INTEGRATION OF VECTOR-VALUED  
PSEUDO-ALMOST PERIODIC FUNCTIONS

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ABSTRACT. A necessary and sufficient condition is given to show that the indefinite integral of a vector-valued pseudo-almost periodic function is again pseudo-almost periodic. Then we use this result to answer a question about weakly almost periodic functions.

Throughout this paper, $X$ denotes a Banach space and $J_a$ stands for $[a, \infty)$ when $a \in \mathbb{R}$ and for $\mathbb{R}$ when $a = -\infty$; $C(J_a, X)$ denotes the space of all bounded continuous functions from $J_a$ to $X$. Also, $m$ denotes Lebesgue measure on $\mathbb{R}$.

Let $f \in C(\mathbb{R}, X)$. The translate of $f$ by $s \in \mathbb{R}$ is the function $R_s f(t) = f(t + s)$, $t \in \mathbb{R}$. $f$ is called (weakly) almost periodic if the set $\{R_s f : s \in \mathbb{R}\}$ is (weakly) relatively compact in $C(\mathbb{R}, X)$ [3]. Denote by $WAP(\mathbb{R}, X)$ all such functions.

In case $X = \mathbb{C}$, we will omit $X$ from our notation and write, for example, $C(J_a)$ for $C(J_a, \mathbb{C})$.

For a function $f \in A P(\mathbb{R})$, the classical Bohl-Bohr integration theorem (cf. [2]) asserts that the indefinite integral $F(t) = \int_0^t f(u) du$ will also be almost periodic on $\mathbb{R}$ whenever $F$ is bounded. For $f \in A P(\mathbb{R}, X)$, there is Kadets's generalized Bohl-Bohr theorem (see [4]). A question arises naturally in $WAP(\mathbb{R}, X)$; that is, if $f \in WAP(\mathbb{R}, X)$, what is a necessary and sufficient condition for $F$ to be again in $WAP(\mathbb{R}, X)$?

Let $f \in WAP(\mathbb{R}, X)$. Then $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ and $\varphi \in WAP_0(\mathbb{R}, X)$, the subspace of $WAP(\mathbb{R}, X)$ whose members have the zero function in the weak closure of the set of translates [3, Theorem 4.11]. The difficult part of answering the question is to handle the function $\varphi$. For this purpose we introduce a new generalization of almost periodic functions, which we call pseudo-almost periodic functions.

In this paper, we first set up some theorems of the indefinite integral of a pseudo-almost periodic function from $J_a$ to $X$. Then we use these theorems to answer the question.

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Definition 1. A subset \( P \) of \( \mathbb{J}_a \) is said to be relatively dense in \( \mathbb{J}_a \) if there exists a number \( l > 0 \) such that \( [t, t + l] \cap P \neq \emptyset \) \( (t \in \mathbb{J}_a) \).

A function \( g \in \mathcal{C}(\mathbb{R}, X) \) is in \( \mathcal{A} \mathcal{P}(\mathbb{R}, X) \) if and only if, for \( \epsilon > 0 \), the set
\[
P(\epsilon) = \{ \tau \in \mathbb{R} : \|g(t + \tau) - g(t)\| < \epsilon \text{ for all } t \in \mathbb{R} \}
\]
is relatively dense in \( \mathbb{R} \) [2, Theorem 6.6]. A function \( g \in \mathcal{A} \mathcal{P}(\mathbb{R}, X) \) is uniformly continuous.

Set
\[
\mathcal{A} \mathcal{P}_0(\mathbb{J}_a, X) = \left\{ \varphi \in \mathcal{C}(\mathbb{J}_a, X) : \lim_{t \to \infty} \frac{1}{t-a} \int_a^t \|\varphi(s)\| ds = 0 \right\}
\]
Definition 2. A function \( f \in \mathcal{C}(\mathbb{J}_a, X) \) is called pseudo-almost periodic if
\[
f = g|_{\mathbb{J}_a} + \varphi,
\]
where \( g \in \mathcal{A} \mathcal{P}(\mathbb{R}, X) \) and \( \varphi \in \mathcal{A} \mathcal{P}_0(\mathbb{J}_a, X) \). Denote by \( \mathcal{A} \mathcal{P}(\mathbb{J}_a, X) \) all such functions. As in [7], \( g \) and \( \varphi \) are called the almost periodic component and the ergodic perturbation of \( f \), respectively.

