GENERALIZED WAVELET DECOMPOSITIONS OF BIVARIATE FUNCTIONS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. The objective of this paper is to introduce an integral transform of wavelet-type on $L^2(R^2)$ that can be applied to decompose the space $L^2(R^2)$ into a direct sum of subspaces, each of which is identified as $L^2(R)$. Projections from $L^2(R^2)$ onto these subspaces are also discussed. Moreover, wavelet expansions for functions in $L^2(R^2)$ are derived in terms of wavelet bases of $L^2(R)$.

1. Introduction

Many well-known integral transforms for time-frequency analysis and phase-space consideration take on the formulation

$$\int_{-\infty}^{\infty} \lambda(ty)g(t) \, dt,$$

where $\lambda$ is a function defined on $(-\infty, \infty)$. These certainly include the Fourier and Laplace transforms. In the analysis of nonstationary signals, for instance, it is sometimes necessary to replace $g(t)$ in (1.1) by functions which depend on more than one variable. In this paper, we study the extension of (1.1) to the formulation

$$\int_{-\infty}^{\infty} \lambda(ty)f(t,y) \, dt, \quad f \in L^2(R^2).$$

Here, the kernel $\lambda$ will always be assumed to be in $L^2(R)$. Under this assumption, it will be shown that $N_\lambda$ is a bounded linear operator from $L^2(R^2)$ onto $L^2(R)$.

On the other hand, associated with any function $\tilde{\lambda} \in L^2(R)$, we introduce another operator $M_{\tilde{\lambda}}$ defined by

$$M_{\tilde{\lambda}}g(x, y) := |y|^{1/2} \tilde{\lambda}(xy)g(y), \quad g \in L^2(R).$$
The importance of these two operators is that they together allow us to construct a "resolution of the identity" that provides a decomposition of the space $L^2(R^2)$ as well as a generalization of the integral wavelet transform. More precisely, let \( \{\lambda_n : n \in I\} \) be a Riesz basis of \( L^2(R) \) with dual basis \( \{\hat{\lambda}_n : n \in I\} \) for some countable index set \( I \). Here, as usual, duality means that
\[
\langle \lambda_m, \hat{\lambda}_n \rangle := \int_{-\infty}^{\infty} \lambda_m(x)\overline{\hat{\lambda}_n(x)} \, dx = \delta_{m,n}, \quad m, n \in I.
\]
We will then prove that
\[
f(x, y) = \sum_{n \in I} (M_n N_n f)(x, y), \quad f \in L^2(R^2),
\]
where, for simplicity, we have used the notation
\[
M_n := M_{\lambda_n}, \quad N_n := N_{\lambda_n}.
\]
As an example, let \( \psi \in L^2(R) \) be a wavelet with dual \( \hat{\psi} \) (cf. [1, p. 14]). That is, both
\[
x_{i \lambda}(x) := \sqrt{2^j} \xi / \sqrt{2^j},
\]
and
\[
\psi_{j, k}(x) := \sqrt{2^j} \psi(2^j x - k), \quad j, k \in Z,
\]
are Riesz bases of \( L^2(R) \) such that
\[
\langle \psi_{j, k}, \psi_{j', k'} \rangle = \delta_{j, j'}\delta_{k, k'}, \quad j, j', k, k' \in Z.
\]
Then by considering \( I = Z \times Z \) and
\[
\{\lambda_n\}_{n \in I} = \{\psi_{j, k}\}_{j, k \in I}, \quad \{\hat{\lambda}_n\}_{n \in I} = \{\psi_{j, k}\}_{j, k \in I},
\]
the resolution of the identity in (1.4) becomes
\[
f(x, y) = \sum_{j, k \in Z} |y| \psi_{j, k}(xy) \int_{-\infty}^{\infty} \psi_{j, k}(sy) f(s, y) \, ds
\]
\[
= \sum_{j, k \in Z} \psi_{j, k}(xy) \int_{-\infty}^{\infty} \psi_{j, k}(t) f(t/y, y) \, dt
\]
\[
= \sum_{j, k \in Z} \psi_{j, k}(xy)^{2j/2} \int_{-\infty}^{\infty} (2jt - k) f(t/y, y) \, dt.
\]
Here, it is noted that
\[
(N_{j, k} f)(y) := (N_{j, k} f)(y) = 2^{j/2} |y|^{-1/2} \int_{-\infty}^{\infty} \psi(2jt - k) f(t/y, y) \, dt
\]
and
\[
(M_{j, k} g)(x, y) := (M_{j, k} g)(x, y) = 2^{j/2} |y|^{1/2} \hat{\psi}(2^j xy - k) g(y).
\]
Of course, the formulation (1.6) extends the notion of the discretized integral wavelet transform (with \( y = 1 \)) to the upper half-plane (cf. [3, 1, 2]).

