CONSTANT CODIMENSION FIXED SETS OF COMMUTING INVOLUTIONS

R. J. SHAKER, JR.

(Communicated by Thomas Goodwillie)

Abstract. The ideals \( \mathcal{J}_{*,k}^r \) of cobordism classes in \( \mathcal{N}_* \) containing a representative admitting a \((\mathbb{Z}_2)^k\)-action with fixed point set of codimension \( r \) are determined for \( 2^k > r \).

1. Introduction

Following [P] let \( \mathcal{J}_{*,k}^r \) denote the ideal in the unoriented cobordism ring \( \mathcal{N}_* \) of classes containing a representative that admits a \((\mathbb{Z}_2)^k\)-action (that is, an action generated by \( k \) commuting involutions) with fixed point set of constant dimension \( n - r \). That article contains references for the cases \( k = 1, r < 8 \) and complete calculations for the cases \( k > 1, r < 2 \).

Let \( \chi : \mathcal{N}_* \rightarrow \mathbb{Z}_2 \) denote the mod 2 Euler characteristic. It is well known [C-F, 27.2] that if an involution on \( M \) has fixed point set \( F \) then \( \chi([M]) = \chi([F]) \).

Along with obvious dimension considerations this enables us to conclude that

\[
\mathcal{J}_{*,k}^r \subseteq \begin{cases} 
\bigoplus_{n=r}^{\infty} \mathcal{N}_n, & r \text{ even}, \\
\bigoplus_{n=r}^{\infty} \mathcal{N}_n \cap \text{Ker } \chi, & r \text{ odd}.
\end{cases}
\]

A result of tom Dieck implies that for equality to hold it is necessary that \( 2^k > r \). The present work shows that the condition is sufficient for \( r > 1 \).

This work is part of my doctoral dissertation at the University of Virginia under the direction of Professor R. E. Stong.

2. Background

Throughout this work \( \mathbb{Z}_2 \) denotes the integers mod 2 and \( \equiv \) denotes congruence mod 2. Binomial coefficients are \( \binom{m}{n} = m! / n! (m-n)! \).

Manifolds are to be smooth, compact, of constant dimension, and without boundary but not necessarily connected. For a particular example \( \phi : (\mathbb{Z}_2)^k \times M^n \rightarrow M^n \) of a smooth action on an \( n \)-dimensional manifold \( T_1, T_2, \ldots, T_k \)

Received by the editors August 5, 1992; portions of this paper were presented in a session of the joint meetings in San Antonio, TX, January 16, 1993.

1991 Mathematics Subject Classification. Primary 57S17; Secondary 57R85.

Key words and phrases. Cobordism class, fixed point set, line bundle, projective space bundle, \((\mathbb{Z}_2)^k\)-action.

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may be used to denote the commuting involutions that generate \( \phi \); the fixed point set of the action is \( F \) (or perhaps \( F^{n-r} \) if it is to be of constant dimension \( n-r \)); \( F_{T_1, \ldots, T_i} \) is the fixed point set of the subgroup generated by \( T_1, \ldots, T_i \).

Let \( \mathcal{N}_* = \bigoplus_{n=0}^{\infty} \mathcal{N}_n \) denote the oriented cobordism ring; this dimension is shown in [T] to be a \( \mathbb{Z}_2 \)-polynomial algebra on a single generator \( x_i \) of each dimension not of the form \( 2^u - 1 \). The equivalence class of \( M \) in \( \mathcal{N}_* \) is denoted by \( [M] \). If \( [M] \) can be expressed \( [M] = \Sigma [M_i][N_i] \) with all \([M_i],[N_i] \neq 1 \) then \( [M] \) is decomposable; otherwise it is indecomposable.

Using the notation of [P] for \( r \geq 0, \ k \geq 1 \) let \( \mathcal{I}_{*,k}^r = \bigoplus_{n=r}^{\infty} \mathcal{I}_{*,k}^r \subseteq \mathcal{N}_* \) be the ideal of classes containing a representative admitting a \( (\mathbb{Z}_2)^k \)-action with fixed set of constant codimension \( r \), i.e., an \( (n-r) \)-dimensional manifold. It is immediate, for example, that \( \mathcal{I}_{*,k}^0 = \mathcal{N}_*, \mathcal{I}_{*,k}^r \subseteq \mathcal{I}_{*,k+1}^r \), and \( \mathcal{I}_{*,k}^r \subseteq \mathcal{I}_{*,k}^{r+1} \). A complete calculation of \( \mathcal{I}_{*,k}^r \) for the cases \( r = 1,2 \) is also found in [P].