Definition 3. A closed subset \( C \) of \( \mathbb{J}_a \) is said to be an ergodic zero set in \( \mathbb{J}_a \) if \( m(C \cap [a, t])/(t-a) \to 0 \) as \( t \to \infty \) \( (m(C \cap [-t, t])/2t \to 0 \) as \( t \to \infty \), when \( a = -\infty \).

The proof of the following proposition is straightforward.

Proposition 4. A function \( \varphi \in \mathcal{C}(\mathbb{J}_a, X) \) is in \( \mathcal{A} \mathcal{P}_0(\mathbb{J}_a, X) \) if and only if, for \( \epsilon > 0 \), the set \( C_\epsilon = \{ t \in \mathbb{J}_a : \|\varphi(t)\| \geq \epsilon \} \) is an ergodic zero subset in \( \mathbb{J}_a \).

Proposition 5. Let \( C \) be an ergodic zero set in \( \mathbb{J}_a \). Then for any \( \delta > 0 \) and \( L > 0 \), there exists an interval \( (u, v) \subset \mathbb{J}_a \) with the properties that \( v - u > L \) and \( m(C \cap (u, v)) < \delta \).

Proof. If such a \( (u, v) \) does not exist, one sees readily that
\[
\lim_{t \to \infty} m(C \cap [a, t])/t = \delta/2L
\]
\[
(\liminf_{t \to \infty} m(C \cap [-t, t])/2t \geq \delta/2L, \text{ when } a = -\infty).
\]

Using Propositions 4 and 5, we can show that, for \( g \in \mathcal{A} \mathcal{P}(\mathbb{R}, X) \), if \( g|_{\mathbb{J}_a} \in \mathcal{A} \mathcal{P}_0(\mathbb{J}_a, X) \), then \( g = 0 \). In fact, since \( g \) is uniformly continuous, for \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( \|g(t') - g(t'')\| < \epsilon \) whenever \( t', t'' \in \mathbb{R} \) with \( |t' - t''| < \delta \). Let \( l > 0 \) be the number for \( P(\epsilon) \) as in Definition 1, let \( C_\epsilon \) be the ergodic zero subset for \( g|_{\mathbb{J}_a} \in \mathcal{A} \mathcal{P}_0(\mathbb{J}_a, X) \) in \( \mathbb{J}_a \) as in Proposition 4, and let \( L = 2l \). By Proposition 5, for \( \delta > 0 \) and \( L > 0 \), there exists an interval \( (u, v) \subset \mathbb{J}_a \) with the properties that \( v - u > L \) and \( m(C \cap (u, v)) < \delta \). It follows that \( \|g(t)\| < 2\epsilon, t \in (u, v) \). For \( t \in \mathbb{R} \), since \( v - u > 2l \), one can find \( \tau \in P(\epsilon) \) such that \( t + \tau \in (u, v) \). Therefore, \( \|g(t)\| \leq \|g(t) - g(t + \tau)\| + \|g(t + \tau)\| < 3\epsilon \). Since \( \epsilon > 0 \) is arbitrary, \( g = 0 \). Thus
\[
\mathcal{A} \mathcal{P}(\mathbb{J}_a, X) = \mathcal{A} \mathcal{P}(\mathbb{R}, X)|_{\mathbb{J}_a} \oplus \mathcal{A} \mathcal{P}_0(\mathbb{J}_a, X).
\]

To show the next theorem, we need the following lemma.
Lemma 6. Let $P$ be relatively dense in $J_a$, and let $C$ be an ergodic zero set in $J_a$. Then, for any given interval $[c, d] \subset \mathbb{R}$ and $\delta > 0$, there exist $(u, v) \subset J_a$ and $\tau \in P$ such that

$$[c, d] + \tau \subset (u, v) \quad \text{and} \quad m(C \cap (u, v)) < \delta.$$  

Proof. Let $l > 0$ be the number for $P$ as in Definition 1, and let $L = l + (d - c)$. By Proposition 5, there exists an interval $(u, v) \subset J_a$ such that $m(C \cap (u, v)) < \delta$ and $L < v - u$. Since we can assume that $u - c \in J_a$, we can choose $\tau \in [u - c, u - c + l] \cap P$. If $t \in [c, d]$,

$$u < c + \tau \leq t + \tau \leq d + \tau \leq d + u - c + l < v;$$

that is, $[c, d] + \tau \subset (u, v)$.

