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A CONVERSE TO STANLEY'S CONJECTURE FOR Sl_2

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ABSTRACT. We prove, in the case of Sl_2, a converse to Stanley's conjecture about Cohen-Macaulayness of invariant modules for reductive algebraic groups.

1. INTRODUCTION

Let $G = \mathrm{Sl}(V)$ where V is a two-dimensional vectorspace over an algebraically closed field k of characteristic zero. Define $W = \bigoplus_{i=1}^m S^{d_i}V$, $d = \dim W = \sum(d_i + 1)$, and $R = SW$, where SW denotes the symmetric algebra of W.

Define for $n \geq 0$

$$s^{(n)} = \begin{cases} n + (n-2) + \cdots + 1 = \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd}, \\ n + (n-2) + \cdots + 2 = \dfrac{n(n+2)}{4} & \text{if } n \text{ is even}, \end{cases}$$

and put $s = \sum_{i=1}^m s^{(d_i)}$.

It follows from a conjecture of Stanley [5] that $(R \otimes S^\mu V)^G$ is Cohen-Macaulay if $\mu < s - 2$. This conjecture was proved partially in [4], and in almost complete generality in [3].

In [1] Broer proved a partial converse to Stanley's conjecture for Sl_2. In this note we will prove a complete converse.

We may always drop all trivial irreducible components of W since the Cohen-Macaulayness of $(R \otimes S^\mu V)^G$ is not affected by them. Hence we assume that all $d_i > 0$. We separate the following cases:

(A) $W = V$, S^2V, $V \oplus V$, $V \oplus S^2V$, $S^2V \oplus S^2V$, S^3V, S^4V.
(B) All d_i are even and u is odd.
(C) All other cases.

In this note we will prove the following theorem.

Theorem 1.1. *In case* (A) $(R \otimes S^\mu V)^G$ *is always Cohen-Macaulay. In case* (B) $(R \otimes S^\mu V)^G = 0$. *In case* (C) *the converse to Stanley's conjecture is true.*

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It should be noted that, in connection with a possible converse to Stanley's conjecture, one cannot expect a nice, succinct statement. See, e.g., [2], and in particular Example 4.5, for the torus case.

Case (B) of Theorem 1.1 is easy to see by looking at the action of the center of G on $(R \otimes S^\mu V)^G$.

The representations listed in case (A) are the so-called "equidimensional" representations, i.e., those for which the quotient map $R \to R^G$ is equidimensional. It is well known that this implies that all $(R \otimes S^\mu V)^G$ are Cohen-Macaulay. One possible argument is given in the beginning of the next section.

The reader should note, however, that more is true. Namely, in case (A), R^G turns out to be always a polynomial ring. This is a special case of the "Russian conjecture" which remains open for general reductive groups. Hence in case (A) all $(R \otimes S^\mu V)^G$ are actually free.

2. The method

Keep the same notation as above. In the sequel $R = SW$ will be equipped with its natural $\mathbb{Z}$-grading. Let $I = R(R^G)^+$, $h = \dim R^G$. Recall from [3] that $(R \otimes S^\mu V)^G$ is Cohen-Macaulay if and only if $S^\mu V$ does not occur as a summand when $H^i_I(R)$ for $i = 0, \dots, h-1$ is decomposed as a sum of irreducible representation of G.

Let $X = \operatorname{Spec} R$. The radical of I is the defining ideal of the G-unstable locus in X, which will be denoted by X^u; that is,

$$X^u = \{x \in X \mid 0 \in \overline{Gx}\}.$$

In particular $H^i_I(R) = H^i_{X^u}(X, \mathscr{O}_X)$ and

$$H^i_{X^u}(X, \mathscr{O}_X) = 0 \quad \text{for } 0 < i < \operatorname{codim}(X_u, X). \tag{1}$$

Fix a basis for V and use this basis to identify $\mathrm{Sl}(V)$ with $\mathrm{Sl}_2(k)$. Let $z \mapsto \operatorname{diag}(z, z^{-1})$ be a one-parameter subgroup of G, and let

$$X_\lambda = \left\{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\right\}.$$

Then it follows from the Hilbert-Mumford criterion that $X^u = GX_\lambda$. Hence we have to compute $H^i_{GX_\lambda}(X, \mathscr{O}_X)$ for $0 \le i < h$. Let

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

be resp. a Borel subgroup and a maximal torus in G. Then B acts on X_λ and it is easy to verify that the standard map $G \times^B (X_\lambda - \{0\}) \to GX_\lambda - \{0\}$ is set-theoretically a bijection.

Hence $\dim X^u = 1 + \dim X_\lambda$. Therefore using (1) we find that if $1 + \dim X_\lambda + h \le d$ then all $(R \otimes S^\mu V)^G$ are Cohen-Macaulay. An easy verification shows that this is precisely the case for the representations in (A).

Having settled cases (A) and (B) we now concentrate on the proof of (C).

Let $[e] \in G/B$ be the class of the unit element. Taking the fiber over $[e]$ defines an equivalence between $\mathscr{O}_{G \times^B X}$-modules with a G-action and $\mathscr{O}_X$-modules with a B-action. The inverse of this functor will be denoted by $\sim$.