**Lemma 2.1.** The following holds for \( \mathcal{I}_{*,k}^r \):

\[
\mathcal{I}_{*,k}^r \subseteq \begin{cases} 
\bigoplus_{n=r}^{\infty} \mathcal{N}_n, & \text{r even,} \\
\bigoplus_{n=r}^{\infty} \mathcal{N}_n \cap \text{Ker} \chi, & \text{r odd.}
\end{cases}
\]

**Proof.** The inclusion \( \mathcal{I}_{*,k}^r \subseteq \bigoplus_{n=r}^{\infty} \mathcal{N}_n \) holds for obvious dimension reasons.

Suppose \( r \) is odd and \( (M^n, T_1, T_2, \ldots, T_k) \) has fixed point set \( F^{n-r} \) so that \([M^n] \in \mathcal{I}_{*,k}^r \). If \( n \) is odd then of course \( \chi([M^n]) = 0 \). If \( n \) is even then \( n-r \) is odd and \( \chi([F^{n-r}]) = 0 \), and since \( T_i \) can be taken as a \( \mathbb{Z}_2 \)-action on \( F_{T_1, \ldots, T_{i-1}} \), the result for a single involution applies successively to give

\[
\chi([M^n]) = \chi([F_{T_1}]) = \cdots = \chi([F_{T_1, \ldots, T_k}]) = \chi([F^{n-r}]) = 0.
\]

Whether equality holds in Lemma 2.1 for a particular choice of \( r, k \) is subject to the following constraint.

**Lemma 2.2 [1D, 1].** If \([M^n] \) is indecomposable and \( n-r < \lfloor n/2^k \rfloor \) then \([M^n] \notin \mathcal{I}_{*,k}^r \).

Consequently in order to get equality in Lemma 2.1 it is necessary that \( 2^k > r \) (otherwise, if \( r \) is of the form \( 2^u - 1 \) then \( x_{r+1} \notin \mathcal{I}_{*,k}^r \) and if \( r \) is not of the form \( 2^u - 1 \) then \( x_r \notin \mathcal{I}_{*,k}^r \)).

The task then is to find examples of actions on indecomposables. They will come from the following source.

**Lemma 2.3 [S, 3.4].** Let \( RP(n_1, n_2, \ldots, n_i) \) be the projective space bundle of \( \lambda_1 \oplus \cdots \oplus \lambda_i \) over \( RP^{n_1} \times \cdots \times RP^{n_i} \), where \( \lambda_i \) is the pullback of the usual line bundle over the \( i \)th factor. Then for \( l > 1 \), \( RP(n_1, n_2, \ldots, n_l) \) is indecomposable in \( \mathcal{N}_* \) if and only if

\[
(l + n_1 - 2) + \cdots + (l + n_l - 2) \equiv 1,
\]

where \( n = n_1 + \cdots + n_l \).  \( \square \)
The manifold thus described has dimension $l + n - 1$. If
\[ n_{i+1} = n_{i+2} = \ldots = n_l = 0 \]
then $RP(n_1, n_2, \ldots, n_l)$ will sometimes be written $RP(n_1, n_2, \ldots, n_l)$.

To calculate binomial coefficients mod 2 the following specific case of the Lucas theorem, first shown by Kummer, is effective.

**Lemma 2.4.** If $m = \sum_{i=0}^{k} m_i 2^i$ and $n = \sum_{i=0}^{k} n_i 2^i$ with $0 \leq m_i, n_i \leq 1$, then $\binom{m}{n} \equiv 1$ if and only if $m_i \geq n_i$ for every $i$. □

### 3. Actions on projective space bundles

To obtain desirable actions on $RP(n_1, \ldots, n_l)$ we devise a method which given line bundles $\lambda_i \to X_i$ and $(\mathbb{Z}_2)^{k_i}$-actions on $\lambda_i$ fixing $F_i$ on $X_i$, $1 \leq i \leq l$, $\sum 2^{k_i} \leq 2^k$, produces a $(\mathbb{Z}_2)^k$-action on the total space of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ over $X_1 \times X_2 \times \cdots \times X_l$ for which the induced action on $RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l)$ has fixed point set $l$ copies of $F_1 \times F_2 \times \cdots \times F_l$.

It is clear that the involution on $RP(\xi \oplus \xi')$ acting as multiplication by $-1$ in the fibers of $\xi$ and as $1$ in those of $\xi'$ has fixed point set $RP(\xi) \cup RP(\xi')$.

For $a > b > c > d$ identify $RP(\xi_{c=b})$ fibering over $X_a \times \cdots \times X_d$ with $X_a \times \cdots \times X_{b-1} \times RP(\xi_{c=b})$ fibering over $X_b \times \cdots \times X_d$. Note that $RP(\lambda_i)$ fibering over $X$ is homeomorphic to $X$, and to distinguish it from other such spaces refer to it as $\{e_i\} \times X$.

As a motivating example suppose we are given $\lambda_i \to X_i$ for $1 \leq i \leq 4$ with a $(\mathbb{Z}_2)^2$-action generated by $T_{1,1}, T_{1,2}$ on $X_1$, a $(\mathbb{Z}_2)^1$-action generated by $T_{2,1}$ on $X_2$, $(\mathbb{Z}_2)^0$-actions on $X_3, X_4$, and involutions $T_{i,j}$ on $\lambda_i$ covering each $T_{i,j}$. Define $T_{3,1}, T_{3,2}, T_{1,1}$ respectively to be the involutions induced on $RP(\lambda_1 \oplus \cdots \oplus \lambda_4)$ by $-1 \times 1 \times 1 \times 1, T_{1,2} \times -1 \times 1 \times 1$, and $T_{1,1} \times T_{i,1} \times -1 \times 1$ on $\lambda_1 \oplus \cdots \oplus \lambda_4$, where $-1$ is scalar multiplication in the fiber. Then

\[
F_{T_3} = RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_3 \oplus \lambda_4) \quad (\text{over } X_1 \times X_2 \times X_3 \times X_4)
\]
\[
= X_1 \times \{e_1\} \times X_2 \times X_3 \times X_4 \cup RP(\lambda_2 \oplus \lambda_3 \oplus \lambda_4) \quad (\text{over } X_2 \times X_3 \times X_4);
\]
\[
F_{T_3}, T_2 = F_{T_{1,2}} \times \{e_1\} \times X_2 \times X_3 \times X_4 \cup RP(\lambda_2) \cup RP(\lambda_3 \oplus \lambda_4) \quad (\text{over } X_3 \times X_4);
\]
\[
F_{T_3}, T_1 = F_{T_{1,2}} \times T_{1,1} \times F_{T_{2,1}} \times \{e_1, e_2\} \times X_3 \times X_4 \cup RP(\lambda_3) \cup RP(\lambda_4) \quad (\text{over } X_1 \times X_2 \times X_3 \times X_4 \times \{e_1, e_2, e_3, e_4\})
\]
\[
= F_{T_{1,2}} \times T_{1,1} \times F_{T_{2,1}} \times X_3 \times X_4 \times \{e_1, e_2, e_3, e_4\}
\]
\[
= F_1 \times F_2 \times F_3 \times F_4 \times \{e_1, e_2, e_3, e_4\}.
\]

If each $(\mathbb{Z}_2)^{k_i}$-action on $X_i$ has fixed set of constant codimension then so too will be the action produced by the general procedure, formalized below.

**Lemma 3.1.** Let $\lambda_i \to X_i$ be line bundles, and let $(\mathbb{Z}_2)^{k_i}$ act on $\lambda_i$ as bundle maps with fixed set $F_i$ on $X_i$ for all $1 \leq i \leq l$ with $\sum_{i=1}^{l} 2^{k_i} \leq 2^k$. Then $RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l)$ over $X_1 \times X_2 \times \cdots \times X_l$ admits a $(\mathbb{Z}_2)^k$-action with fixed point set $F_1 \times F_2 \times \cdots \times F_l \times E$ with $E$ a set of $l$ points.

**Proof.** Since any $(\mathbb{Z}_2)^{k_i}$-action can be extended to a $(\mathbb{Z}_2)^{k_i+1}$-action with the same fixed set by including a new generator acting as the identity on $X_i$, assume...
Let \( \sum_{i=1}^{l} 2^{k_i} = 2^k \) without loss of generality. Let each \( (\mathbb{Z}_2)^{k_i} \)-action on \( X_i \) be generated by \( T_{i,1}, \ldots, T_{i,k_i} \) with covering involutions \( \overline{T}_{i,1}, \ldots, \overline{T}_{i,k_i} \) on \( \lambda_i \).

Let \( k_1 \geq k_2 \geq \cdots \geq k_l \), and define \( l_j \) for each \( j \) so that \( k_i \geq j \) if and only if \( i \leq l_j \).

For each \( 1 \leq j \leq k \) define \( T_j \) to be the involution induced on the bundle \( RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l) \) by \( \overline{T}_{i,j} \times \cdots \times \overline{T}_{i,j} \times (-1)^{a_{j+1}} \times \cdots \times (-1)^{a_l} \) on \( \lambda_1 \oplus \cdots \oplus \lambda_l \), where \(-1\) is scalar multiplication in the fiber and \( a_i = 1 \) if \( 2^{k_1} + 2^{k_2} + \cdots + 2^{k_i} \equiv 1, 2, 3, \ldots, 2^j - 1 \mod 2^j \), \( a_i = 0 \) otherwise.

To verify that the fixed point set of the action thus defined is as desired again identify \( RP(\bigoplus_{i=b}^{c} \lambda_i) \) fibering over \( X_a \times \cdots \times X_d \) with \( X_a \times \cdots \times X_{b-1} \times \bigoplus_{i=b}^{c} \lambda_i \) fibering over \( X_b \times \cdots \times X_d \) and \( RP(\lambda_i) \) fibering over \( X \) with \( \{e_i\} \times X \).

First let \( j \) be the greatest \( k_i \) (that is, \( j = k_1 \)). For \( j' > j \) each \( T_j \) acts exclusively in the fibers; if \( j' \neq k \) then \( F_{T_k} = RP(\bigoplus_{i=1}^{l} \lambda_i) \setminus RP(\bigoplus_{i=l+1}^{r} \lambda_i) \) where \( \sum_{i=1}^{l} 2^{k_i} = \sum_{i=l+1}^{r} 2^{k_i} = 2^{k-1} \) and in fact \( F_{T_k}, \ldots, F_{T_{j+1}} = \bigcup F(\xi_i) \) where each \( \xi_i = \bigoplus_{i \in I} \lambda_i \) with \( \sum_{i \in I} 2^{k_i} = 2^j \). Each \( \lambda_i \) for which \( k_i = j \) is itself an \( \xi_i \), so by the preceding identifications write

\[
F_{T_k, \ldots, T_{j+1}} = \left( \bigcap_{i=1}^{l} X_i \right) \times \left[ \bigcup_{i=1}^{l} \{e_i\} \right] \times \left( \bigcup_{i=l+1}^{r} X_i \right) \cup \left( \bigcup F(\xi_i) \right)
\]

where each \( \xi_i \) contains at least two \( \lambda_i \) summands and fibers over \( X_{j+1} \times \cdots \times X_l \).

Thus it is seen that the action of \( \overline{T}_{i,j} \) is confined to the exposed factor \( \bigcap_{i=1}^{l} X_i \) which combined with the fiber action of \( T_j \) gives

\[
F_{T_k, \ldots, T_j} = \left( \bigcap_{i=1}^{l} X_i \right) \times \left( \bigcup_{i=l+1}^{r} X_i \right) \cup \left( \bigcup F(\xi_i) \right)
\]

where each \( \xi_i = \bigoplus_{i \in I} \lambda_i \) with \( \sum_{i \in I} 2^{j-1} \) and \( \xi_i \) fibers over \( X_{j+1} \times \cdots \times X_l \).

