CENTRAL EXTENSIONS OF NONSYMMETrIZABLE KAC-MOODY ALGEBRAS OVER COMMUTATIVE ALGEBRAS

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Abstract. For a commutative algebra $R$ over a field $k$ of characteristic zero and a nonsymmetrizable Kac-Moody algebra $g(A)$, we prove that the Lie algebra $g_R(A) = R \otimes_k g(A)$ is centrally closed. Consequently, we get a characterization of the symmetrizability of $g(A)$ by the second homology group of the Kac-Moody algebra over Laurent polynomials. Also a presentation of $g_R(A)$ is given when $A$ is of nonaffine type.

1. Introduction

Let $k$ be a field of characteristic zero and $R$ be an associative and commutative $k$-algebra with identity. Given an $l \times l$ generalized Cartan matrix (GCM, for short) $A = (A_{ij})$, the associated Lie algebra $g(A)$ over $k$ is called a Kac-Moody algebra and has the following standard presentation:

generators: $e_i, f_i, h_i, 1 \leq i \leq l;$
relations: $[h_i, e_j] = A_{ij} e_j, [h_i, f_j] = -A_{ij} f_j, [h_i, h_j] = 0, [e_i, f_i] = h_i,$
and
for $i \neq j$, $[e_i, f_j] = (ade_i)^{-A_{ij}+1} e_j = (adf_i)^{-A_{ij}+1} f_j = 0$.

Consider the $k$-vector space $g_R(A) = R \otimes_k g(A)$ and define a Lie bracket on $g_R(A)$ by

$[a \otimes x, b \otimes y] = ab \otimes [x, y],$

where $a, b \in R, x, y \in g(A)$. Then $g_R(A)$ is a perfect Lie algebra over $k$.

Kassel [7] studied the universal central extensions of $g_R(A)$ when $A$ is of finite type. An elegant approach to this work, which follows Wilson [10], has been given by Moody, Rao, and Yokonuma in [9]. Some of these results are used in [3]. More generally, if $A$ is indecomposable and symmetrizable, Haddi [5] computed the second homology groups of $g_R(A)$ as follows:

$H_2(g_R(A)) = \Omega^1_R/dR$, if $A$ is of nonaffine type,

$H_2(g_R(A)) = \Omega^1_R/dR \oplus (\Omega^1_R \otimes I)$, if $A$ is of affine type,

where $\Omega^1_R/dR$ is the module of Kähler differentials of $R$ over $k$ modulo exact forms and $I$ is the augmentation ideal of the algebra $k[t, t^{-1}]$ of Laurent polynomials which is generated by $(t - 1)$.

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In this note, we will give a presentation of \( g_R(A) \) for a nonaffine GCM \( A \) and show that \( H_2(g_R(A)) = 0 \) when \( A \) is indecomposable and nonsymmetric. In particular, it follows that if \( A \) is an indecomposable GCM, and \( R = k[t, t^{-1}] \), then \( H_2(g_R(A)) \neq 0 \) if and only if \( A \) is symmetrizable. This means that the second homology group of a Kac-Moody algebra over Laurent polynomials characterizes the symmetrizability of the Kac-Moody algebra.

2. Presentations for \( g_R(A) \)

All Lie algebras under consideration will be regarded as \( k \)-Lie algebras. We write \( x(a) = a \otimes x \) for \( a \in R, \ x \in g(A) \), so then \( g_R(A) \) is a Lie algebra generated by \( e_i(a), f_i(a), h_i(a) \) for all \( a \in R, \ 1 \leq i \leq l \). As in [4, 6, or 8], for each tuple \( (n_1, \ldots, n_l) \) of nonnegative (resp. nonpositive) integers not all zero, let \( g(n_1, \ldots, n_l) \) be the subspace of \( g(A) \) spanned by the elements

\[
[e_{i_1}, \ldots, e_{i_{m-1}}, e_{i_m}] \quad \text{(resp. } [f_{i_1}, \ldots, f_{i_{m-1}}, f_{i_m}] \text{)}
\]

where \( e_j \) (resp. \( f_j \)) occurs \( |n_j| \) times. Also let \( g(0, \ldots, 0) \) be the linear span of \( h_1, \ldots, h_l \) and \( g(n_1, \ldots, n_l) = 0 \) for any other tuple of integers. Then

\[
g(A) = \bigoplus_{(n_1, \ldots, n_l) \in \mathbb{Z}^l} g(n_1, \ldots, n_l),
\]

which is a Lie algebra gradation of \( g(A) \). Obviously, this induces a gradation of \( g_R(A) \),

\[
g_R(A) = \bigoplus_{(n_1, \ldots, n_l) \in \mathbb{Z}^l} g_R(n_1, \ldots, n_l),
\]

where \( g_R(n_1, \ldots, n_l) = R \otimes g(n_1, \ldots, n_l) = \{x(a) : x \in g(n_1, \ldots, n_l), \ a \in R\} \).

