INTEGRATION OF VECTOR-VALUED PSEUDO-ALMOST PERIODIC FUNCTIONS

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Abstract. A necessary and sufficient condition is given to show that the indefinite integral of a vector-valued pseudo-almost periodic function is again pseudo-almost periodic. Then we use this result to answer a question about weakly almost periodic functions.

Throughout this paper, $X$ denotes a Banach space and $J_a$ stands for $[a, \infty)$ when $a \in \mathbb{R}$ and for $\mathbb{R}$ when $a = -\infty$; $C(J_a, X)$ denotes the space of all bounded continuous functions from $J_a$ to $X$. Also, $m$ denotes Lebesgue measure on $\mathbb{R}$.

Let $f \in C(\mathbb{R}, X)$. The translate of $f$ by $s \in \mathbb{R}$ is the function $R_s f(t) = f(t + s), \ t \in \mathbb{R}$. $f$ is called (weakly) almost periodic if the set $\{R_s f : s \in \mathbb{R}\}$ is (weakly) relatively compact in $C(\mathbb{R}, X)$ [3]. Denote by $(WAP(\mathbb{R}, X))_0 A P(\mathbb{R}, X)$ all such functions.

In case $X = C$, we will omit $X$ from our notation and write, for example, $C(J_a)$ for $C(J_a, X)$.

For a function $f \in A P(\mathbb{R})$, the classical Bohl-Bohr integration theorem (cf. [2]) asserts that the indefinite integral $F(t) = \int_0^t f(u) \, du$ will also be almost periodic on $\mathbb{R}$ whenever $F$ is bounded. For $f \in A P(\mathbb{R}, X)$, there is Kadets's generalized Bohl-Bohr theorem (see [4]). A question arises naturally in $WAP(\mathbb{R}, X)$; that is, if $f \in WAP(\mathbb{R}, X)$, what is a necessary and sufficient condition for $F$ to be again in $WAP(\mathbb{R}, X)$?

Let $f \in WAP(\mathbb{R}, X)$. Then $f = g + \phi$, where $g \in A P(\mathbb{R}, X)$ and $\phi \in WAP_0(\mathbb{R}, X)$, the subspace of $WAP(\mathbb{R}, X)$ whose members have the zero function in the weak closure of the set of translates [3, Theorem 4.11]. The difficult part of answering the question is to handle the function $\phi$. For this purpose we introduce a new generalization of almost periodic functions, which we call pseudo-almost periodic functions.

In this paper, we first set up some theorems of the indefinite integral of a pseudo-almost periodic function from $J_a$ to $X$. Then we use these theorems to answer the question.

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Definition 1. A subset $P$ of $J_a$ is said to be relatively dense in $J_a$ if there exists a number $l > 0$ such that $[t, t + l] \cap P \neq \emptyset \quad (t \in J_a)$.

A function $g \in \mathcal{C}(\mathbb{R}, X)$ is in $\mathcal{AP}(\mathbb{R}, X)$ if and only if, for $\epsilon > 0$, the set

$$P(\epsilon) = \{ \tau \in \mathbb{R} : \|g(t + \tau) - g(t)\| < \epsilon \quad \text{for all} \quad t \in \mathbb{R} \}$$

is relatively dense in $\mathbb{R}$ [2, Theorem 6.6]. A function $g \in \mathcal{AP}(\mathbb{R}, X)$ is uniformly continuous.

Set

$$\mathcal{AP}_0(J_a, X) = \left\{ \varphi \in \mathcal{C}(J_a, X) : \lim_{t \to -\infty} \frac{1}{t - a} \int_a^t \|\varphi(s)\| \, ds = 0 \right\} \setminus \left\{ \varphi \in \mathcal{C}(J_a, X) : \lim_{t \to -\infty} \frac{1}{2t} \int_{-t}^t \|\varphi(s)\| \, ds = 0, \text{ when } a = -\infty \right\}.$$

Definition 2. A function $f \in \mathcal{C}(J_a, X)$ is called pseudo-almost periodic if

$$f = g|_{J_a} + \varphi,$$

where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{AP}_0(J_a, X)$. Denote by $\mathcal{AP}(J_a, X)$ all such functions. As in [7], $g$ and $\varphi$ are called the almost periodic component and the ergodic perturbation of $f$, respectively.