To discuss the decomposition of \( L^2(R^2) \), we consider the range
\[
S_\lambda := \{M_{j} g : g \in L^2(R)\}
\]
of the operator \( M_{\lambda} \), \( \lambda \in L^2(R) \). It will be shown in the next section that \( S_\lambda \) is isometrically isomorphic to \( L^2(R) \). The main result in this paper is
Theorem 1. Suppose that \{\lambda_n : n \in I\} is a Riesz basis of \(L^2(R)\) with dual basis \{\check{\lambda}_n : n \in I\}. Then \(L^2(R^2)\) has the direct-sum decomposition

\[
L^2(R^2) = \bigoplus_{n \in I} S_n,
\]

where \(S_n := S_{\lambda_n}\) as defined in (1.8). Moreover, if \(\{\lambda_n : n \in I\}\) is an orthonormal basis of \(L^2(R)\) so that \(\check{\lambda}_n = \lambda_n\), then \(P_n := M_n N_n\) is the orthogonal projection from \(L^2(R^2)\) onto \(S_n\), and (1.9) becomes an orthogonal decomposition; namely, \(S_n \perp S_m, \ n \neq m\).

The decomposition of functions of two variables into a sum of functions of one variable was also considered in Jiang and Peng [5], again motivated by the integral wavelet transform and the results in Paul [6].

2. RESULTS ON INTEGRAL TRANSFORMS OF WAVELET-TYPE

Before showing that the integral transform \(N_\lambda\) in (1.2) is a bounded linear operator from \(L^2(R^2)\) onto \(L^2(R)\), we make the following simple observation: For any \(\lambda \in L^2(R)\),

\[
\|M_\lambda g\|_{L^2(R^2)} = \|\lambda\|_{L^2(R)} \|g\|_{L^2(R)}, \quad g \in L^2(R).
\]

Therefore, \(M_\lambda\) is a linear operator from \(L^2(R)\) to \(L^2(R^2)\), and the range \(S_\lambda\) of \(M_\lambda\), as defined in (1.8), is a closed subspace of \(L^2(R^2)\). In particular, if \(\|\lambda\|_{L^2(R)} = 1\), then \(S_\lambda\) is isometrically isomorphic to \(L^2(R)\). Furthermore, for any \(\lambda, \mu \in L^2(R)\), the identity

\[
(N_\lambda M_\mu g)(y) = \langle \mu, \lambda \rangle_{L^2(R)} g(y), \quad g \in L^2(R),
\]

can be easily verified.

Lemma 1. For any \(\lambda \in L^2(R)\), the operator \(N_\lambda\) is a bounded linear operator from \(L^2(R^2)\) onto \(L^2(R)\) with \(\|N_\lambda\| = \|\lambda\|_{L^2(R)}\).

Proof. For any \(f \in L^2(R^2)\),

\[
\|N_\lambda f\|_{L^2(R)}^2 = \int_{-\infty}^{\infty} dy \left| \int_{-\infty}^{\infty} \left| y \right|^{1/2} \overline{\lambda(ty)} f(t, y) dt \right|^2
\]

\[
\leq \int_{-\infty}^{\infty} dy \left( \left| y \right| \int_{-\infty}^{\infty} |\lambda(ty)|^2 dt \right) \int_{-\infty}^{\infty} |f(t, y)|^2 dt
\]

\[
= \|f\|_{L^2(R^2)} \|\lambda\|_{L^2(R)}^2.
\]

By taking \(\lambda = \mu\) in (2.2), we have

\[
(N_\lambda M_\lambda g)(y) = \|\lambda\|_{L^2(R)}^2 g(y)
\]

for any \(g \in L^2(R)\). Thus, this lemma is established. □

Let \(I\) be a countable index set. Suppose that a Riesz basis \(\{\lambda_n : n \in I\}\) of \(L^2(R)\) with dual basis \(\{\check{\lambda}_n : n \in I\}\) is given. Assume that \(\{\check{\lambda}_n : n \in I\}\) has Riesz bounds \(A\) and \(B\); i.e.,

\[
A \|\{c_n\}\|_I^2 \leq \left\| \sum_{n \in I} c_n \check{\lambda}_n \right\|_{L^2(R)}^2 \leq B \|\{c_n\}\|_I^2
\]
for any \( \{c_n\} \in l_2^2 \). It then follows from the expansion
\[
h(x) = \sum_{n \in \mathbb{I}} (h, \lambda_n) \hat{\lambda}_n(x) = \sum_{n \in \mathbb{I}} \hat{\lambda}_n(x) \int_{-\infty}^{\infty} \overline{\lambda_n(s)} h(s) \, ds, \quad h \in L^2(\mathbb{R}),
\]
that
\[
A \sum_{n \in \mathbb{I}} |(h, \lambda_n)|^2 \leq \|h\|_{L^2(\mathbb{R})}^2 \leq B \sum_{n \in \mathbb{I}} |(h, \lambda_n)|^2.
\]

Lemma 2. Let \( \{\lambda_n : n \in \mathbb{I}\} \) be a Riesz basis of \( L^2(\mathbb{R}) \) with dual basis \( \{\hat{\lambda}_n : n \in \mathbb{I}\} \). Then
\[
f(x, y) = \sum_{n \in \mathbb{I}} (M_n N_n f)(x, y), \quad f \in L^2(\mathbb{R}^2),
\]
where the notation in (1.5) is used and the convergence is in \( L^2(\mathbb{R}^2) \).

Proof. Suppose that \( h \in L^2(\mathbb{R}) \) is given. For any \( z \neq 0 \), let \( l(x) = h(x/z) \). Then
\[
h(x) = l(xz) = \sum_{n \in \mathbb{I}} \hat{\lambda}_n(xz) \int_{-\infty}^{\infty} \overline{\lambda_n(t)} l(t) \, dt
\]
\[
= \sum_{n \in \mathbb{I}} \hat{\lambda}_n(xz) \int_{-\infty}^{\infty} \overline{\lambda_n(t)} h(t/z) \, dt
\]
\[
= \sum_{n \in \mathbb{I}} \hat{\lambda}_n(xz) |z| \int_{-\infty}^{\infty} \overline{\lambda_n(sz)} h(s) \, ds.
\]
For any \( f \in L^2(\mathbb{R}^2) \), we have \( f(\cdot, y) \in L^2(\mathbb{R}) \) for almost all \( y \), so that
\[
f(x, y) = \sum_{n \in \mathbb{I}} \hat{\lambda}_n(xz) |z| \int_{-\infty}^{\infty} \overline{\lambda_n(sz)} f(s, y) \, ds,
\]
which becomes (2.5) by taking \( z = y \). To complete the proof of the lemma, we need to prove that the convergence is in \( L^2(\mathbb{R}^2) \). So, let us consider an increasing nested sequence of finite subsets \( I_k \) of \( \mathbb{I} \) with \( \bigcup I_k = \mathbb{I} \), and consider
\[
g_k(y) := \left( \int_{-\infty}^{\infty} \left[ f(x, y) - \sum_{n \in I_k} (M_n N_n f)(x, y) \right]^2 \, dx \right)^{1/2},
\]
from (2.6), by taking \( z = y \), we have
\[
\lim_{k \to \infty} g_k(y) = 0 \quad \text{a.e.}
\]
Write
\[
g_k(y) = \int_{-\infty}^{\infty} \left| f(x, y) - \sum_{n \in I_k} \hat{\lambda}_n(xy) |y| \int_{-\infty}^{\infty} \overline{\lambda_n(sy)} f(s, y) \, ds \right|^2 \, dx
\]
\[
= \int_{-\infty}^{\infty} \left| f(x, y) - \sum_{n \in I_k} \hat{\lambda}_n(xy) \int_{-\infty}^{\infty} \overline{\lambda_n(t)} f(t/y, y) \, dt \right|^2 \, dx.
\]
Suppose that $A$ and $B$ are Riesz bounds of $\{\hat{\lambda}_n : n \in I\}$ as described in (2.3). Then, it follows from the Cauchy-Schwarz inequality, (2.3), and (2.4) that

\[
\left( g_k(y) \right)^{1/2} \leq \left( \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \right)^{1/2} + \left( \frac{1}{|y|} \int_{-\infty}^{\infty} \left| \sum_{|n| \leq k} \left( f\left( \frac{\cdot}{y}, y \right), \lambda_n(\cdot) \right) \hat{\lambda}_n(s) \right|^2 \, ds \right)^{1/2}
\]

\[
\leq \left( \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \right)^{1/2} + \left( B \right)^{1/2} \left( \frac{1}{|y|} \int_{-\infty}^{\infty} \left| f\left( \frac{x}{y}, y \right) \right|^2 \, dx \right)^{1/2},
\]

or equivalently

\[
g_k(y) \leq (1 + (B/A)^{1/2})^2 \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx.
\]

Hence, with (2.8) and (2.9), the Dominated Convergence Theorem applies, yielding

\[
\lim_{k \to \infty} \int_{-\infty}^{\infty} g_k(y) \, dy = 0.
\]

Since this holds for any $f \in L^2(\mathbb{R}^2)$, the convergence in (2.5) is in the $L^2(\mathbb{R}^2)$, completing the proof of the lemma. 

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** For any $f \in L^2(\mathbb{R}^2)$, we have, by Lemma 1, $N_n f \in L^2(\mathbb{R})$ so that $M_n N_n f \in S_n$ for any $n \in I$. Therefore, it follows from Lemma 2 that $f \in \sum_{n \in I} S_n$, or $L^2(\mathbb{R}^2) = \sum_{n \in I} S_n$. This decomposition of $L^2(\mathbb{R}^2)$ is actually a direct-sum decomposition. Indeed, if there are two sequences $\{g_n\}, \{h_n\} \subset L^2(\mathbb{R})$ such that

\[
\sum_{n \in I} M_n g_n = \sum_{n \in I} M_n h_n,
\]

or equivalently

\[
\sum_{n \in I} |y|^{1/2} \hat{\lambda}_n(xy) g_n(y) = \sum_{n \in I} |y|^{1/2} \hat{\lambda}_n(xy) h_n(y),
\]

then we have

\[
\sum_{n \in I} \hat{\lambda}_n(xy) \hat{\lambda}_n(xy) g_n(y) = \sum_{n \in I} \hat{\lambda}_n(xy) \hat{\lambda}_n(xy) h_n(y), \quad m \in I.
\]

Integrating both sides of the above identity with respect to $x$ yields $g_m = h_m$ for all $m \in I$. Hence, the first part of the theorem is proved.