For convenience, we write \([x_1, [x_2, \ldots, [x_{m-1}, x_m] \cdots] = [x_1, x_2, \ldots, x_{m-1}, x_m] \cdots]] = [x_1, x_2, \ldots, x_{m-1}, x_m] \cdots]\.

Let \( \overline{g_R(A)} \) be the Lie algebra over \( k \) with the following presentation:

**generators:** \( E_i(a), F_i(a), H_i(a), a \in R, \ 1 \leq i \leq l \);

**relations:**

(1) \( E_i(\alpha a + \beta b) = \alpha E_i(a) + \beta E_i(b), \quad F_i(\alpha a + \beta b) = \alpha F_i(a) + \beta F_i(b) \),

(2) \( [H_i(a), E_j(b)] = A_{ij} E_j(ab) \),

(3) \( [H_i(a), F_j(b)] = -A_{ij} F_j(ab) \),

(4) \( [E_i(a), F_i(b)] = H_i(ab) \),

(5) \( [E_i(a), E_i(b)] = 0 \),

(6) \( [F_i(a), F_i(b)] = 0 \),

(7) \[
[E_{i_1}(a_1), \ldots, [E_{i_m}(a_{m-1}), E_{i_m}(a_m)] \cdots]
= [E_{i_1}(1), \ldots, [E_{i_{m-1}}(1), E_{i_m}(a_1 \cdots a_{m-1} a_m)] \cdots]
\]

(8) \[
[F_{i_1}(a_1), \ldots, [F_{i_m}(a_{m-1}), F_{i_m}(a_m)] \cdots]
= [F_{i_1}(1), \ldots, [F_{i_{m-1}}(1), F_{i_m}(a_1 \cdots a_{m-1} a_m)] \cdots]
\]
for all \( a_1, \ldots, a_m, a, b \in R, \alpha, \beta \in k, 1 \leq i_1, \ldots, i_m, i, j \leq l, \) and if \( i \neq j, \)

\begin{align*}
(9) \quad [E_i(a), F_j(b)] &= 0, \\
(10) \quad (ad E_i(1))^{-A_{ij}+1} E_j(a) &= 0, \\
(11) \quad (ad F_i(1))^{-A_{ij}+1} F_j(a) &= 0,
\end{align*}

where \( a, b \in R, 1 \leq i \neq j \leq l. \)

From the definition, it is not hard to check that \( H_i(\alpha a + \beta b) = \alpha H_i(a) + \beta H_i(b), \) and

\begin{align*}
(12) \quad [H_i(a), H_j(b)] &= 0, \\
(13) \quad [H_j(a), [E_i(1), \ldots, E_{i_{m-1}}(1), E_{im}(b)]]
&= (A_{ji_1} + \cdots + A_{jim})[E_i(1), \ldots, E_{im-1}(1), E_{im}(ab)], \\
(14) \quad [H_i(a), [F_i(1), \ldots, F_{i_{m-1}}(1), F_{im}(b)]]
&= -(A_{ji_1} + \cdots + A_{jim})[F_i(1), \ldots, F_{im-1}(1), F_{im}(ab)],
\end{align*}

where \( a, b \in R, \alpha, \beta \in k, 1 \leq i_1, \ldots, i_m, i, j \leq l. \)

One can easily see that the Lie algebra \( g_R(A) \) satisfies the above defining relations of \( g_R(A) \) (replacing \( E_i(a), F_i(a), H_i(a) \) by \( e_i(a), f_i(a), h_i(A) \), respectively). By the universal property of \( g_R(A) \), there exists a homomorphism \( \phi \) from \( g_R(A) \) onto \( g_R(A) \) such that

\begin{align*}
\phi(e_i(a)) &= e_i(a), \quad \phi(f_i(a)) = f_i(a), \quad \phi(h_i(a)) = h_i(A)
\end{align*}

for all \( a \in R, 1 \leq i \leq l. \)

Let \( \overline{g}(A) \) be the subalgebra of \( \overline{g}_R(A) \) generated by \( \{E_i(1), F_i(1), H_i(1), 1 \leq i \leq l\} \). The restriction of \( \phi \) on \( \overline{g}(A) \) induces a homomorphism from \( \overline{g}(A) \) onto \( g(A) \) which we shall denote by \( \psi \). Since \( g(A) \) has a universal property, there exists a homomorphism \( \rho \) from \( g(A) \) onto \( \overline{g}(A) \) such that

\begin{align*}
\rho(e_i) &= E_i(1), \quad \rho(f_i) = F_i(1), \quad \rho(h_i) = H_i(1), \quad \text{for } 1 \leq i \leq l.
\end{align*}

It follows that \( \rho \circ \psi \) is the identity map, proving that \( \psi \) is an isomorphism.