If \( l_j = l_j-1 \), the factor \( \bigcap_{i=l_j-1}^{l_j} X_i \) is ignored. For smaller \( j \) we obtain a partial fixed set for which the first factor is \( \bigcap_{i=1}^{l_j} F_{T_{i,k_i}, \ldots, F_{i,j}} \) so finally

\[
F_{T_k, \ldots, T_j} = \left( \bigcap_{i=1}^{l_j} F_{T_{i,k_i}, \ldots, F_{i,j}} \right) \times \bigcup_{i=1}^{l_j} \{e_i\}. \quad \square
\]

**Corollary 3.2.** If \( 2^k > r \) then \( \mathcal{F}_{*,k} \) \( \subseteq \) \( \{RP(n_1, \ldots, n_{r+1})\} \).

**Proof.** Let \( (\mathbb{Z}_2)^0 \) act as the identity on each \( RP(n_i) \) in the base. \( \square \)

**Corollary 3.3.** If \( 2^k > r \) then \( \mathcal{F}_{*,k} \) \( \subseteq \) \( \{RP(n_1 = 1, \ldots, n_r)\} \).
Proof. Let \((Z_2)^1\) act on \(RP^{n_1} = RP^1\) with \(F_1\) two copies of \(RP^0\) and \((Z_2)^0\) act as the identity on the rest of the base. □

**Corollary 3.4.** If \(M^n = RP(n_1, n_2, \ldots, n_I)\) with \(n_1 = n_2 = \cdots = n_d = 1\) and each \(n_i < 2^{k_i}\) with \(\sum_{i=1}^l 2^{k_i} \leq 2^k\) then \([M^n] \in \mathcal{F}_{*k}\) for \(n - d \leq r \leq n\).

Proof. Let \((Z_2)^{k_i}\) act on \(RP^{n_i}\) with \(F_i = RP^{n_i}\) for \(i \leq n - r\) and \(F_i\) a set of \(n_i + 1\) points for \(i > n - r\). □

4. Existence of indecomposables

The following combinatorial proofs establish the existence of indecomposable manifolds \(RP(n_1, n_2, \ldots, n_I)\) mandated by the preceding.

**Lemma 4.1.** Suppose \(l + n - 1 \geq 2^k \geq l \geq 3\). Then there exists an indecomposable \(RP(n_1, n_2, \ldots, n_I)\) if \(l\) is odd and \(l + n - 1\) is not of the form \(2^u - 1\) or if \(l\) is even and \(l + n - 1\) is not of the form \(2^u\) or \(2^u - 1\).

Proof. Let \(m = l + n - 2\), and express \(m = \sum_{i=0}^{k} m_i 2^i\) with \(0 \leq m_i \leq 1\). Let \(s = \sum_{i=1}^{l} \binom{m_i}{n_i}\). According to Lemma 2.3 the manifold is indecomposable if and only if \(s \equiv 1\). To construct representatives it is necessary to treat cases.

**Case l odd, n even.** Take \(RP(n/2, n/2; l)\) for which

\[
s = 2\left(\frac{m}{n/2}\right) + (l - 2)\left(\frac{m}{0}\right) \equiv 1.
\]

due to disagreement in the \(j\)th, 0th places respectively, and \(RP(n - 2^j, 2^j; l)\) has

\[
s = \left(\frac{m}{2^j}\right) + \left(\frac{m}{n - 2^j}\right) + (l - 2)\left(\frac{m}{0}\right) \equiv 1.
\]

**Case l even, n even.** \(RP(2^j + 1, (n - 2^j - 1)/2, (n - 2^j - 1)/2; l)\) has

\[
s = \left(\frac{m}{2^j + 1}\right) + 2\left(\frac{m}{(n - 2^j - 1)/2}\right) + (l - 3)\left(\frac{m}{0}\right) \equiv 1.
\]

**Lemma 4.2.** If \(l\) is odd then \(RP(1, 2^u - 1; l)\) is indecomposable for every \(2^u > l\).

Proof. Since here \(m = 2^u - 1\) and so \(\binom{m}{n_i} \equiv 1\) for any \(n_i\), \(s \equiv l \equiv 1\). □

**Lemma 4.3.** If \(2^{k-1} \leq 2^{k-1} + d < 2^k - 1\) then there exists an indecomposable \(M^{2^{k-1} + d} = RP(n_1, \ldots, n_I)\) with \(n_1 = n_2 = \cdots = n_d = 1\) and each \(n_i < 2^{k_i}\) with \(\sum_{i=1}^l 2^{k_i} \leq 2^k\).
\textbf{Proof.} Let \( m = 2^{k-1} + d - 1 \). If \( d \) is even consider \( RP(n_1 = 1, \ldots, n_d = 1; 2^{k-1} + 1) \) for which
\[
s = d \binom{m}{1} + (2^{k-1} - d + 1) \binom{m}{0} \equiv 0 \cdot 1 + 1 \cdot 1 \equiv 1.
\]
This qualifies since \( d \cdot 2 + (2^{k-1} - d + 1) \cdot 1 = 2^{k-1} + d + 1 \leq 2^k \).