Theorem 7. Let $a \in \mathbb{R}$, and let $f \in \mathcal{PA}(J_a, X)$. Define $F : J_a \rightarrow X$ by $F(t) = \int_a^t f(u)\, du$. Then $F \in \mathcal{PA}(J_a, X)$ if and only if there is a vector $A \in X$ such that $F - A \in \mathcal{PA}(J_a, X)$.

Proof. The sufficiency is obvious. Now we show the necessity.

Since $F \in \mathcal{PA}(J_a, X)$, $F = G|_{J_a} + \Phi$, where $G \in \mathcal{AP}(\mathbb{R}, X)$ and $\Phi \in \mathcal{PA}(J_a, X)$. To show the necessity, we need to show that $G$ is a constant vector in $X$.

If it is not, there are $t', t'' \in \mathbb{R}$ with $t' < t''$ such that

$$\|G(t') - G(t'')\| = \epsilon > 0.$$  

Since $G \in \mathcal{AP}(\mathbb{R}, X)$, for any $\tau \in P(\epsilon/4)$,

$$\|G(t') - G(t' + \tau)\| < \epsilon/4 \quad \text{and} \quad \|G(t'') - G(t'' + \tau)\| < \epsilon/4.$$  

Combining these three inequalities, we have

$$\|G(t' + \tau) - G(t'' + \tau)\| > \epsilon/2 \quad (\tau \in P(\epsilon/4)).$$  

$\Phi$ is uniformly continuous on $J_a$ since $F$ and $G$ are. Let $\delta > 0$ be such that $\|f\|\delta < \epsilon/8$ and

$$\|\Phi(t_1) - \Phi(t_2)\| < \epsilon/16 \quad (t_1, t_2 \in J_a, \ |t_1 - t_2| < \delta).$$

Set

$$C_1 = \left\{ t \in J_a : \|\Phi(t)\| \geq \min \left\{ \frac{\epsilon}{8(t'' - t')}, \frac{\epsilon}{16} \right\} \right\},$$

$$C_2 = \left\{ t \in J_a : \|f(t)\| \geq \min \left\{ \frac{\epsilon}{8(t'' - t')}, \frac{\epsilon}{16} \right\} \right\},$$

and $C_e = C_1 \cup C_2$. It follows from Proposition 4 that $C_e$ is an ergodic zero subset in $J_a$. By Lemma 6, there exist a $\tau_0 \in P(\epsilon/4)$ and $(u, v) \subset J_a$ such that $[t', t''] + \tau_0 \subset (u, v)$ and $m((u, v) \cap C_e) < \delta$.

We claim that $\|\Phi(t' + \tau_0)\| < \epsilon/8$ and $\|\Phi(t'' + \tau_0)\| < \epsilon/8$. In fact, if $t' + \tau_0 \in (u, v) \setminus C_e$, then $\|\Phi(t' + \tau_0)\| < \epsilon/16$; if $t' + \tau_0 \in (u, v) \cap C_e$, then by (2)

$$\|\Phi(t' + \tau_0)\| \leq \|\Phi(t' + \tau_0) - \Phi(t)\| + \|\Phi(t)\| < \epsilon/8,$$

where $t \in (u, v) \setminus C_e$ is such that $|t - (t' + \tau_0)| < \delta$.

