. Cobordism class, fixed point set, line bundle, projective space bundle, $(\mathbf{Z}_2)^k$-action.

may be used to denote the commuting involutions that generate ϕ; the fixed point set of the action is F (or perhaps F^{n-r} if it is to be of constant dimension $n-r$); $F_{T_1,\dots,T_i}$ is the fixed point set of the subgroup generated by $T_1,\dots,T_i$.

Let $\mathcal{N}_* = \bigoplus_{n=0}^{\infty}\mathcal{N}_n$ denote the unoriented cobordism ring; this is shown in [T] to be a $\mathbf{Z}_2$-polynomial algebra on a single generator x_i of each dimension not of the form 2^u-1. The equivalence class of M in $\mathcal{N}_*$ is denoted by $[M]$. If $[M]$ can be expressed $[M]=\Sigma[M_i][N_i]$ with all $[M_i],[N_i]\neq 1$ then $[M]$ is *decomposable*; otherwise it is *indecomposable*.

Using the notation of [P] for $r\geq 0$, $k\geq 1$ let $\mathcal{J}^r_{*,k} = \bigoplus_{n=r}^{\infty}\mathcal{J}^r_{n,k} \subseteq \mathcal{N}_*$ be the ideal of classes containing a representative admitting a $(\mathbf{Z}_2)^k$-action with fixed set of constant codimension r, i.e., an $(n-r)$-dimensional manifold. It is immediate, for example, that $\mathcal{J}^0_{*,k}=\mathcal{N}_*$, $\mathcal{J}^r_{*,k}\subseteq\mathcal{J}^r_{*,k+1}$, and $\mathcal{J}^r_{*,k}\mathcal{J}^{r'}_{*,k}\subseteq \mathcal{J}^{r+r'}_{*,k}$. A complete calculation of $\mathcal{J}^r_{*,k}$ for the cases $r=1,2$ is also found in [P].

Lemma 2.1. *The following holds for* $\mathcal{J}^r_{*,k}$:

$$\mathcal{J}^r_{*,k}\subseteq\begin{cases}\bigoplus_{n=r}^{\infty}\mathcal{N}_n, & r \text{ even},\\ \bigoplus_{n=r}^{\infty}\mathcal{N}_n\cap\operatorname{Ker}\chi, & r \text{ odd}.\end{cases}$$

Proof. The inclusion $\mathcal{J}^r_{*,k}\subseteq\bigoplus_{n=r}^{\infty}\mathcal{N}_n$ holds for obvious dimension reasons.

Suppose r is odd and $(M^n, T_1, T_2,\dots,T_k)$ has fixed point set F^{n-r} so that $[M^n]\in\mathcal{J}^r_{*,k}$. If n is odd then of course $\chi([M^n])=0$. If n is even then $n-r$ is odd and $\chi([F^{n-r}])=0$, and since T_i can be taken as a $\mathbf{Z}_2$-action on $F_{T_1,\dots,T_{i-1}}$, the result for a single involution applies successively to give

$$\chi([M^n])\equiv\chi([F_{T_1}])\equiv\cdots\equiv\chi([F_{T_1,\dots,T_k}])\equiv\chi([F^{n-r}])\equiv 0. \quad \square$$

Whether equality holds in Lemma 2.1 for a particular choice of r,k is subject to the following constraint.

Lemma 2.2 [tD, 1]. *If* $[M^n]$ *is indecomposable and* $n-r<[n/2^k]$ *then* $[M^n]\notin\mathcal{J}^r_{*,k}$. $\square$

Consequently in order to get equality in Lemma 2.1 it is necessary that $2^k>r$ (otherwise, if r is of the form 2^u-1 then $x_{r+1}\notin\mathcal{J}^r_{r,k}$ and if r is not of the form 2^u-1 then $x_r\notin\mathcal{J}^r_{r,k}$).

The task then is to find examples of actions on indecomposables. They will come from the following source.

Lemma 2.3 [S, 3.4]. *Let* $RP(n_1,n_2,\dots,n_l)$ *be the projective space bundle of* $\lambda_1\oplus\cdots\oplus\lambda_l$ *over* $RP^{n_1}\times\cdots\times RP^{n_l}$, *where* λ_i *is the pullback of the usual line bundle over the* i*th factor. Then for* $l>1$, $RP(n_1,n_2,\dots,n_l)$ *is indecomposable in* $\mathcal{N}_*$ *if and only if*

$$\binom{l+n-2}{n_1}+\cdots+\binom{l+n-2}{n_l}\equiv 1,$$

where $n=n_1+\cdots+n_l$. $\square$

The manifold thus described has dimension $l+n-1$. If

$$n_{i+1} = n_{i+2} = \cdots = n_l = 0$$

then $RP(n_1, n_2, \ldots, n_l)$ will sometimes be written $RP(n_1, n_2, \ldots, n_i; l)$.

To calculate binomial coefficients mod 2 the following specific case of the Lucas theorem, first shown by Kummer, is effective.

Lemma 2.4. *If* $m = \sum_{i=0}^{k} m_i 2^i$ *and* $n = \sum_{i=0}^{k} n_i 2^i$ *with* $0 \le m_i, n_i \le 1$, *then* $\binom{m}{n} \equiv 1$ *if and only if* $m_i \ge n_i$ *for every* i. □

3. Actions on projective space bundles

To obtain desirable actions on $RP(n_1, \ldots, n_l)$ we devise a method which given line bundles $\lambda_i \to X_i$ and $(\mathbf{Z}_2)^{k_i}$-actions on λ_i fixing F_i on X_i, $1 \le i \le l$, $\sum 2^{k_i} \le 2^k$, produces a $(\mathbf{Z}_2)^k$-action on the total space of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ over $X_1 \times X_2 \times \cdots \times X_l$ for which the induced action on $RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l)$ has fixed point set l copies of $F_1 \times F_2 \times \cdots \times F_l$.

It is clear that the involution on $RP(\xi \oplus \xi')$ acting as multiplication by -1 in the fibers of ξ and as 1 in those of ξ' has fixed point set $RP(\xi) \cup RP(\xi')$. For $a > b \ge c \ge d$ identify $RP(\bigoplus_{i=b}^{c} \lambda_i)$ fibering over $X_a \times \cdots \times X_d$ with $X_a \times \cdots \times X_{b-1} \times RP(\bigoplus_{i=b}^{c} \lambda_i)$ fibering over $X_b \times \cdots \times X_d$. Note that $RP(\lambda_i)$ fibering over X is homeomorphic to X, and to distinguish it from other such spaces refer to it as $\{e_i\} \times X$.

As a motivating example suppose we are given $\lambda_i \to X_i$ for $1 \le i \le 4$ with a $(\mathbf{Z}_2)^2$-action generated by $T_{1,1}, T_{1,2}$ on X_1, a $(\mathbf{Z}_2)^1$-action generated by $T_{2,1}$ on X_2, $(\mathbf{Z}_2)^0$-actions on X_3, X_4, and involutions $\widetilde{T_{i,j}}$ on λ_i covering each $T_{i,j}$. Define T_3, T_2, T_1 respectively to be the involutions induced on $RP(\lambda_1 \oplus \cdots \oplus \lambda_4)$ by $-1 \times 1 \times 1 \times 1$, $\widetilde{T_{1,2}} \times -1 \times 1 \times 1$, and $\widetilde{T_{1,1}} \times \widetilde{T_{1,2}} \times -1 \times 1$ on $\lambda_1 \oplus \cdots \oplus \lambda_4$, where -1 is scalar multiplication in the fiber. Then

$$\begin{aligned}
F_{T_3} &= RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_3 \oplus \lambda_4) \quad (\text{over } X_1 \times X_2 \times X_3 \times X_4) \\
&= X_1 \times [\{e_1\} \times X_2 \times X_3 \times X_4 \cup RP(\lambda_2 \oplus \lambda_3 \oplus \lambda_4)] \\
&\qquad\qquad\qquad\qquad (\text{over } X_2 \times X_3 \times X_4); \\
F_{T_3, T_2} &= F_{T_{1,2}} \times [\{e_1\} \times X_2 \times X_3 \times X_4 \cup RP(\lambda_2) \cup RP(\lambda_3 \oplus \lambda_4)] \\
&= F_{T_{1,2}} \times X_2 \times [\{e_1, e_2\} \times X_3 \times X_4 \cup RP(\lambda_3 \oplus \lambda_4)] \quad (\text{over } X_3 \times X_4); \\
F_{T_3, T_2, T_1} &= F_{T_{1,2}, T_{1,1}} \times F_{T_{2,1}} \times [\{e_1, e_2\} \times X_3 \times X_4 \cup RP(\lambda_3) \cup RP(\lambda_4)] \\
&= F_{T_{1,2}, T_{1,1}} \times F_{T_{2,1}} \times X_3 \times X_4 \times \{e_1, e_2, e_3, e_4\} \\
&= F_1 \times F_2 \times F_3 \times F_4 \times \{e_1, e_2, e_3, e_4\}.
\end{aligned}$$

If each $(\mathbf{Z}_2)^{k_i}$-action on X_i has fixed set of constant codimension then so too will be the action produced by the general procedure, formalized below.

Lemma 3.1. *Let* $\lambda_i \to X_i$ *be line bundles, and let* $(\mathbf{Z}_2)^{k_i}$ *act on* λ_i *as bundle maps with fixed set* F_i *on* X_i *for all* $1 \le i \le l$ *with* $\sum_{i=1}^{l} 2^{k_i} \le 2^k$. *Then* $RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l)$ *over* $X_1 \times X_2 \times \cdots \times X_l$ *admits a* $(\mathbf{Z}_2)^k$*-action with fixed point set* $F_1 \times F_2 \times \cdots \times F_l \times E$ *with* E *a set of* l *points.*

Proof. Since any $(\mathbf{Z}_2)^{k_i}$-action can be extended to a $(\mathbf{Z}_2)^{k_i+1}$-action with the same fixed set by including a new generator acting as the identity on X_i, assume

$\sum_{i=1}^{l} 2^{k_i} = 2^k$ without loss of generality. Let each $(\mathbf{Z}_2)^{k_i}$-action on X_i be generated by $T_{i,1}, \dots, T_{i,k_i}$ with covering involutions $\widetilde{T_{i,1}}, \dots, \widetilde{T_{i,k_i}}$ on λ_i. Let $k_1 \geq k_2 \geq \cdots \geq k_l$, and define l_j for each j so that $k_i \geq j$ if and only if $i \leq l_j$.

For each $1 \leq j \leq k$ define T_j to be the involution induced on the bundle $RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l)$ by $\widetilde{T_{1,j}} \times \cdots \times \widetilde{T_{1,l_j}} \times (-1)^{a_{l_j+1}} \times \cdots \times (-1)^{a_l}$ on $\lambda_1 \oplus \cdots \oplus \lambda_l$, where -1 is scalar multiplication in the fiber and $a_i = 1$ if $2^{k_1} + 2^{k_2} + \cdots + 2^{k_i} \equiv 1, 2, 3, \dots, 2^{j-1} \bmod 2^j$, $a_i = 0$ otherwise.

To verify that the fixed point set of the action thus defined is as desired again identify $RP(\bigoplus_{i=b}^{c} \lambda_i)$ fibering over $X_a \times \cdots \times X_d$ with $X_a \times \cdots \times X_{b-1} \times RP(\bigoplus_{i=b}^{c} \lambda_i)$ fibering over $X_b \times \cdots \times X_d$ and $RP(\lambda_i)$ fibering over X with $\{e_i\} \times X$.

First let j be the greatest k_i (that is, $j = k_1$). For $j' > j$ each $T_{j'}$ acts exclusively in the fibers; if $j \neq k$ then $F_{T_k} = RP(\bigoplus_{i=1}^{l'} \lambda_i) \cup RP(\bigoplus_{i=l'+1}^{l} \lambda_i)$ where $\sum_{i=1}^{l'} 2^{k_i} = \sum_{i=l'+1}^{l} 2^{k_i} = 2^{k-1}$ and in fact $F_{T_k, \dots, T_{j+1}} = \bigcup_I RP(\xi_I)$ where each $\xi_I = \bigoplus_{i \in I} \lambda_i$ with $\sum_{i \in I} 2^{k_i} = 2^j$. Each λ_i for which $k_i = j$ is itself an ξ_I, so by the preceding identifications write

$$F_{T_k, \dots, T_{j+1}} = \left(\mathop{\times}_{i=1}^{l_j} X_i \right) \times \left[\left(\bigcup_{i=1}^{l_j} \{e_i\} \right) \times \left(\mathop{\times}_{i=l_j+1}^{l} X_i \right) \cup \left(\bigcup_I RP(\xi_I) \right) \right],$$

where each ξ_I contains at least two λ_i summands and fibers over $X_{l_j+1} \times \cdots \times X_l$.

Thus it is seen that the action of $\widetilde{T_{i,j}}$ is confined to the exposed factor $(\mathop{\times}_{i=1}^{l_j} X_i)$ which combined with the fiber action of T_j gives

$$F_{T_k, \dots, T_j} = \left(\mathop{\times}_{i=1}^{l_j} F_{T_{i,j}} \right) \times \left(\mathop{\times}_{i=l_j+1}^{l_{j-1}} X_i \right) \times \left[\left(\bigcup_{i=1}^{l_{j-1}} \{e_i\} \right) \times \left(\mathop{\times}_{i=l_{j-1}+1}^{l} X_i \right) \cup \left(\bigcup_I RP(\xi_I) \right) \right],$$

where each $\xi_I = \bigoplus_{i \in I} \lambda_i$ with $\sum_{i \in I} 2^{j-1}$ and ξ_I fibers over $X_{l_{j-1}+1} \times \cdots \times X_l$. (If $l_j = l_{j-1}$, the factor $(\mathop{\times}_{i=l_j+1}^{l_{j-1}} X_i)$ is ignored.) For smaller j we obtain a partial fixed set for which the first factor is $(\mathop{\times}_{i=1}^{l_j} F_{T_{i,k_i}, \dots, F_{i,j}})$ so finally

$$F_{T_k, \dots, T_1} = \left(\mathop{\times}_{i=1}^{l} F_{T_{i,k_i}, \dots, F_{i,1}} \right) \times \bigcup_{i=1}^{l} \{e_i\}. \quad \square$$

Corollary 3.2. *If* $2^k > r$ *then* $\mathscr{J}^r_{*,k} \ni [RP(n_1, \dots, n_{r+1})]$.

Proof. Let $(\mathbf{Z}_2)^0$ act as the identity on each RP^{n_i} in the base. $\square$

Corollary 3.3. *If* $2^k > r$ *then* $\mathscr{J}^r_{*,k} \ni [RP(n_1 = 1, \dots, n_r)]$.

Proof. Let $(\mathbf{Z}_2)^1$ act on $RP^{n_1} = RP^1$ with F_1 two copies of RP^0 and $(\mathbf{Z}_2)^0$ act as the identity on the rest of the base. □

Corollary 3.4. *If $M^n = RP(n_1, n_2, \dots, n_l)$ with $n_1 = n_2 = \cdots = n_d = 1$ and each $n_i < 2^{k_i}$ with $\sum_{i=1}^{l} 2^{k_i} \le 2^k$ then $[M^n] \in \mathscr{I}^r_{*,k}$ for $n - d \le r \le n$.*

Proof. Let $(\mathbf{Z}_2)^{k_i}$ act on RP^{n_i} with $F_i = RP^{n_i}$ for $i \le n - r$ and F_i a set of $n_i + 1$ points for $i > n - r$. □

4. Existence of indecomposables

The following combinatorial proofs establish the existence of indecomposable manifolds $RP(n_1, n_2, \dots, n_l)$ mandated by the preceding.

Lemma 4.1. *Suppose $l + n - 1 \ge 2^k \ge l \ge 3$. Then there exists an indecomposable $RP(n_1, n_2, \dots, n_l)$ if l is odd and $l + n - 1$ is not of the form $2^u - 1$ or if l is even and $l + n - 1$ is not of the form 2^u or $2^u - 1$.*

Proof. Let $m = l + n - 2$, and express $m = \sum_{i=0}^{k} m_i 2^i$ with $0 \le m_i \le 1$. Let $s = \sum_{i=1}^{l} \binom{m}{n_i}$. According to Lemma 2.3 the manifold is indecomposable if and only if $s \equiv 1$. To construct representatives it is necessary to treat cases.

Case l odd, n even. Take $RP(n/2, n/2; l)$ for which

$$s = 2\binom{m}{n/2} + (l-2)\binom{m}{0} \equiv 1.$$

Remaining cases. One may assume m is not of the form $2^u - 1$ or $2^u - 2$, so one can find a minimal $j \ge 1$ so that $m_j = 0$ and thus $l + n - 2 \ge 2^k + (2^j - 2)$; since $2^k \ge l$, it follows that $n \ge 2^j$.

Case l odd, n odd. One has

$$\binom{m}{2^j} \equiv \binom{m}{n - 2^j} \equiv 0$$

due to disagreement in the jth, 0th places respectively, and $RP(n - 2^j, 2^j; l)$ has

$$s = \binom{m}{2^j} + \binom{m}{n - 2^j} + (l-2)\binom{m}{0} \equiv 1.$$

Case l even, n even. $RP(2^j, (n - 2^j)/2, (n - 2^j)/2; l)$ has

$$s = \binom{m}{2^j} + 2\binom{m}{(n - 2^j)/2} + (l-3)\binom{m}{0} \equiv 1.$$

Case l even, n odd. $RP(2^j + 1, (n - 2^j - 1)/2, (n - 2^j - 1)/2; l)$ has

$$s = \binom{m}{2^j + 1} + 2\binom{m}{(n - 2^j - 1)/2} + (l-3)\binom{m}{0} \equiv 1. \quad \square$$

Lemma 4.2. *If l is odd then $RP(1, 2^u - l; l)$ is indecomposable for every $2^u > l$.*

Proof. Since here $m = 2^u - 1$ and so $\binom{m}{n_i} \equiv 1$ for any n_i, $s \equiv l \equiv 1$. □

Lemma 4.3. *If $2^{k-1} \le 2^{k-1} + d < 2^k - 1$ then there exists an indecomposable $M^{2^{k-1}+d} = RP(n_1, \dots, n_l)$ with $n_1 = n_2 = \cdots = n_d = 1$ and each $n_i < 2^{k_i}$ with $\sum_{i=1}^{l} 2^{k_i} \le 2^k$.*

Proof. Let $m = 2^{k-1} + d - 1$. If d is even consider $RP(n_1 = 1, \dots, n_d = 1; 2^{k-1} + 1)$ for which

$$s = d\binom{m}{1} + (2^{k-1} - d + 1)\binom{m}{0} \equiv 0 \cdot 1 + 1 \cdot 1 \equiv 1.$$

This qualifies since $d \cdot 2 + (2^{k-1} - d + 1) \cdot 1 = 2^{k-1} + d + 1 \leq 2^k$.

If d is odd then let $1 < 2^j < 2^{k-1}$ be such that $\binom{m}{2^j} \equiv 0$ as justifiable by the fact that $2^{k-1} - 2 < m < 2^k - 2$ and consider the manifolds $RP(n_1 = 1, \dots, n_d = 1, 2^j; 2^{k-1} - 2^j + 1)$ for which

$$s = d\binom{m}{1} + \binom{m}{2^j} + (2^{k-1} - 2^j - d)\binom{m}{0} \equiv 1 \cdot 0 + 0 + 1 \cdot 1 \equiv 1.$$

This qualifies since $d \cdot 2 + 2^{j+1} + (2^{k-1} - 2^j - d) \cdot 1 = 2^{k-1} + 2^j + d \leq 2^k$. □

5. Determination of ideals

We now arrive at our main result.

Proposition 5.1. *If* $2^k > r$ *then*

$$\mathscr{J}^r_{*,k} = \begin{cases} (0), & r = 1, \\ \bigoplus_{n=r}^{\infty} \mathscr{N}_n, & r \text{ even}, \\ \bigoplus_{n=r}^{\infty} \mathscr{N}_n \cap \operatorname{Ker} \chi, & r \text{ odd}, \ r \geq 3. \end{cases}$$

Proof. The case $r = 1$ is [P, 3.1].

For $r \geq 2$ Lemma 2.1 gives one inclusion; we wish to show the reverse. The first task is to show that $\mathscr{J}^r_{*,k}$ contains an indecomposable in each dimension $d \geq r$. For $r \leq d \leq 2^k$ this is handled by Corollary 3.4 and Lemma 4.3. For $d > 2^k$ and either r even or d not of the form 2^u, the indecomposable is provided by Corollary 3.2 and Lemma 4.1. For $d > 2^k$, r odd, and d of the form 2^u it is provided by Corollary 3.3 and Lemma 4.2.

An inductive argument shows $\mathscr{J}^r_{*,k}$ contains all specified decomposables, i.e. of dimension $d \geq r$ and, for r odd, in Ker χ. The cases $r = 2, 3$ follow simply from the fact that $\mathscr{J}^r_{*,k}$ contains indecomposables of dimension $d \geq r$ and is an ideal. Therefore assume that $2^k > r$ and $\mathscr{J}^{r_i}_{*,k}$ contains all specified decomposables for $r_i < r$.

Suppose $r \leq n$ and $[M^n] = [M_1^{n_1}][M_2^{n_2}]$. We wish to find $r_i \leq n_i$, $i = 1, 2$, with $r_1 + r_2 = r$, so as to exploit the inductive hypothesis and $\mathscr{J}^{r_1}_{*,k}\mathscr{J}^{r_2}_{*,k} \subseteq \mathscr{J}^r_{*,k}$. If $\chi([M^n]) = 0$ then simply choose $r_1 = n_1$, $r_2 = n_2 - (n - r)$. If r is even and $\chi([M^n]) = 1$, hence n_1, n_2 are even, the same assignment works because it makes r_1, r_2 even. □

Corollary 5.2. $\mathscr{J}^2_{*,k} = \bigoplus_{n=2}^{\infty} \mathscr{N}_n$ *for* $k \geq 2$. □

Corollary 5.3. $\mathscr{J}^3_{*,k} = \operatorname{Ker} \chi$ *for* $k \geq 2$. □

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