Fix a basis \( \{r_\lambda\}_{\lambda \in \Lambda} \) of \( R \) with \( 1 \in \{r_\lambda\}_{\lambda \in \Lambda} \), where \( \Lambda \) is an index set. Then

\begin{equation}
\overline{g}_R(A) = \bigoplus_{(n_1, \ldots, n_l) \in \mathbb{Z}^l} \left( \bigoplus_{\lambda \in \Lambda} r_\lambda \otimes g(n_1, \ldots, n_l) \right).
\end{equation}

Now we assume that \( A \) is of nonaffine type and recall the following well-known fact (see Theorem 4.3 in [6] or Proposition 3.6.5 in [8]).

**Lemma 1.** If \( A = (A_{ij}) \) is an indecomposable and nonaffine GCM, and \( n_1, \ldots, n_l \) are nonnegative integers such that

\begin{align*}
A_{i_1} n_1 + \cdots + A_{i_l} n_l &= 0 \\
&\vdots \\
A_{i_1} n_1 + \cdots + A_{i_l} n_l &= 0,
\end{align*}

then \( n_1 = \cdots = n_l = 0. \)

To define a linear mapping from \( g_R(A) \) to \( \overline{g}_R(A) \), we need the following elementary result.
Lemma 2. Suppose $V$ is any vector space over $k$ with $\{x_i\}_{i \in I}$ spanning $V$ and let $W$ be another vector space over $k$ with elements $\{y_i\}_{i \in I} \subseteq W$. If $\sum_i \alpha_i x_i = 0$ implies $\sum_i \alpha_i y_i = 0$, then there is a unique linear map $\phi$ from $V$ to $W$ satisfying $\phi(\sum_i \alpha_i x_i) = \sum_i \alpha_i y_i$.

We define a mapping $\phi: g(\mathcal{A}) \rightarrow g(\mathcal{A})$ given by

$$\phi \left( \sum_{i=1}^l \alpha_i h_i(r_\lambda) \right) = \sum_{i=1}^l \alpha_i H_i(r_\lambda),$$

where $\alpha_i \in k$, and $S_m$ is the standard permutation group on $\{1, \ldots, m\}$.

We claim that $\phi$ is a well-defined $k$-linear mapping. Note that by (15) and Lemma 2, it suffices to show that if

$$\sum_{\sigma \in S_m} \alpha_\sigma [e_{i_\sigma(1)}(1), \ldots, e_{i_{\sigma(m-1)}(1)}, e_{i_{\sigma(m)}(r_\lambda)}] = 0,$$

where $[e_1(1), \ldots, e_{i_{\sigma(m-1)}(1)}, e_{i_{\sigma(m)}(r_\lambda)}] \in r_\lambda \otimes g(n_1, \ldots, n_I)$ and $n_1, \ldots, n_I$ are nonnegative integers not all zero, then

$$\sum_{\sigma \in S_m} \alpha_\sigma [E_{i_\sigma(1)}(1), \ldots, E_{i_{\sigma(m-1)}(1)}, E_{i_{\sigma(m)}(r_\lambda)}] = 0.$$

Indeed, (16) gives us

$$\sum_{\sigma \in S_m} \alpha_\sigma [e_1(1), \ldots, e_{i_{\sigma(m-1)}}, e_{i_{\sigma(m)}(r_\lambda)}] = 0.$$

Since $\psi$ is an isomorphism from $g(\mathcal{A})$ onto $g(\mathcal{A})$, we have

$$\sum_{\sigma \in S_m} \alpha_\sigma [E_{i_{\sigma(1)}(1), \ldots, E_{i_{\sigma(m-1)}(1)}, E_{i_{\sigma(m)}(1)}] = 0.$$

Let $C = \sum_{\sigma \in S_m} \alpha_\sigma [E_{i_{\sigma(1)}(1), \ldots, E_{i_{\sigma(m-1)}(1)}, E_{i_{\sigma(m)}(r_\lambda)}]$. Then for any $j$, by (13), we get

$$[H_j(1), C] = \left[ H_j(1), \sum_{\sigma \in S_m} \alpha_\sigma [E_{i_{\sigma(1)}(1), \ldots, E_{i_{\sigma(m-1)}(1)}, E_{i_{\sigma(m)}(r_\lambda)}] \right]$$

$$= \sum_{\sigma \in S_m} \alpha_\sigma [H_j(1), [E_{i_{\sigma(1)}(1), \ldots, E_{i_{\sigma(m-1)}(1)}, E_{i_{\sigma(m)}(r_\lambda)}]$$

$$= \sum_{\sigma \in S_m} \alpha_\sigma (A_{j_1}n_1 + \cdots + A_{j_I}n_I)[E_{i_{\sigma(1)}(1), \ldots, E_{i_{\sigma(m-1)}(1)}, E_{i_{\sigma(m)}(r_\lambda)}]$$

$$= (A_{j_1}n_1 + \cdots + A_{j_I}n_I)C.$$
Also,

\[ [H_j(1), C] = \left[ H_j(1), \sum_{\sigma \in S_m} \alpha_\sigma [E_{i_\sigma(1)}(1), \ldots, E_{i_\sigma(m-1)}(1), E_{i_\sigma(m)}(r_\lambda)] \right] \]

\[ = \sum_{\sigma \in S_m} \alpha_\sigma [H_j(r_\lambda), [E_{i_\sigma(1)}(1), \ldots, E_{i_\sigma(m-1)}(1), E_{i_\sigma(m)}(1)]] \]

\[ = \left[ H_j(r_\lambda), \sum_{\sigma \in S_m} \alpha_\sigma [E_{i_\sigma(1)}(1), \ldots, E_{i_\sigma(m-1)}(1), E_{i_\sigma(m)}(1)] \right] = 0. \]

Thus \( C = 0 \) which verifies (17), for otherwise \( A_{j_1}n_1 + \cdots + A_{j_l}n_l = 0 \) for all \( 1 \leq j \leq l \) and this contradicts Lemma 1.

**Lemma 3.** Let \( L_1 \) and \( L_2 \) be two Lie algebras over \( k \). Suppose \( L_1 \) is generated by \( \{x_i\}_{i \in I} \). If \( \phi: L_1 \to L_2 \) is a \( k \)-linear mapping and \( \phi([x_i, y]) = [\phi(x_i), \phi(y)] \) for all \( i \in I, y \in L_1 \), then \( \phi \) is a Lie algebra homomorphism.

**Proof.** This is a routine matter.

Clearly, \( g_R(A) \) is actually generated by \( e_i(a), f_i(a), a \in R, 1 \leq i \leq l \). We will show

\[ \phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)], \quad \phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)] \]

for all \( a \in R, y \in g_R(A), 1 \leq j \leq l \).

**Case 1.** Let \( y \in g_R(0, \ldots, 0) \). Using (2), we obtain

\[ \phi([e_j(a), h_i(b)]) = \phi(-A_{ij}e_j(ab)) = -A_{ij} \phi(e_j(ab)) \]

\[ = -A_{ij}E_j(ab) = [E_j(a), H_i(b)] = [\phi(e_j(a)), \phi(h_i(b))]. \]

Hence \( \phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)] \) for all \( y \in g_R(0, \ldots, 0) \).

**Case 2.** Similar to Case 1, \( \phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)] \) for all \( y \in g_R(0, \ldots, 0) \).

**Case 3.** Using (7), we get

\[ \phi([e_j(a), e_i(b)]) = \phi([e_j(1), e_i(ab)]) = [E_j(1), E_i(ab)] \]

\[ = [E_j(a), E_i(b)] = [\phi(e_j(a)), \phi(e_i(b))]. \]

By induction on \( m \), one can see

\[ \phi([e_j(a), e_{i_1}(1), \ldots, e_{i_{m-1}}(1), e_{i_m}(b)]) \]

\[ = [\phi(e_j(a)), \phi([e_{i_1}(1), \ldots, e_{i_{m-1}}(1), e_{i_m}(b)])]. \]

Hence \( \phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)] \) for all \( y \in g_R(n_1, \ldots, n_l) \), where \( n_1, \ldots, n_l \) are nonnegative integers not all zero.

**Case 4.** Similar to Case 3, \( \phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)] \) for all \( y \in g_R(n_1, \ldots, n_l) \), where \( n_1, \ldots, n_l \) are nonpositive integers not all zero.