Similarly we can show that $\Phi(t'' + \tau_0) < \epsilon/8$.
Now
\[ ||G(t' + \tau_0) - G(t'' + \tau_0)|| \]
\[ \leq \left| \int_a^{t' + \tau_0} - \int_a^{t'' + \tau_0} f(u) \, du \right| + ||\Phi(t' + \tau_0)|| + ||\Phi(t'' + \tau_0)|| \]
\[ < \int_a^{t' + \tau_0} f(u) \, du + \varepsilon \]
\[ = \left( \int_{[t' + \tau_0, t'' + \tau_0] \cap C_c} f(u) \, du + \int_{[t' + \tau_0, t'' + \tau_0] \cap C_c} f(u) \, du \right) + \frac{\varepsilon}{4} \]
\[ \leq |t'' - t'| \cdot \frac{\varepsilon}{8(t'' - t')} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon, \]
which contradicts (1).

Denote by \( \mathcal{C}_0(J_a, X) \) the set of functions \( f \in \mathcal{C}(J_a, X) \) such that \( f(t) \to 0 \) as \( t \to \infty \). A function \( f \in \mathcal{C}(J_a, X) \) is called asymptotically almost periodic if \( f = g|_{J_a} + \varphi \), where \( g \in \mathcal{A}(\mathbb{R}, X) \) and \( \varphi \in \mathcal{C}_0(J_a, X) \) [6]. Denote by \( \mathcal{A}(J_a, X) \) all such functions \( f \). Since \( \mathcal{C}_0(J_a, X) \subset \mathcal{P}(J_a, X) \), \( \mathcal{A}(J_a, X) \subset \mathcal{P}(J_a, X) \). We have

**Corollary 8 [5, 4.2]**. Let \( a \in \mathbb{R} \), let \( f \in \mathcal{C}(J_a, X) \), and let \( F : J_a \to X \) be defined by \( F(t) = \int_a^t f(u) \, du \). Then \( F \in \mathcal{A}(J_a, X) \) if and only if \( \lim_{t \to \infty} F(t) \) exists.

**Proof.** Necessity. Since \( F \in \mathcal{A}(J_a, X) \), \( F = G|_{J_a} + \Phi' \), where \( G \in \mathcal{A}(\mathbb{R}, X) \) and \( \Phi' \in \mathcal{C}_0(J_a, X) \). By Theorem 7, \( F = A + \Phi \), where \( A \in X \) and \( \Phi \in \mathcal{P}(J_a, X) \). The uniqueness of the decomposition implies that \( G = A \) and \( \Phi \in \mathcal{C}_0(J_a, X) \). Therefore, \( F(t) = \int_a^t f(u) \, du \to A \) as \( t \to \infty \).

The sufficiency is easy to prove.

The following example of [5, 4.1] can also be used here to show that the bounded integral of a pseudo-almost periodic function may fail to be pseudo-almost periodic even in the numerical case.

**Example 9.** Consider the function \( f : J_1 \to \mathbb{R} \) defined by
\[ f(t) = \left( \frac{1}{t} \right) \cos(\log t) \quad (t \in J_1) \]
Since \( f(t) \to 0 \) as \( t \to \infty \), \( f(t) \in \mathcal{P}(J_1) \). The corresponding indefinite integral
\[ F(t) = \int_1^t f(u) \, du = \sin(\log t) \quad (t \in J_1) \]
defines a bounded function on \( J_1 \); however, \( F \notin \mathcal{A}(J_a, X) \). That is because
\[ \frac{1}{r} \int_1^r |\sin(\log t)| \, dt \neq 0 \]
as \( r \to \infty \) and neither does \( \frac{1}{r} \int_1^r |\sin(\log t)| - A |dt \) for any \( A \in \mathbb{C} \). But, if \( F \in \mathcal{P}(J_a, X) \), \( F \) differs from some member of \( \mathcal{P}(J_a, X) \) by a constant (Theorem 7).

Let \( f \in \mathcal{P}(J_a, X) \), and define \( F(t) = \int_a^t f(u) \, du \) for \( t \in J_a \). To show \( F \in \mathcal{P}(J_a, X) \), we need to treat two cases. Theorem 7 allows us to control
the ergodic perturbation of $f$. We utilize the work of Kadets [4] in treating the almost periodic part of $f$. Also, as in [5], we need the following lemmas. In the lemmas, for a subset $B$ of $X$, $\overline{conv}(B)$ denotes the closed convex circled hull of $B$.

**Lemma 10.** