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GENERALIZED WAVELET DECOMPOSITIONS OF BIVARIATE FUNCTIONS

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ABSTRACT. The objective of this paper is to introduce an integral transform of wavelet-type on $L^2(R^2)$ that can be applied to decompose the space $L^2(R^2)$ into a direct sum of subspaces, each of which is identified as $L^2(R)$. Projections from $L^2(R^2)$ onto these subspaces are also discussed. Moreover, wavelet expansions for functions in $L^2(R^2)$ are derived in terms of wavelet bases of $L^2(R)$.

1. INTRODUCTION

Many well-known integral transforms for time-frequency analysis and phase-space consideration take on the formulation

$$\int_{-\infty}^{\infty} \overline{\lambda(ty)} g(t)\, dt, \tag{1.1}$$

where λ is a function defined on $(-\infty, \infty)$. These certainly include the Fourier and Laplace transforms. In the analysis of nonstationary signals, for instance, it is sometimes necessary to replace $g(t)$ in (1.1) by functions which depend on more than one variable. In this paper, we study the extension of (1.1) to the formulation

$$(N_\lambda f)(y) := |y|^{1/2} \int_{-\infty}^{\infty} \overline{\lambda(ty)} f(t, y)\, dt, \qquad f \in L^2(R^2). \tag{1.2}$$

Here, the kernel λ will always be assumed to be in $L^2(R)$. Under this assumption, it will be shown that N_λ is a bounded linear operator from $L^2(R^2)$ onto $L^2(R)$.

On the other hand, associated with any function $\tilde{\lambda} \in L^2(R)$, we introduce another operator $M_{\tilde{\lambda}}$ defined by

$$(M_{\tilde{\lambda}} g)(x, y) := |y|^{1/2} \tilde{\lambda}(xy) g(y), \qquad g \in L^2(R). \tag{1.3}$$

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The importance of these two operators is that they together allow us to construct a "resolution of the identity" that provides a decomposition of the space $L^2(R^2)$ as well as a generalization of the integral wavelet transform. More precisely, let $\{\lambda_n : n \in I\}$ be a Riesz basis of $L^2(R)$ with dual basis $\{\tilde{\lambda}_n : n \in I\}$ for some countable index set I. Here, as usual, duality means that

$$\langle \lambda_m , \tilde{\lambda}_n \rangle := \int_{-\infty}^{\infty} \lambda_m(x)\overline{\tilde{\lambda}_n(x)}\, dx = \delta_{m,n}, \qquad m, n \in I.$$

We will then prove that

$$f(x, y) = \sum_{n\in I} (M_n N_n f)(x, y), \qquad f \in L^2(R^2), \tag{1.4}$$

where, for simplicity, we have used the notation

$$M_n := M_{\tilde{\lambda}_n}, \qquad N_n := N_{\lambda_n}. \tag{1.5}$$

As an example, let $\psi \in L^2(R)$ be a wavelet with dual $\tilde{\psi}$ (cf. [1, p. 14]). That is, both

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k),$$

and

$$\tilde{\psi}_{j,k}(x) := 2^{j/2}\tilde{\psi}(2^j x - k), \qquad j, k \in Z,$$

are Riesz bases of $L^2(R)$ such that

$$\langle \psi_{j,k} , \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'}\delta_{k,k'}, \qquad j, j', k, k' \in Z.$$

Then by considering $I = Z \times Z$ and

$$\{\lambda_n\}_{n\in I} = \{\psi_{j,k}\}_{j,k\in I}, \qquad \{\tilde{\lambda}_n\}_{n\in I} = \{\tilde{\psi}_{j,k}\}_{j,k\in I},$$

the resolution of the identity in (1.4) becomes

$$\begin{aligned} f(x, y) &= \sum_{j,k\in Z} |y|\tilde{\psi}_{j,k}(xy) \int_{-\infty}^{\infty} \overline{\psi_{j,k}(sy)} f(s, y)\, ds \\ &= \sum_{j,k\in Z} \tilde{\psi}_{j,k}(xy) \int_{-\infty}^{\infty} \overline{\psi_{j,k}(t)} f(t/y, y)\, dt \\ &= \sum_{j,k\in Z} \tilde{\psi}_{j,k}(xy) 2^{j/2} \int_{-\infty}^{\infty} \overline{\psi(2^j t - k)} f(t/y, y)\, dt. \end{aligned}$$

Here, it is noted that

$$(N_{j,k} f)(y) := (N_{\psi_{j,k}} f)(y) = 2^{j/2}|y|^{-1/2} \int_{-\infty}^{\infty} \overline{\psi(2^j t - k)} f(t/y, y)\, dt \tag{1.6}$$

and

$$(M_{j,k} g)(x, y) := (M_{\tilde{\psi}_{j,k}} g)(x, y) = 2^{j/2}|y|^{1/2}\tilde{\psi}(2^j xy - k) g(y). \tag{1.7}$$

Of course, the formulation (1.6) extends the notion of the discretized integral wavelet transform (with $y = 1$) to the upper half-plane (cf. [3, 1, 2]).

To discuss the decomposition of $L^2(R^2)$, we consider the range

$$S_{\tilde{\lambda}} := \{M_{\tilde{\lambda}} g : g \in L^2(R)\} \tag{1.8}$$

of the operator $M_{\tilde{\lambda}}$, $\tilde{\lambda} \in L^2(R)$. It will be shown in the next section that $S_{\tilde{\lambda}}$ is isometrically isomorphic to $L^2(R)$. The main result in this paper is

Theorem 1. *Suppose that* $\{\lambda_n : n \in I\}$ *is a Riesz basis of* $L^2(R)$ *with dual basis* $\{\tilde{\lambda}_n : n \in I\}$. *Then* $L^2(R^2)$ *has the direct-sum decomposition*

$$L^2(R^2) = \bigoplus_{n\in I} S_n, \tag{1.9}$$

where $S_n := S_{\tilde{\lambda}_n}$ *as defined in* (1.8). *Moreover, if* $\{\lambda_n : n \in I\}$ *is an orthonormal basis of* $L^2(R)$ *so that* $\tilde{\lambda}_n = \lambda_n$, *then* $P_n := M_n N_n$ *is the orthogonal projection from* $L^2(R^2)$ *onto* S_n, *and* (1.9) *becomes an orthogonal decomposition*; *namely,* $S_n \perp S_m$, $n \neq m$.

The decomposition of functions of two variables into a sum of functions of one variable was also considered in Jiang and Peng [5], again motivated by the integral wavelet transform and the results in Paul [6].

2. Results on integral transforms of wavelet-type

Before showing that the integral transform N_λ in (1.2) is a bounded linear operator from $L^2(R^2)$ onto $L^2(R)$, we make the following simple observation: For any $\tilde{\lambda} \in L^2(R)$,

$$\|M_{\tilde{\lambda}} g\|_{L^2(R^2)} = \|\tilde{\lambda}\|_{L^2(R)} \|g\|_{L^2(R)}, \qquad g \in L^2(R). \tag{2.1}$$

Therefore, $M_{\tilde{\lambda}}$ is a linear operator from $L^2(R)$ to $L^2(R^2)$, and the range $S_{\tilde{\lambda}}$ of $M_{\tilde{\lambda}}$, as defined in (1.8), is a closed subspace of $L^2(R^2)$. In particular, if $\|\tilde{\lambda}\|_{L^2(R)} = 1$, then $S_{\tilde{\lambda}}$ is isometrically isomorphic to $L^2(R)$. Furthermore, for any λ, $\mu \in L^2(R)$, the identity

$$(N_\lambda M_\mu g)(y) = \langle \mu, \lambda \rangle_{L^2(R)} g(y), \qquad g \in L^2(R), \tag{2.2}$$

can be easily verified.

Lemma 1. *For any* $\lambda \in L^2(R)$, *the operator* N_λ *is a bounded linear operator from* $L^2(R^2)$ *onto* $L^2(R)$ *with* $\|N_\lambda\| = \|\lambda\|_{L^2(R)}$.

Proof. For any $f \in L^2(R^2)$,

$$\begin{aligned}
\|N_\lambda f\|^2_{L^2(R)} &= \int_{-\infty}^{\infty} dy \left| \int_{-\infty}^{\infty} |y|^{1/2} \overline{\lambda(ty)} f(t, y)\, dt \right|^2 \\
&\leq \int_{-\infty}^{\infty} dy \left(|y| \int_{-\infty}^{\infty} |\lambda(ty)|^2\, dt \right) \int_{-\infty}^{\infty} |f(t, y)|^2\, dt \\
&= \|f\|^2_{L^2(R^2)} \|\lambda\|^2_{L^2(R)}.
\end{aligned}$$

By taking $\lambda = \mu$ in (2.2), we have

$$(N_\lambda M_\lambda g)(y) = \|\lambda\|^2_{L^2(R)} g(y)$$

for any $g \in L^2(R)$. Thus, this lemma is established. □

Let I be a countable index set. Suppose that a Riesz basis $\{\lambda_n : n \in I\}$ of $L^2(R)$ with dual basis $\{\tilde{\lambda}_n : n \in I\}$ is given. Assume that $\{\tilde{\lambda}_n : n \in I\}$ has Riesz bounds A and B; i.e.,

$$A \|\{c_n\}\|^2_{l^2_I} \leq \left\| \sum_{n\in I} c_n \tilde{\lambda}_n \right\|^2_{L^2(R)} \leq B \|\{c_n\}\|^2_{l^2_I} \tag{2.3}$$

for any $\{c_n\} \in l_I^2$. It then follows from the expansion

$$h(x) = \sum_{n\in I} \langle h, \lambda_n\rangle \tilde{\lambda}_n(x) = \sum_{n\in I} \tilde{\lambda}_n(x) \int_{-\infty}^{\infty} \overline{\lambda_n(s)} h(s)\, ds, \qquad h \in L^2(R),$$

that

$$A \sum_{n\in I} |\langle h, \lambda_n\rangle|^2 \le \|h\|^2_{L^2(R)} \le B \sum_{n\in I} |\langle h, \lambda_n\rangle|^2. \tag{2.4}$$

Lemma 2. *Let $\{\lambda_n : n \in I\}$ be a Riesz basis of $L^2(R)$ with dual basis $\{\tilde{\lambda}_n : n \in I\}$. Then*

$$f(x, y) = \sum_{n\in I} (M_n N_n f)(x, y), \qquad f \in L^2(R^2), \tag{2.5}$$

where the notation in (1.5) *is used and the convergence is in $L^2(R^2)$.*

Proof. Suppose that $h \in L^2(R)$ is given. For any $z \neq 0$, let $l(x) = h(x/z)$. Then

$$\begin{aligned} h(x) = l(xz) &= \sum_{n\in I} \tilde{\lambda}_n(xz) \int_{-\infty}^{\infty} \overline{\lambda_n(t)} l(t)\, dt \\ &= \sum_{n\in I} \tilde{\lambda}_n(xz) \int_{-\infty}^{\infty} \overline{\lambda_n(t)} h(t/z)\, dt \\ &= \sum_{n\in I} \tilde{\lambda}_n(xz)|z| \int_{-\infty}^{\infty} \overline{\lambda_n(sz)} h(s)\, ds. \end{aligned} \tag{2.6}$$

For any $f \in L^2(R^2)$, we have $f(\cdot, y) \in L^2(R)$ for almost all y, so that

$$f(x, y) = \sum_{n\in I} \tilde{\lambda}_n(xz)|z| \int_{-\infty}^{\infty} \overline{\lambda_n(sz)} f(s, y)\, ds, \tag{2.7}$$

which becomes (2.5) by taking $z = y$. To complete the proof of the lemma, we need to prove that the convergence is in $L^2(R^2)$. So, let us consider an increasing nested sequence of finite subsets I_k of I with $\bigcup I_k = I$, and consider

$$g_k(y) := \int_{-\infty}^{\infty} \left| f(x, y) - \sum_{n\in I_k} (M_n N_n f)(x, y) \right|^2 dx, \qquad y \neq 0.$$

From (2.6), by taking $z = y$, we have

$$\lim_{k\to\infty} g_k(y) = 0 \quad \text{a.e.} \tag{2.8}$$

Write

$$\begin{aligned} g_k(y) &= \int_{-\infty}^{\infty} \left| f(x, y) - \sum_{n\in I_k} \tilde{\lambda}_n(xy)|y| \int_{-\infty}^{\infty} \overline{\lambda_n(sy)} f(s, y)\, ds \right|^2 dx \\ &= \int_{-\infty}^{\infty} \left| f(x, y) - \sum_{n\in I_k} \tilde{\lambda}_n(xy) \int_{-\infty}^{\infty} \overline{\lambda_n(t)} f(t/y, y)\, dt \right|^2 dx. \end{aligned}$$

Suppose that A and B are Riesz bounds of $\{\tilde{\lambda}_n : n \in I\}$ as described in (2.3). Then, it follows from the Cauchy-Schwarz inequality, (2.3), and (2.4) that

$$\begin{aligned}(g_k(y))^{1/2} &\le \left(\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\right)^{1/2} \\ &\quad + \left(\frac{1}{|y|}\int_{-\infty}^{\infty}\left|\sum_{|n|\le k}\left(f\left(\frac{\cdot}{y},y\right),\lambda_n(\cdot)\right)\tilde{\lambda}_n(s)\right|^2 ds\right)^{1/2} \\ &\le \left(\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\right)^{1/2} + \left|\frac{B}{y}\right|^{1/2}\left(\sum_{|n|\le k}\left|\left(f\left(\frac{\cdot}{y},y\right),\lambda_n(\cdot)\right)\right|^2\right)^{1/2} \\ &\le \left(\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\right)^{1/2} + \left(\frac{B}{A}\right)^{1/2}\left(\frac{1}{|y|}\int_{-\infty}^{\infty}\left|f\left(\frac{x}{y},y\right)\right|^2 dx\right)^{1/2},\end{aligned}$$

or equivalently

$$g_k(y) \le (1+(B/A)^{1/2})^2 \int_{-\infty}^{\infty} |f(x,y)|^2\,dx. \tag{2.9}$$

Hence, with (2.8) and (2.9), the Dominated Convergence Theorem applies, yielding

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} g_k(y)\,dy = 0.$$

Since this holds for any $f \in L^2(R^2)$, the convergence in (2.5) is in the $L^2(R^2)$, completing the proof of the lemma. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. For any $f \in L^2(R^2)$, we have, by Lemma 1, $N_n f \in L^2(R)$ so that $M_n N_n f \in S_n$ for any $n \in I$. Therefore, it follows from Lemma 2 that $f \in \sum_{n\in I} S_n$, or $L^2(R^2) = \sum_{n\in I} S_n$. This decomposition of $L^2(R^2)$ is actually a direct-sum decomposition. Indeed, if there are two sequences $\{g_n\}$, $\{h_n\} \subset L^2(R)$ such that

$$\sum_{n\in I} M_n g_n = \sum_{n\in I} M_n h_n,$$

or equivalently

$$\sum_{n\in I} |y|^{1/2}\tilde{\lambda}_n(xy) g_n(y) = \sum_{n\in I} |y|^{1/2}\tilde{\lambda}_n(xy) h_n(y),$$

then we have

$$\sum_{n\in I} \overline{\lambda_m(xy)}\tilde{\lambda}_n(xy) g_n(y) = \sum_{n\in I} \overline{\lambda_m(xy)}\tilde{\lambda}_n(xy) h_n(y), \qquad m \in I.$$

Integrating both sides of the above identity with respect to x yields $g_m = h_m$ for all $m \in I$. Hence, the first part of the theorem is proved.

Now if $\{\lambda_n : n \in I\}$ is an orthonormal basis, then it is self-dual. For any $h \in S_n$, then $h = M_n g$, for some $g \in L^2(R)$, and according to (2.2), we have

$$P_n h = M_n N_n(M_n g) = M_n(N_n M_n g) = \langle \lambda_n, \lambda_n\rangle M_n g = h.$$

On the other hand, for any f orthogonal to S_n and $\beta \in L^2(R^2)$, we have

$$\begin{aligned}
&\langle M_n N_n f, \beta\rangle_{L^2(R^2)} \\
&\quad = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \left(|y|\lambda_n(xy) \int_{-\infty}^{\infty} \overline{\lambda_n(sy)} f(s, y)\, ds\right) \overline{\beta(x, y)}\, dx \\
&\quad = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(s, y) \overline{\left(|y|\lambda_n(sy) \int_{-\infty}^{\infty} \overline{\lambda_n(xy)} \beta(x, y)\, dx\right)}\, ds \\
&\quad = \langle f, M_n N_n \beta\rangle_{L^2(R^2)} = 0.
\end{aligned}$$