Definition 3. A closed subset $C$ of $J_a$ is said to be an ergodic zero set in $J_a$ if

$$m(C \cap [a, t])/(t - a) \to 0 \quad \text{as} \quad t \to \infty \quad (m(C \cap [-t, t])/2t \to 0 \quad \text{as} \quad t \to \infty, \text{ when } a = -\infty).$$

The proof of the following proposition is straightforward.

Proposition 4. A function $\varphi \in \mathcal{C}(J_a, X)$ is in $\mathcal{AP}_0(J_a, X)$ if and only if, for $\epsilon > 0$, the set $C_\epsilon = \{ t \in J_a : \|\varphi(t)\| \geq \epsilon \}$ is an ergodic zero subset in $J_a$.

Proposition 5. Let $C$ be an ergodic zero set in $J_a$. Then for any $\delta > 0$ and $L > 0$, there exists an interval $(u, v) \subset J_a$ with the properties that $v - u > L$ and $m(C \cap (u, v)) < \delta$.

Proof. If such a $(u, v)$ does not exist, one sees readily that

$$\liminf_{t \to -\infty} m(C \cap [a, t])/t \geq \delta/2L$$

$$(\liminf_{t \to -\infty} m(C \cap [-t, t])/2t \geq \delta/2L, \text{ when } a = -\infty).$$

Using Propositions 4 and 5, we can show that, for $g \in \mathcal{AP}(\mathbb{R}, X)$, if $g|_{J_a} \in \mathcal{AP}_0(J_a, X)$, then $g = 0$. In fact, since $g$ is uniformly continuous, for $\epsilon > 0$ there is $\delta > 0$ such that $\|g(t') - g(t'')\| < \epsilon$ whenever $t'$, $t'' \in \mathbb{R}$ with $|t' - t''| < \delta$. Let $l > 0$ be the number for $P(\epsilon)$ as in Definition 1, let $C_\epsilon$ be the ergodic zero subset for $g|_{J_a} \in \mathcal{AP}_0(J_a, X)$ in $J_a$ as in Proposition 4, and let $L = 2l$. By Proposition 5, for $\delta > 0$ and $L > 0$, there exists an interval $(u, v) \subset J_a$ with the properties that $v - u > L$ and $m(C \cap (u, v)) < \delta$. It follows that $\|g(t)\| < 2\epsilon$, $t \in (u, v)$. For $t \in \mathbb{R}$, since $v - u > 2l$, one can find $\tau \in P(\epsilon)$ such that $t + \tau \in (u, v)$. Therefore, $\|g(t)\| \leq \|g(t) - g(t + \tau)\| + \|g(t + \tau)\| < 3\epsilon$. Since $\epsilon > 0$ is arbitrary, $g = 0$. Thus

$$\mathcal{AP}(J_a, X) = \mathcal{AP}(\mathbb{R}, X)|_{J_a} \oplus \mathcal{AP}_0(J_a, X).$$

To show the next theorem, we need the following lemma.
Lemma 6. Let $P$ be relatively dense in $I_a$, and let $C$ be an ergodic zero set in $I_a$. Then, for any given interval $[c, d] \subset \mathbb{R}$ and $\delta > 0$, there exist $(u, v) \subset I_a$ and $\tau \in P$ such that

$$[c, d] + \tau \subset (u, v) \quad \text{and} \quad m(C \cap (u, v)) < \delta.$$ 

Proof. Let $l > 0$ be the number for $P$ as in Definition 1, and let $L = l + (d - c)$. By Proposition 5, there exists an interval $(u, v) \subset I_a$ such that $m(C \cap (u, v)) < \delta$ and $L < v - u$. Since we can assume that $u - c \in I_a$, we can choose $\tau \in [u - c, u - c + l] \cap P$. If $t \in [c, d]$,

$$u < c + \tau \leq t + \tau \leq d + \tau \leq d + u - c + l < v;$$

that is, $[c, d] + \tau \subset (u, v)$.

Theorem 7. Let $a \in \mathbb{R}$, and let $f \in \mathcal{P}_0(I_a, X)$. Define $F : I_a \rightarrow X$ by $F(t) = \int_a^t f(u) \, du$. Then $F \in \mathcal{P}_0(I_a, X)$ if and only if there is a vector $A \in X$ such that $F - A \in \mathcal{P}_0(I_a, X)$.

Proof. The sufficiency is obvious. Now we show the necessity. Since $F \in \mathcal{P}_0(I_a, X)$, $F = G|_{I_a} + \Phi$, where $G \in \mathcal{A}(\mathbb{R}, X)$ and $\Phi \in \mathcal{P}_0(I_a, X)$. To show the necessity, we need to show that $G$ is a constant vector in $X$.

If it is not, there are $t', t'' \in \mathbb{R}$ with $t' < t''$ such that

$$\|G(t') - G(t'')\| = \epsilon > 0.$$ 

Since $G \in \mathcal{A}(\mathbb{R}, X)$, for any $t \in P(\epsilon/4)$,

$$\|G(t') - G(t' + \tau)\| < \epsilon/4 \quad \text{and} \quad \|G(t'') - G(t'' + \tau)\| < \epsilon/4.$$ 

Combining these three inequalities, we have

1. $\|G(t' + \tau) - G(t'' + \tau)\| > \epsilon/2 \quad (\tau \in P(\epsilon/4))$.

$\Phi$ is uniformly continuous on $I_a$ since $F$ and $G$ are. Let $\delta > 0$ be such that $\|f\| \delta < \epsilon/8$ and

2. $\|\Phi(t_1) - \Phi(t_2)\| < \epsilon/16 \quad (t_1, t_2 \in I_a, \ |t_1 - t_2| < \delta).$

Set

$$C_1 = \left\{ t \in I_a : \|\Phi(t)\| \geq \min \left\{ \frac{\epsilon}{8(t' - t')}, \frac{\epsilon}{16} \right\} \right\},$$

$$C_2 = \left\{ t \in I_a : \|f(t)\| \geq \min \left\{ \frac{\epsilon}{8(t'' - t'')}, \frac{\epsilon}{16} \right\} \right\},$$

and $C_e = C_1 \cup C_2$. It follows from Proposition 4 that $C_e$ is an ergodic zero subset in $I_a$. By Lemma 6, there exist a $\tau_0 \in P(\epsilon/4)$ and $(u, v) \subset I_a$ such that $[t', t''] + \tau_0 \subset (u, v)$ and $m((u, v) \cap C_e) < \delta$.

We claim that $\|\Phi(t' + \tau_0)\| < \epsilon/8$ and $\|\Phi(t'' + \tau_0)\| < \epsilon/8$. In fact, if $t' + \tau_0 \in (u, v) \setminus C_e$, then $\|\Phi(t' + \tau_0)\| < \epsilon/16$; if $t' + \tau_0 \in (u, v) \cap C_e$, then by (2)

$$\|\Phi(t' + \tau_0)\| \leq \|\Phi(t' + \tau_0) - \Phi(t)\| + \|\Phi(t)\| < \epsilon/8,$$

where $t \in (u, v) \setminus C_e$ is such that $|t - (t' + \tau_0)| < \delta$.

Similarly we can show that $\Phi(t'' + \tau_0) < \epsilon/8$.  

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Now
\[ \| G(t' + \tau_0) - G(t'' + \tau_0) \| \]
\[ \leq \left\| \int_a^{t' + \tau_0} f(u) \, du - \int_a^{t'' + \tau_0} f(u) \, du \right\| + \| \Phi(t' + \tau_0) \| + \| \Phi(t'' + \tau_0) \| \]
\[ \leq \left\| \int_a^{t'' + \tau_0} f(u) \, du \right\| + \left\| \int_a^{t'' + \tau_0} f(u) \, du \right\| + \epsilon \quad \frac{\epsilon}{4} \]
\[ = \int_{[t'+\tau_0,t''+\tau_0] \cap C} \| f(u) \| \, du + \int_{[t'+\tau_0,t''+\tau_0] \cap C} \| f(u) \| \, du + \frac{\epsilon}{4} \]
\[ \leq |t'' - t'| \cdot \frac{\epsilon}{8(t'' - t')} + \| f \| \delta + \frac{\epsilon}{4} < \frac{\epsilon}{2}, \]
which contradicts (1).

Denote by \( C_0(\mathbb{J}_a, X) \) the set of functions \( f \in \mathcal{C}(\mathbb{J}_a, X) \) such that \( f(t) \to 0 \) as \( t \to \infty \). A function \( f \in \mathcal{C}(\mathbb{J}_a, X) \) is called asymptotically almost periodic if \( f = g + \varphi \), where \( g \in \mathcal{A}P(\mathbb{R}, X) \) and \( \varphi \in C_0(\mathbb{J}_a, X) \) [6]. Denote by \( \mathcal{A}P(\mathbb{J}_a, X) \) all such functions \( f \). Since \( C_0(\mathbb{J}_a, X) \subset \mathcal{P}A \mathcal{P}_0(\mathbb{J}_a, X) \), \( \mathcal{A}P(\mathbb{J}_a, X) \subset \mathcal{P}A \mathcal{P}(\mathbb{J}_a, X) \). We have

**Corollary 8** [5, 4.2]. Let \( a \in \mathbb{R} \), let \( f \in C_0(\mathbb{J}_a, X) \), and let \( F : \mathbb{J}_a \to X \) be defined by \( F(t) = \int_a^t f(u) \, du \). Then \( F \in \mathcal{A} \mathcal{P}(\mathbb{J}_a, X) \) if and only if \( \lim_{t \to \infty} F(t) \) exists.

**Proof.** Necessity. Since \( F \in \mathcal{A} \mathcal{P}(\mathbb{J}_a, X) \), \( F = G + \Phi' \), where \( G \in \mathcal{A} \mathcal{P}(\mathbb{R}, X) \) and \( \Phi' \in C_0(\mathbb{J}_a, X) \). By Theorem 7, \( F = A + \Phi \), where \( A \in X \) and \( \Phi \in \mathcal{P}A \mathcal{P}_0(\mathbb{J}_a, X) \). The uniqueness of the decomposition implies that \( G = A \) and \( \Phi \in C_0(\mathbb{J}_a, X) \). Therefore, \( F(t) = \int_a^t f(u) \, du \to A \) as \( t \to \infty \).

The sufficiency is easy to prove.

The following example of [5, 4.1] can also be used here to show that the bounded integral of a pseudo-almost periodic function may fail to be pseudo-almost periodic even in the numerical case.

**Example 9.** Consider the function \( f : \mathbb{J}_1 \to \mathbb{R} \) defined by
\[ f(t) = \left( \frac{1}{r} \right) \cos(\log t) \quad (t \in \mathbb{J}_1). \]
Since \( f(t) \to 0 \) as \( t \to \infty \), \( f(t) \in \mathcal{P}A \mathcal{P}_0(\mathbb{J}_1). \) The corresponding indefinite integral
\[ F(t) = \int_1^t f(u) \, du = \sin(\log t) \quad (t \in \mathbb{J}_1) \]
defines a bounded function on \( \mathbb{J}_1 \); however, \( F \notin \mathcal{P}A \mathcal{P}(\mathbb{J}_1) \). That is because
\[ \frac{1}{r} \int_1^r \sin(\log t) \, dt \neq 0 \]
as \( r \to \infty \) and neither does \( \frac{1}{r} \int_1^r |\sin(\log t)| \, dt \) for any \( A \in \mathbb{C} \). But, if \( F \in \mathcal{P}A \mathcal{P}(\mathbb{J}_a) \), \( F \) differs from some member of \( \mathcal{P}A \mathcal{P}_0(\mathbb{J}_a) \) by a constant (Theorem 7).

Let \( f \in \mathcal{P}A \mathcal{P}(\mathbb{J}_a, X) \), and define \( F(t) = \int_a^t f(u) \, du \) for \( t \in \mathbb{J}_a \). To show \( F \in \mathcal{P}A \mathcal{P}(\mathbb{J}_a, X) \), we need to treat two cases. Theorem 7 allows us to control
the ergodic perturbation of \( f \). We utilize the work of Kadets [4] in treating the almost periodic part of \( f \). Also, as in [5], we need the following lemmas. In the lemmas, for a subset \( B \) of \( X \), \( \overline{\text{co}}(B) \) denotes the closed convex circled hull of \( B \).

**Lemma 10.** Let \( a \in \mathbb{R} \), and let \( f \in \mathcal{AP}(\mathbb{J}_a, X) \). Let \( g \) and \( \varphi \) be the almost periodic component and the ergodic perturbation of \( f \), respectively. If \( F(t) = \int_a^t f(u) \, du \) and \( G_a(t) = \int_a^t g(u) \, du \) for \( t \in \mathbb{J}_a \), then

\[
G_a(\mathbb{J}_a) \subset 2\overline{\text{co}}(F(\mathbb{J}_a)).
\]

**Proof.** Suppose that the conclusion does not hold; we can find a \( t_0 \in \mathbb{J}_a \) such that \( G_a(t_0) \notin 2\overline{\text{co}}(F(\mathbb{J}_a)) \). In this case, we point out that \( \|\varphi\| \neq 0 \); otherwise we will have the conclusion. Also, \( t_0 \neq a \) because \( G_a(a) = 0 \in 2\overline{\text{co}}F(\mathbb{J}_a) \). By the Hahn-Banach theorem, there is an \( x^* \in X^* \) such that

\[
\min_{y \in 2\overline{\text{co}}(F(\mathbb{J}_a))} |x^*(G_a(t_0) - y)| = \epsilon > 0.
\]

Let

\[
0 < \delta = \min \left\{ \frac{\epsilon}{3\|x^*\|(t_0 - a)} : \frac{\epsilon}{3\|x^*\|\|\varphi\|} \right\}
\]

and

\[
C = \{ t \in \mathbb{J}_a : \|\varphi(t)\| \geq \delta \}.
\]

By Proposition 4, \( C \) is an ergodic zero subset of \( \mathbb{J}_a \). For \( \delta > 0 \) and \( g \in \mathcal{AP}(\mathbb{R}, X) \), let \( P(\delta) \) be the relatively dense subset of \( \mathbb{R} \). By Lemma 6, there are \( (u, v) \subset \mathbb{J}_a \) and a \( \tau \in P(\delta) \) such that \( [a, t_0 + \tau] \subset (u, v) \) and \( m((u, v) \cap C) < \delta \). Now,

\[
eq \min_{y \in 2\overline{\text{co}}(F(\mathbb{J}_a))} |x^*(G_a(t_0) - y)|
\]

\[
\leq \left| x^* \left\{ G_a(t_0) - \int_a^{t_0} g(u + \tau) \, du \right\} \right|
\]

\[
+ \left| x^* \left\{ \int_a^{t_0} g(u + \tau) \, du - \int_{a + \tau}^{t_0 + \tau} f(u) \, du \right\} \right|
\]

\[
\leq \|x^*\| \int_a^{t_0} \|g(u) - g(u + \tau)\| \, du + \left| x^* \left\{ \int_a^{t_0} \varphi(u + \tau) \, du \right\} \right|
\]

\[
\leq \|x^*\|(t_0 - a)\delta + \int_a^{t_0} |x^*\{\varphi(u + \tau)\}| \, du
\]

\[
\leq \|x^*\|(t_0 - a)\delta + \int_{[a + \tau, t_0 + \tau] \cap C} |x^*(\varphi(u))| \, du + \int_{[a + \tau, t_0 + \tau] \cap C} |x^*(\varphi(u))| \, du
\]

\[
< \|x^*\|(t_0 - a)\delta + \delta\|\varphi\|\|x^*\| < \epsilon,
\]

a contradiction.

**Lemma 11** [5, 4.5]. Let \( a \in \mathbb{R} \), and let \( g \in \mathcal{AP}(\mathbb{R}, X) \). Put \( G_a(t) = \int_a^t g(u) \, du \) for \( t \in \mathbb{J}_a \), and set

\[
G(t) = \begin{cases} 
\int_a^t g(u) \, du, & t \in \mathbb{J}_0, \\
- \int_0^t g(u) \, du, & t \in \mathbb{R} \setminus \mathbb{J}_0.
\end{cases}
\]
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Then

\[ G(\mathbb{R}) \subset G([0, |a|]) + 2\varepsilon \delta(G_a(\mathbb{J}_a)). \]

Taken together, Lemmas 10 and 11 yield the following result.

**Lemma 12.** Let a, f, g, \( \varphi \), F, and G\(_a\) be as in Lemma 10. If \( F(\mathbb{J}_a) \) is bounded \{weakly relatively compact\} \{relatively compact\} in \( X \), then the same is true for \( G(\mathbb{R}) \).

Let \( \mathcal{B}(\mathbb{J}_a, X) \) denote the set of all bounded functions from \( \mathbb{J}_a \) to \( X \). Now we have the following theorem.

**Theorem 13.** Let a, f, g, \( \varphi \), and F be as in Lemma 10. Suppose either

(i) \( F \in \mathcal{B}(\mathbb{J}_a, X) \) and \( X \) does not contain an isomorphic copy of \( c_0 \), or

(ii) \( F(\mathbb{J}_a) \) is weakly relatively compact in \( X \).

Then \( F \in \mathcal{WAP}(\mathbb{J}_a, X) \) if and only if there is an \( A \in X \) such that \( \Phi \), defined by

\[ \Phi(x) = \int_0^x \varphi(u) \, du - A, \]

is in \( \mathcal{WAP}(\mathbb{J}_a, X) \).

**Proof.** Necessity. We define the indefinite integrals \( G_a : \mathbb{J}_a \to X \) and \( G : \mathbb{R} \to X \) of \( g \) as in Lemma 11.

Since (i) or (ii) is satisfied, Lemma 12 shows that at least one of the following holds: \( c_0 \not\subset X \), \( G(\mathbb{R}) \) is bounded, and \( G(\mathbb{R}) \) is weakly relatively compact in \( X \). By [4, Theorem 1] in case (i), or by [4, Theorem 2] in case (ii), \( G \) is almost periodic. Therefore, \( G_a = G|_{\mathbb{J}_a} - G(a) \in \mathcal{WAP}(\mathbb{J}_a, X) \), as is \( \psi \), where

\[ \psi(t) = \int_a^t \varphi(u) \, du = F(t) - \int_a^t g(u) \, du. \]

Now the necessity follows from Theorem 7.

The sufficiency is easy to prove; we omit the proof.

The following theorem was shown in [5, Theorem 4.11].

**Theorem 14.** Let \( f = g + \varphi \), where \( g \in \mathcal{AP}(\mathbb{R}, X) \) and \( \varphi \in \mathcal{C}_0(\mathbb{R}, X) \). Define \( F : \mathbb{R} \to X \) by \( F(t) = \int_0^t f(u) \, du \). Then \( F \) is in \( \mathcal{WAP}(\mathbb{R}, X) \) if and only if either

(i) \( F(\mathbb{R}) \) is weakly relatively compact in \( X \) or

(ii) \( c_0 \not\subset X \) and \( F \in \mathcal{B}(\mathbb{R}, X) \),

and the following limits exist and satisfy

\[
\lim_{t \to \infty} \int_0^t \varphi(u) \, du = \lim_{t \to -\infty} \int_0^t \varphi(u) \, du.
\]

Since \( \mathcal{C}_0(\mathbb{R}, X) \subset \mathcal{WAP}(\mathbb{R}, X) \), Ruess and Summers pointed out in [5, p. 33] that Theorem 14 does not answer the question: when is the integral of an \( f \in \mathcal{WAP}(\mathbb{R}, X) \) again in \( \mathcal{WAP}(\mathbb{R}, X) \) ?

The next theorem answers this question. Before stating the theorem, we show that if \( f \in \mathcal{WAP}(\mathbb{R}, X) \) and \( x^* \in X^* \), then \( x^* f|_{\mathbb{J}_0} \in \mathcal{PS}(\mathbb{J}_0) \), where \( \mathbb{J}_0 = [0, \infty) \). In fact, \( f = g + \varphi \), where \( g \in \mathcal{AP}(\mathbb{R}, X) \) and \( \varphi \in \mathcal{WAP}(\mathbb{R}, X) \) [3, Theorem 4.11]. \( x^* g \in \mathcal{AP}(\mathbb{R}) \), and \( x^* \varphi \in \mathcal{WAP}(\mathbb{R}) \). Since \( \mathcal{WAP}(\mathbb{R}) \subset \mathcal{PS}(\mathbb{R}) \) [1, 4.3.13], \( x^* \varphi \in \mathcal{PS}(\mathbb{R}) \). So \( x^* f|_{\mathbb{J}_0} \in \mathcal{PS}(\mathbb{J}_0) \) and \( x^* f|_{\mathbb{J}_0} \in \mathcal{PS}(\mathbb{J}_0) \).
Theorem 15. Let $f \in \mathcal{WAP}(\mathbb{R}, X)$. Define $F : \mathbb{R} \to X$ by $F(t) = \int_0^t f(u) \, du$. Then $F$ is in $\mathcal{WAP}(\mathbb{R}, X)$ if and only if either

(i) $F(\mathbb{R})$ is weakly relatively compact in $X$, or

(ii) $c_0 \not\subseteq X$ and $F \in \mathcal{B}(\mathbb{R}, X)$,

and there is a vector $A \in X$ such that

(4) $\psi - A \in \mathcal{WAP}_0(\mathbb{R}, X),$

where $\psi : \mathbb{R} \to X$ is defined by $\psi(t) = \int_0^t \varphi(u) \, du$.

Proof. Set $G(t) = \int_0^t g(u) \, du$ for $t \in \mathbb{R}$. With a proof similar to that for Lemma 12, we can show that if $F(\mathbb{R})$ is bounded (weakly relatively compact) in $X$, then the same is true for $G(\mathbb{R})$.

Necessity. Since $F \in \mathcal{WAP}(\mathbb{R}, X)$, $F(\mathbb{R})$ is weakly relatively compact, as is $G(\mathbb{R})$. So $G \in \mathcal{AP}(\mathbb{R}, X)$ [4, Theorem 2].

Now we show that (4) holds for some $A \in X$. Since $\psi \in \mathcal{WAP}(\mathbb{R}, X)$,

(5) $\psi = G_1 + \Phi_1,$

where $G_1 \in \mathcal{AP}(\mathbb{R}, X)$ and $\Phi_1 \in \mathcal{WAP}_0(\mathbb{R}, X)$. We claim that $G_1$ is a constant function. For, suppose that there are $t_1, t_2 \in \mathbb{R}$ such that $G_1(t_1) \neq G_1(t_2)$. The almost periodicity of $G_1$ makes it possible to assume that $t_1$ and $t_2$ are in $J_0$. Then we can find a $x^* \in X^*$ such that $x^*G_1(t_1) \neq x^*G_1(t_2)$. Since $\psi \in \mathcal{WAP}(\mathbb{R}, X)$, $x^*\psi|_{J_0} \in \mathcal{AP}(J_0)$. By Theorem 7 there are an $A_{x^*} \in \mathbb{C}$ and a $\Phi_{x^*} \in \mathcal{AP}_0(J_0)$ such that

(6) $x^*\psi(t) = \int_0^t x^*\varphi(u) \, du = A_{x^*} + \Phi_{x^*}(t) \quad (t \in J_0).$

Comparing (6) with (5) and using the uniqueness of the decomposition, we conclude that $x^*G_1$ is a constant function, a contradiction.

Sufficiency. It is obvious that $\psi \in \mathcal{WAP}(\mathbb{R}, X)$ if (4) holds for some $A \in X$. By the first paragraph in the proof and [4, Theorems 1 and 2], either (i) or (ii) is a sufficient condition for $G$ to be in $\mathcal{AP}(\mathbb{R}, X)$, so $F \in \mathcal{WAP}(\mathbb{R}, X)$.

Remark 16. Theorem 14 is a corollary of Theorem 15. In fact, in Theorem 14, $A = \lim_{t \to \infty} \int_0^t \varphi(u) \, du$.

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Bibliography


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