**Case 5.** Let \( y \in g_R(n_1, \ldots, n_l) \), where \( n_1, \ldots, n_l \) are nonpositive integers not all zero. When \( n_1 + \cdots + n_l = n = 1 \), using (4) and (9), we obtain

\[ \phi([e_j(a), f_i(b)]) = [\phi(e_j(a)), \phi(f_i(b))]. \]

Next, if \( n_1 + \cdots + n_l = n \geq 2 \), we have

\[ \phi([e_j(a), f_i(1), \ldots, f_{i_{m-1}}(1), f_{i_m}(b)]) \]

\[ = \phi([e_j(a), f_i(1)], [f_{i_2}(1), \ldots, f_{i_{m-1}}(1), f_{i_m}(b)]) \]

\[ + \phi([f_{i_1}(1), [e_j(a), f_{i_2}(1), \ldots, f_{i_{m-1}}(1), f_{i_m}(b)]]). \]
Now, using (14),
\[ \phi([[e_j(a), f_i(1)], [f_i(1), \ldots, f_{im-1}(1), f_{im}(b)]]) = \phi([[\delta_{ji}(h_j(a), [f_i(1), \ldots, f_{im-1}(1), f_{im}(b)])]) = \delta_{ji}(A_{jj} + \cdots + A_{jm})(\phi([f_i(1), \ldots, f_{im-1}(1), f_{im}(b)]))
\]
\[ = \phi([f_i(1), \ldots, f_{im-1}(1), f_{im}(b)])
\]
By Case 2, Case 4, and induction on \( n \), we get
\[ \phi([[f_i(1), [e_j(a), [f_i(1), \ldots, f_{im-1}(1), f_{im}(b)]])]) = \phi([f_i(1), [e_j(a), [f_i(1), \ldots, f_{im-1}(1), f_{im}(b)]])]
\]
So, we obtain
\[ \phi([[e_j(a), [f_i(1), \ldots, f_{im-1}(1), f_{im}(b)]]) = [[E_j(a), F_i(1)], [F_i(1), \ldots, F_{im-1}(1), F_{im}(b)]])
\]
Therefore \( \phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)] \) for all \( y \in g_R(n_1, \ldots, n_l) \) where \( n_1, \ldots, n_l \) are nonpositive integers not all zero.

**Case 6.** Similar to Case 5, \( \phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)] \) for all \( y \in g_R(n_1, \ldots, n_l) \), where \( n_1, \ldots, n_l \) are nonnegative integers not all zero.

By Lemma 3, \( \phi \) is a homomorphism from \( g_R(A) \) onto \( \overline{g}_R(A) \). Clearly \( \phi \circ \phi = id_{\overline{g}_R(A)} \), so \( \phi \) is injective.

Summarizing the above, we have proved

**Theorem 1.** If \( A \) is an indecomposable and nonaffine GCM, then \( g_R(A) \) is isomorphic to \( \overline{g}_R(A) \).

### 3. Central extensions of \( g_R(A) \)

Some ideas of this section are motivated by the work of Benkart and Moody [1].

**Theorem 2.** If \( A \) is an indecomposable and nonsymmetrizable GCM, then each central extension of the Lie algebra \( g_R(A) \) splits.

**Proof.** Suppose that \( 0 \to V \to L \xrightarrow{\pi} g_R(A) \to 0 \) is a central extension of \( g_R(A) \).

Let \( \tilde{e}_i(a) \), \( \tilde{h}_i(a) \), and \( \tilde{f}_i(a) \), for \( 1 \leq i \leq l \), be preimages of \( e_i(a) \), \( h_i(a) \), and \( f_i(a) \), respectively, in \( L \) under \( \pi \). It is sufficient to choose \( \tilde{e}_i(a) \), \( \tilde{h}_i(a) \), \( \tilde{f}_i(a) \) for \( 1 \leq i \leq l \), \( a \in \{r_i \}_{i \in \Lambda} \), and then extend our choices to all \( a \in R \) by linearity.

Clearly, \( [\tilde{h}_i(1), \tilde{e}_i(a)] = 2\tilde{e}_i(a) + c_i(a) \) for some \( c_i(a) \in V \). Replacing \( \tilde{e}_i(a) \) by \( \tilde{e}_i(a) + \frac{1}{2}c_i(a) \), we get
\[ [\tilde{h}_i(1), \tilde{e}_i(a)] = 2\tilde{e}_i(a) \quad \text{for } 1 \leq i \leq l, \ a \in R. \]
Similarly we may take $\tilde{f}_i(a), \tilde{h}_i(a)$ so that

\begin{align}
(19) \quad [\tilde{h}_i(1), \tilde{f}_i(a)] &= -2\tilde{f}_i(a), \\
(20) \quad [\tilde{e}_i(1), \tilde{f}_i(a)] &= \tilde{h}_i(a)
\end{align}

for $1 \leq i \leq l, \ a \in R$.

The proof of the theorem will be carried out in steps.

**Step 1.** We claim

\begin{align}
(21) \quad [\tilde{h}_i(a), \tilde{e}_j(b)] &= A_{ij}\tilde{e}_j(ab), \\
(22) \quad [\tilde{h}_i(a), \tilde{f}_j(b)] &= -A_{ij}\tilde{f}_j(ab), \\
(23) \quad [\tilde{e}_i(a), \tilde{e}_i(b)] &= 0, \\
(24) \quad [\tilde{f}_i(a), \tilde{f}_i(b)] &= 0,
\end{align}

for all $a, b \in R, \ 1 \leq i, j \leq l$, and if $i \neq j$,

\begin{align}
(25) \quad [\tilde{e}_i(a), \tilde{f}_j(b)] &= 0, \\
(26) \quad (ad\tilde{e}_i(1))^{-A_{ij}+1}\tilde{e}_j(ab) &= 0, \\
(27) \quad (ad\tilde{f}_i(1))^{-A_{ij}+1}\tilde{f}_j(ab) &= 0
\end{align}

where $a, b \in R, \ 1 \leq i \neq j \leq l$.

Using (18) and the Jacobi identity, we have

\[ [\tilde{h}_i(a), \tilde{e}_j(b)] = \frac{1}{2}[[\tilde{h}_i(1), \tilde{e}_j(1)], \tilde{e}_j(b)] + \frac{1}{2}[\tilde{h}_i(1), [\tilde{h}_i(a), \tilde{e}_j(b)]] \]

Applying the central trick, namely, $[x, y] = [x', y']$ if $\pi(x) = \pi(x'), \pi(y) = \pi(y')$ for $x, y, x', y' \in L$, we obtain $[[\tilde{h}_i(a), \tilde{h}_j(1)], \tilde{e}_j(b)] = 0$ and $[[\tilde{h}_j(1), [\tilde{h}_i(a), \tilde{e}_j(b)]] = [\tilde{h}_j(1), A_{ij}\tilde{e}_j(ab)] = 2A_{ij}\tilde{e}_j(ab)$.

Thus $[\tilde{h}_i(a), \tilde{e}_j(b)] = A_{ij}\tilde{e}_j(ab)$ which gives (21). Similarly (22) holds.

Let $i \neq j$. Since $[\tilde{e}_i(a), \tilde{f}_j(b)] = c \in \mathbb{V}$, we see that

\[ 0 = [\tilde{h}_i(1), [\tilde{e}_i(a), \tilde{f}_j(b)]] = [[\tilde{h}_i(1), \tilde{e}_i(a)], \tilde{f}_j(b)] + [\tilde{e}_i(a), [\tilde{h}_i(1), \tilde{f}_j(b)]] = [2\tilde{e}_i(a), \tilde{f}_j(b)] - [\tilde{e}_i(a), A_{ij}\tilde{f}_j(ab)] = (2 - A_{ij})c \]

which implies $c = 0$ as $A_{ij} \leq 0$, so (25) follows.

To prove (23), (24), (26), and (27), we need the following formulas which follow easily from our definition and induction.

\begin{align}
(28) \quad [\tilde{h}_j(1), [\tilde{e}_{i_1}(a_1), \ldots, \tilde{e}_{i_m}(am)]] = (A_{j_1i_1} + \cdots + A_{j_mi_m})[\tilde{e}_{i_1}(a_1), \ldots, \tilde{e}_{i_m}(am)]
\end{align}

\begin{align}
(29) \quad [\tilde{h}_j(1), [\tilde{f}_{i_1}(a_1), \ldots, \tilde{f}_{i_m}(am)]] = -(A_{j_1i_1} + \cdots + A_{j_mi_m})[\tilde{f}_{i_1}(a_1), \ldots, \tilde{f}_{i_m}(am)]
\end{align}

where $1 \leq j, \ i_1, \ldots, i_m \leq l$ and $a_1, \ldots, am \in R$.

Now it is easy to see that $[\tilde{e}_i(a), \tilde{e}_i(b)] \in \mathbb{V}$, so

\[ 0 = [\tilde{h}_i(1), [\tilde{e}_i(a), \tilde{e}_i(b)]] = 4[\tilde{e}_i(a), \tilde{e}_i(b)] \]

which gives (23). Similarly we have (24).
Also, if \( i \neq j \), \((ad\hat{e}_i(1))^{-A_{ij}+1}\hat{e}_j(a) = c \in V\), then

\[
0 = [\hat{h}_i(1), c] = (2(-A_{ij} + 1) + A_{ij})c = (-A_{ij} + 2)c
\]

which implies \( c = 0 \) as \( A_{ij} \leq 0 \). Thus (26) holds and so does (27).

**Step 2.** We claim

\[
[\hat{e}_i(a_1), \ldots, \hat{e}_{i_{m-1}}(a_{m-1}), \hat{e}_{i_m}(a_m)] \\
= [\hat{e}_i(1), \ldots, \hat{e}_{i_{m-1}}(1), \hat{e}_{i_m}(a_1 \cdots a_m)],
\]

\[
[\hat{f}_i(a_1), \ldots, \hat{f}_{i_{m-1}}(a_{m-1}), \hat{f}_{i_m}(a_m)] \\
= [\hat{f}_i(1), \ldots, \hat{f}_{i_{m-1}}(1), \hat{f}_{i_m}(a_1 \cdots a_m)]
\]

for all \( a_1, \ldots, a_m \in R \), \( 1 \leq i_1, \ldots, i_m \leq l \), and \( m \geq 1 \).

In fact, suppose \( j \) occurs \( n_j \) times among \( \{i_1, \ldots, i_m\} \). Obviously, we have

\[
[\hat{e}_i(a_1), \ldots, \hat{e}_{i_{m-1}}(a_{m-1}), \hat{e}_{i_m}(a_m)] - [\hat{e}_i(1), \ldots, \hat{e}_{i_{m-1}}(1), \hat{e}_{i_m}(a_1 \cdots a_m)] = c
\]

for some \( c \in V \). Letting \( \hat{h}_j(1) \) act on both sides, by (28), we get

\[
(A_{j1}n_1 + \cdots + A_{jj}n_j)c = 0 \quad \text{for all } j.
\]

Since \( (A_{ij}) \) is nonsymmetrizable; it is, of course, of nonaffine type, so Lemma 1 forces \( c = 0 \). Thus (30) holds and so does (31).

**Step 3.** We claim

\[
[\hat{h}_i(a), \hat{h}_j(b)] = 0,
\]

\[
[\hat{e}_i(a), \hat{f}_i(b)] = \hat{h}_i(ab)
\]

for all \( a, b \in R \), \( 1 \leq i, j \leq l \).

Let \([\hat{h}_i(a), \hat{h}_j(b)] = H_{ij}(a, b)\), evidently \( H_{ij}(a, b) \in V \), and

\[
H_{ij}(a, b) + H_{ji}(b, a) = 0.
\]

By (20), (21), and the Jacobi identity, one has

\[
[\hat{e}_i(a), \hat{f}_i(b)] = [\frac{1}{2}[\hat{h}_i(a), \hat{e}_i(1)], \hat{f}_i(b)]
\]

\[
= \frac{1}{2}[\hat{h}_i(a), [\hat{e}_i(1), \hat{f}_i(b)]] - \frac{1}{2}[\hat{e}_i(1), [\hat{h}_i(a), \hat{f}_i(b)]]
\]

\[
= \frac{1}{2}[\hat{h}_i(a), \hat{h}_j(b)] + [\hat{e}_i(1), \hat{f}_i(ab)] = \hat{h}_i(ab) + \frac{1}{2}H_{ii}(a, b).
\]

From (35), we find

\[
H_{ij}(c, ab) = [\hat{h}_i(c), \hat{h}_j(ab)] = [\hat{h}_i(c), [\hat{e}_j(a), \hat{f}_j(b)]]
\]

\[
= [[\hat{h}_i(c), \hat{e}_j(a)], \hat{f}_j(b)] + [\hat{e}_j(a), [\hat{h}_i(c), \hat{f}_j(b)]]
\]

\[
= A_{ij}[\hat{e}_j(ca), \hat{f}_j(b)] - A_{ij}[\hat{e}_j(a), \hat{f}_j(cb)]
\]

\[
= A_{ij}([\hat{h}_j(abc) + \frac{1}{2}H_{jj}(ca, b)] - A_{ij}([\hat{h}_j(abc) + \frac{1}{2}H_{jj}(a, cb)])
\]

\[
= \frac{1}{2}A_{ij}H_{jj}(ca, b) - \frac{1}{2}A_{ij}H_{jj}(a, cb).
\]
In (36), setting \( c = 1, \ b = 1, \ i = j \) we get

\[
H_{ii}(1, a) = 0,
\]

and setting \( a = 1 \) and using (37), we have

\[
H_{ij}(c, b) = \frac{1}{2}A_{ij}H_{jj}(c, b).
\]