If \( d \) is odd then let \( 1 < 2^j < 2^{k-1} \) be such that \( \binom{m}{2^j} \equiv 0 \) as justifiable by the fact that \( 2^{k-1} - 2 < m < 2^k - 2 \) and consider the manifolds \( RP(n_1 = 1, \ldots, n_d = 1, 2^j; 2^{k-1} - 2^j + 1) \) for which
\[
s = d \binom{m}{1} + \binom{m}{2^j} + (2^{k-1} - 2^j - d) \binom{m}{0} \equiv 1 \cdot 0 + 0 + 1 \cdot 1 \equiv 1.
\]
This qualifies since \( d \cdot 2 + 2^{j+1} + (2^{k-1} - 2^j - d) \cdot 1 = 2^{k-1} + 2^j + d \leq 2^k \).

\section*{5. Determination of ideals}

We now arrive at our main result.

\textbf{Proposition 5.1.} If \( 2^k > r \) then
\[
\mathcal{I}_{*\cdot k} = \begin{cases} 
(0), & r = 1, \\
\bigoplus_{n=r}^{\infty} \mathcal{N}_n, & r \text{ even}, \\
\bigoplus_{n=r}^{\infty} \mathcal{N}_n \cap \text{Ker } \chi, & r \text{ odd}, \ r \geq 3.
\end{cases}
\]

\textbf{Proof.} The case \( r = 1 \) is [P, 3.1].

For \( r \geq 2 \) Lemma 2.1 gives one inclusion; we wish to show the reverse. The first task is to show that \( \mathcal{I}_{*\cdot k} \) contains an indecomposable in each dimension \( d \geq r \). For \( r \leq d \leq 2^k \) this is handled by Corollary 3.4 and Lemma 4.3. For \( d > 2^k \) and either \( r \) even or \( d \) not of the form \( 2^u \), the indecomposable is provided by Corollary 3.2 and Lemma 4.1. For \( d > 2^k, r \) odd, and \( d \) of the form \( 2^u \) it is provided by Corollary 3.3 and Lemma 4.2.

An inductive argument shows \( \mathcal{I}_{*\cdot k} \) contains all specified decomposables, i.e. of dimension \( d \geq r \) and, for \( r \) odd, in \( \text{Ker } \chi \). The cases \( r = 2, 3 \) follow simply from the fact that \( \mathcal{I}_{*\cdot k} \) contains indecomposables of dimension \( d \geq r \) and is an ideal. Therefore assume that \( 2^k > r \) and \( \mathcal{I}_{*\cdot k} \) contains all specified decomposables for \( r_i < r \).

Suppose \( r \leq n \) and \( [M^n] = [M_1^n][M_2^n] \). We wish to find \( r_i \leq n_i, \ i = 1, 2, \) with \( r_1 + r_2 = r \), so as to exploit the inductive hypothesis and \( \mathcal{I}_{*\cdot k}, \mathcal{I}_{*\cdot k} \subseteq \mathcal{I}_{*\cdot k} \). If \( \chi([M^n]) = 0 \) then simply choose \( r_1 = n_1, r_2 = n_2 - (n - r) \). If \( r \) is even and \( \chi([M^n]) = 1 \), hence \( n_1, n_2 \) are even, the same assignment works because it makes \( r_1, r_2 \) even.

\textbf{Corollary 5.2.} \( \mathcal{I}^2_{*\cdot k} = \bigoplus_{n=2}^{\infty} \mathcal{N}_n \) for \( k \geq 2 \).

\textbf{Corollary 5.3.} \( \mathcal{I}^3_{*\cdot k} = \text{Ker } \chi \) for \( k \geq 2 \).

\textbf{References}


10614 FABLE ROW, COLUMBIA, MARYLAND 20144
Current address: Union Bank of Switzerland, 80 Raffles Place #36-00, UOB Plaza 1, Singapore 0104