Assume that W is not V or S^2V. (These cases are included in (A).) In that case X has a G-stable point and hence $h = d - 3$. There is a long exact sequence

$$H^i_{\{0\}}(X, \mathscr{O}_X) \to H^i_{GX_\lambda}(X, \mathscr{O}_X) \to H^i_{GX_\lambda - \{0\}}(X - \{0\}, \mathscr{O}_X) \to H^{i+1}_{\{0\}}(X, \mathscr{O}_X).$$

But $H^{i(+1)}_{\{0\}}(X, \mathscr{O}_X) = 0$ if $i(+1) \neq d$. Hence it suffices to compute

$$H^i_{GX_\lambda - \{0\}}(X - \{0\}, \mathscr{O}_X) \quad \text{for } 0 \leq i < d - 3.$$

Using [4, Lemma 3.2], together with the definition of algebraic De Rham homology we obtain that

$$H^{i-2}_{GX_\lambda - \{0\}}(X - \{0\}, \mathscr{O}_X) = \mathbb{H}^i_{G\times(X_\lambda - \{0\})}(G \times^B X, \Omega^\bullet).$$

Here $\Omega^\bullet$ denotes the relative De Rham complex of $G \times^B X/X$ and $\mathbb{H}^*_?$ denotes hypercohomology with support. Hence we obtain a spectral sequence

$$E_1^{pq}: H^q_{G\times^B(X_\lambda - \{0\})}\left(G \times^B (X - \{0\}), \bigwedge^p \Omega\right) \Rightarrow H^{p+q-2}_{GX_\lambda - \{0\}}(X - \{0\}, \mathscr{O}_X).$$

First note that $E_1^{pq} = 0$ unless $p = 0, 1$. We will compute the terms in this spectral sequence under the hypothesis

$$p + q - 2 < d - 3. \tag{2}$$

There is a long exact sequence

$$\begin{aligned}\cdots \to H^q_{G\times^B\{0\}}\left(G \times^B X, \bigwedge^p \Omega\right) &\to H^q_{G\times^B X_\lambda}\left(G \times^B X, \bigwedge^p \Omega\right)\\ \to H^q_{G\times^B(X_\lambda - \{0\})}&\left(G \times^B (X - \{0\}), \bigwedge^p \Omega\right)\\ \to H^{q+1}_{G\times^B\{0\}}&\left(G \times^B X, \bigwedge^p \Omega\right) \to \cdots.\end{aligned}$$

But $H^{q(+1)}_{G\times^B\{0\}}(G \times^B X, \bigwedge^p \Omega) = 0$ unless $q(+1) \geq d$.

Hence under hypothesis (2)

$$E_1^{pq} = H^q_{G\times^B X_\lambda}\left(G \times^B X, \bigwedge^p \Omega\right).$$

We now employ the composite functor spectral sequence

$$E_2^{q'q''}: H^{q'}\left(\mathscr{H}^{q''}_{G\times^B X_\lambda}\left(G \times^B X, \bigwedge^p \Omega\right)\right) \Rightarrow H^{q'+q''}_{G\times^B X_\lambda}\left(G \times^B X, \bigwedge^p \Omega\right).$$

$G \times^B X_\lambda$ is a local complete intersection in $G \times^B X$ and hence

$$\mathscr{H}^{q''}_{G\times^B X_\lambda}\left(G \times^B X, \bigwedge^p \Omega\right) = 0$$

unless $q'' = d_\lambda$ where $d_\lambda = \operatorname{codim}(X_\lambda, X) = \sum_{i=1,\ldots,m} \lceil (d_i + 1)/2 \rceil$.

Furthermore

$$\mathscr{H}^{d_\lambda}_{G\times^B X_\lambda}\left(G \times^B X, \bigwedge^p \Omega\right) = H^{d_\lambda}_{X_\lambda}(X, \mathscr{O}_X)^\sim \otimes_{\mathscr{O}_{G/B}} \bigwedge^p \Omega_{G/B}.$$

Put $Z = H^{d_\lambda}_{X_\lambda}(X, \mathscr{O}_X)$. Then we obtain

$$E_1^{pq} = H^{q-d_\lambda}\left(G/B, \tilde{Z} \otimes \bigwedge^p \Omega_{G/B}\right).$$

Hence (still under hypothesis (2)) $E_1^{pq} = 0$ unless $q = d_\lambda$, $d_\lambda + 1$, and $p = 0, 1$. For simplicity we put

$$A_{i,j} = H^j\left(G/B, \tilde{Z} \otimes \bigwedge^i \Omega_{G/B}\right).$$

To estimate $A_{i,j}$ we define Z' to be the B-representation on which the unipotent part of B acts trivially but which has the same T-weights as Z. $A'_{i,j}$ will be defined as $A_{i,j}$ but with Z replaced by Z'.