Also, using (34) twice,

\[
H_{ij}(c, b) = -H_{ji}(b, c) = -\frac{1}{2}A_{ji}H_{ii}(b, c) = \frac{1}{2}A_{ji}H_{ii}(c, b).
\]

Therefore

\[
A_{ij}H_{jj}(c, b) = A_{ji}H_{ii}(c, b) \quad \text{for all } b, c \in R, 1 \leq i, j \leq l.
\]

Suppose that \( H_{j_0j_0}(c, b) \neq 0 \) for some \( j_0 \) and some \( c, b \in R \). Since \( A = (A_{ij}) \) is indecomposable, for any \( j \), there exist \( j_1, \ldots, j_m \) where \( j_m = j \) so that \( A_{j_0j_1}A_{j_1j_2} \cdots A_{j_{m-1}j_m} \neq 0 \). Iterating (39), we get \( H_{jj}(c, b) = \varepsilon_jH_{j_0j_0}(c, b) \) for some \( \varepsilon_j \in k, \varepsilon_j \neq 0 \). Substitution into (39) yields \( A_{ij}\varepsilon_jH_{j_0j_0}(c, b) = A_{ji}\varepsilon_iH_{j_0j_0}(c, b) \). As \( H_{j_0j_0}(c, b) \neq 0 \) it follows that \( A_{ij}\varepsilon_j = A_{ji}\varepsilon_i \). This contradicts the fact that \( A \) is nonsymmetrizable.

We have shown that \( H_{jj}(c, b) = 0 \) for all \( c, b \in R, 1 \leq j \leq l \). Then (32) follows from (38) and (33) follows from (35).

**Step 4.** By Theorem 1, there exists a unique homomorphism \( \tau \) from \( g_R(A) \) to \( L \) so that \( \tau(e_i(a)) = \hat{e}_i(a), \tau(f_i(a)) = \hat{f}_i(a), \) and \( \tau(h_i(a)) = \hat{h}_i(a) \). Evidently, \( \pi \circ \tau = id_{g_R(A)} \) which says that the original sequence splits.

Combining Theorem 2 with Haddi's result, we immediately have

**Corollary 1.** If \( A \) is an indecomposable GCM, and if \( R \) satisfies \( \Omega^1_R \neq dR \) (for instance, \( R = k[t, t^{-1}] \)), then \( H_2(g_R(A)) \neq 0 \) if and only if \( A \) is symmetrizable.

## 4. Remarks

**Remark 1.** Theorem 1 has already been worked out by Kassel [7] when \( A \) is of finite type. Our proof is more direct.

**Remark 2.** By further analyzing the proof of Theorem 2, letting \( i = j \) in (36), we get

\[
H_{ii}(c, ab) = H_{ii}(ca, b) - H_{ii}(a, cb).
\]

From (34) and (37), we have

\[
H_{ii}(1, a) = H_{ii}(a, 1) = 0,
\]

\[
H_{ii}(a, b) + H_{ii}(b, a) = 0,
\]

\[
H_{ii}(c, ab) + H_{ii}(a, bc) + H_{ii}(b, ca) = 0.
\]

We thus get another proof for the fact that \( H_2(g_R(A)) = \Omega^1_R/dR \) when \( A \) is an indecomposable and symmetrizable nonaffine GCM.

**Remark 3.** If the Lie algebra \( g(A) \) is replaced by the Lie algebra \( \hat{g}(A) \) which is defined by the relations: \([h_i, h_j] = 0, \ [h_i, e_j] = A_{ij}e_j, \ [h_i, f_j] = -A_{ij}f_j, \ [e_i, f_j] = \delta_{ij}h_i, \) see [2, 6], then our methods also yield that \( R \otimes_k \hat{g}(A) \) is centrally closed when \( A \) is indecomposable and nonsymmetrizable. Also, \( H_2(R \otimes_k \hat{g}(A)) = \Omega^1_R/dR \) if \( A \) is an indecomposable and symmetrizable nonaffine GCM.
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