Now if $\{\lambda_n : n \in I\}$ is an orthonormal basis, then it is self-dual. For any $h \in S_n$, then $h = M_n g$, for some $g \in L^2(\mathbb{R})$, and according to (2.2), we have

\[
P_n h = M_n N_n (M_n g) = M_n (N_n M_n g) = (\lambda_n, \hat{\lambda}_n) M_n g = h.
\]
On the other hand, for any \( f \) orthogonal to \( S_n \) and \( \beta \in L^2(R^2) \), we have
\[
\langle M_nN_nf, \beta \rangle_{L^2(R^2)} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \left( |y|\lambda_n(xy) \int_{-\infty}^{\infty} \frac{\lambda_n(sy)f(s, y)}{\int_{-\infty}^{\infty} \lambda_n(sy)s} \right) \beta(x, y) dx
\]
\[
= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(s, y) \left( |y|\lambda_n(sy) \int_{-\infty}^{\infty} \frac{\lambda_n(xy)\beta(x, y)}{\int_{-\infty}^{\infty} \lambda_n(sy)s} \right) ds
\]
\[
= \langle f, M_nN_n\beta \rangle_{L^2(R^2)} = 0.
\]

Hence, \( P_n = M_nN_n \) is the orthogonal projection from \( L^2(R^2) \) onto \( S_n \). Finally, for any \( h_1 \in S_m, h_2 \in S_n, m \neq n \), we have \( h_1 = S_mg_1, h_2 = S_ng_2 \), for some \( g_1, g_2 \in L^2(R) \), and
\[
\langle h_1, h_2 \rangle_{L^2(R^2)} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} (M_ng_1)(x, y)(M_mg_2)(x, y) dx
\]
\[
= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |y|\lambda_n(xy)\lambda_m(xy)g_1(y)g_2(y) dy
\]
\[
= \int_{-\infty}^{\infty} g_1(y)g_2(y) dy \int_{-\infty}^{\infty} \lambda_n(x)\lambda_m(x) dx = 0.
\]
This completes the proof of the theorem. \( \Box \)

More generally, even if \( \{\lambda_n : n \in I\} \) is not orthonormal but only a Riesz basis of \( L^2(R) \) with dual \( \{\hat{\lambda}_n : n \in I\} \), it is easy to verify that the operator \( P_n = M_nN_n \) is an affine projection from \( L^2(R^2) \) onto \( S_n \) in the sense that \( P_nf = f \) for any \( f \in S_n \), and \( P_nf = 0 \) for all \( f \in S_m \), where \( m \neq n \).

We end this paper by making two remarks.

1. The decomposition result in Theorem 1 can be applied in signal analysis to decompose a nonstationary signal \( f(w, t) \), where \( w \) is the frequency variable. By selecting an appropriate Riesz basis \( \{\lambda_n : n \in I\} \) with dual \( \{\hat{\lambda}_n : n \in I\} \), the signal \( f(w, t) \) is mapped to a sequence of stationary signals
\[
g_n(t) := (N_nf)(t) \in L^2(R), \quad n \in I,
\]
with certain desirable properties. Of course, the original signal can be reconstructed from \( \{g_n : n \in I\} \) by using the identity
\[
f(w, t) = \sum_{n \in I} (M_ng_n)(w, t).
\]

The schematic diagram for this decomposition and reconstruction procedure is shown in the following figure, where \( I = \{n_i : i \in Z\} \):

In practice, the components \( g_n \) of \( f \) are "filtered" by certain operators \( H_{n_i} \). For instance, Hankel-type operators \( H_{n_i} \) can be used for problems in
systems theory such as systems reduction and identification. Of course, the reconstruction operators $M_{n}$ are then applied to $H_{n}g_{n}$ instead of $g_{n}$.

(2) It is also worth mentioning that nonstationary signals are obtained by applying the Zak transform to stationary signals, namely,

$$f(w, t) = (Zg)(t, w) := a^{1/2} \sum_{k \in \mathbb{Z}} g(ta + ka)e^{2\pi i kw}, \quad g \in L^{2}(R).$$

This transformation was introduced by Zak [7] in the solid state physics literature. It can easily be verified that the Zak transform is a unitary map from $L^{2}(R)$ into $L^{2}([0, 1] \times [0, 1))$. For more details, the interested reader is referred to [7, 4].

REFERENCES


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