Let χ: $\operatorname{diag}(z, z^{-1}) \mapsto z$ be the generator of $X(T)$ and let $(\chi^{u_i})_{i=1,\dots,d}$ be the T-weights of W. Then the T-weights of Z are [2]

$$\chi^{-\sum_{u_i \geq 0}(a_i+1)u_i + \sum_{u_i<0} b_i u_i} \tag{3}$$

where $(a_i)_i, (b_i)_i \in \mathbb{N}$, and such a weight occurs in degree $\sum b_i - \sum(a_i + 1)$. Now note that $G/B \cong \mathbb{P}^1$. We claim that $\tilde{\chi} = \mathscr{O}(-1)$, or equivalently $\chi = \mathscr{O}(-1)_e$ where e is the fixpoint for the B-action on $\mathbb{P}^1$. Then $\mathscr{O}(-1) = \mathscr{O}(-e)$, and hence $\mathscr{O}(-1)_e \cong m_e/m_e^2$ with m_e the maximal ideal of $\mathscr{O}_{\mathbb{P}^1, e}$. A local computation now shows what we want.

Lemma 2.1. $A_{i,1} = 0$.

Proof. Z is a rational representation of B and therefore we may construct a left limited ascending filtration on Z such that $\operatorname{gr} Z = Z'$. Hence it suffices to prove the lemma for $A'_{i,1}$. By the above we have to show that

$$\begin{aligned} H^1&\left(G/B, \mathscr{O}\left(\sum_{u_i \geq 0}(a_1 + 1)u_i - \sum_{u_i<0} b_i u_i - 2\right)\right) \\ &= H^0\left(G/B, \mathscr{O}\left(-\sum_{u_i \geq 0}(a_i + 1)u_1 + \sum_{u_i<0} b_i u_i\right)\right) = 0. \end{aligned}$$

It is clear that this is always the case. □

Lemma 2.2. (1) *The arrow from position* $(0, d_\lambda)$ *to position* $(1, d_\lambda)$ *in* E_1 *is injective.*

(2) *The position* $(1, d_\lambda)$ *lies strictly below the line* $p + q - 2 = d - 3$ *if and only if we are not in case* (A).

Proof. (1) This follows from $\operatorname{codim}(X^u, X) = d_\lambda - 1$ and hence $H^i_{X^u}(X, \mathscr{O}_X) = 0$ if $i < d_\lambda - 1$. If the arrow were not injective then $H^{d_\lambda - 2}_{X^u}(X, \mathscr{O}_X) \neq 0$.

(2) This is a simple verification. □

Assume that U is a $\mathbb{Z}$-graded G-representation. We define

$$P(U, x, t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \operatorname{Mult}_{S^r V}(U_s) x^r t^s.$$

In the sequel such an expression is supposed to define an element of $k((t^{-1}))[[x]]$. Let e be the number of even d_i's.

Lemma 2.3.

$$(4) \quad P(A_{0,0}, x, t) = \frac{t^{d_\lambda}}{(1-t^{-1})^e} x^s \frac{1}{\prod_{u_i>0}(1-x^{u_i}t^{-1}) \prod_{u_i<0}(1-x^{-u_i}t)},$$

$$(5) \quad P(A_{1,0}, x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})^e} x^{s-2} \frac{1}{\prod_{u_i>0}(1-x^{u_i}t^{-1}) \prod_{u_i<0}(1-x^{-u_i}t)}.$$

Proof. Since $A'_{0,1} = 0$ it is easy to see that $P(A_{0,0}, x, t) = P(A'_{0,0}, x, t)$. From (3) it follows that

$$P(A'_{0,0}, x, t) = \sum_{(a_i),(b_i)} x^{(\sum_{u_i\geq 0}(a_i+1)u_i - \sum_{u_i<0} b_i u_i)} t^{(\sum_{u_i<0} b^i - \sum_{u_i\geq 0}(a_i+1))}$$

which evaluates to the right-hand side of (4).

The proof for (5) is similar. □

We are now ready to prove the following theorem.

Theorem 2.4. *Assume that we are not in case* (A). *Then* $H_I^i(R) = 0$ *unless* $i = d_\lambda - 1$, $d-3$. *Furthermore*

$$P(H_i^{d_\lambda-1}(R), x, t) = \frac{t^{-d_\lambda}}{(1-t^{-1})^e} x^{s-2} \frac{1-x^2}{\prod_{u_i>0}(1-x^{u_i}t^{-1}) \prod_{u_i<0}(1-x^{-u_i}t)}.$$

Proof. That $H_I^i(R) = 0$ unless $i = d_\lambda - 1$, $d-3$ follows from Lemmas 2.1, 2.2. The statement about the Poincaré series follows from the fact that there is an exact sequence

$$0 \to A_{0,0} \to A_{1,0} \to H_I^{d_\lambda-1}(R) \to 0$$

and hence

$$P(H_I^{d_\lambda-1}(R), x, t) = P(A_{1,0}, x, t) - P(A_{0,0}, x, t).$$

We then apply Lemma 2.3. □

Proof of Theorem 1.1. It is easy to see that all powers of x appear in the expansion of

$$\frac{1-x^2}{\prod_{u_i>0}(1-x^{u_i}t^{-1}) \prod_{u_i<0}(1-x^{-u_i}t)}$$

unless all d_i are even. In that case all even powers appear. □

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