Hence, $P_n = M_n N_n$ is the orthogonal projection from $L^2(R^2)$ onto S_n. Finally, for any $h_1 \in S_m$, $h_2 \in S_n$, $m \neq n$, we have $h_1 = S_m g_1$, $h_2 = S_n g_2$, for some $g_1, g_2 \in L^2(R)$, and

$$\begin{aligned}
\langle h_1, h_2\rangle_{L^2(R^2)} &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} (M_n g_1)(x, y) \overline{(M_m g_2)(x, y)}\, dx \\
&= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |y| \lambda_n(xy) \overline{\lambda_m(xy)} g_1(y) \overline{g_2(y)}\, dy \\
&= \int_{-\infty}^{\infty} g_1(y) \overline{g_2(y)}\, dy \int_{-\infty}^{\infty} \lambda_n(x) \overline{\lambda_m(x)}\, dx = 0.
\end{aligned}$$

This completes the proof of the theorem. □

More generally, even if $\{\lambda_n : n \in I\}$ is not orthonormal but only a Riesz basis of $L^2(R)$ with dual $\{\tilde{\lambda}_n : n \in I\}$, it is easy to verify that the operator $P_n = M_n N_n$ is an affine projection from $L^2(R^2)$ onto S_n in the sense that $P_n f = f$ for any $f \in S_n$, and $P_n f = 0$ for all $f \in S_m$, where $m \neq n$.

We end this paper by making two remarks.

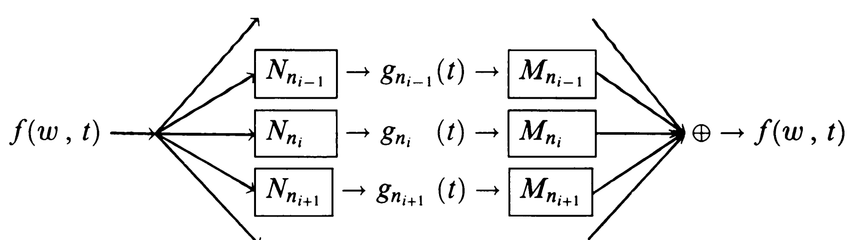
(1) The decomposition result in Theorem 1 can be applied in signal analysis to decompose a nonstationary signal $f(w, t)$, where w is the frequency variable. By selecting an appropriate Riesz basis $\{\lambda_n : n \in I\}$ with dual $\{\tilde{\lambda}_n : n \in I\}$, the signal $f(w, t)$ is mapped to a sequence of stationary signals

$$g_n(t) := (N_n f)(t) \in L^2(R), \qquad n \in I,$$

with certain desirable properties. Of course, the original signal can be reconstructed from $\{g_n : n \in I\}$ by using the identity

$$f(w, t) = \sum_{n \in I} (M_n g_n)(w, t).$$

The schemetic diagram for this decomposition and reconstruction procedure is shown in the following figure, where $I = \{n_i : i \in Z\}$:



In practice, the components g_{n_i} of f are "filtered" by certain operators H_{n_i}. For instance, Hankel-type operators H_{n_i} can be used for problems in systems theory such as systems reduction and identification. Of course, the reconstruction operators M_{n_i} are then applied to $H_{n_i} g_{n_i}$ instead of g_{n_i}.

(2) It is also worth mentioning that nonstationary signals are obtained by applying the Zak transform to stationary signals, namely,

$$f(w, t) = (Zg)(t, w) := a^{1/2} \sum_{k \in Z} g(ta + ka) e^{2\pi i k w}, \qquad g \in L^2(R).$$

This transformation was introduced by Zak [7] in the solid state physics literature. It can easily be verified that the Zak transform is a unitary map from $L^2(R)$ into $L^2([0, 1) \times [0, 1))$. For more details, the interested reader is referred to [7, 4].

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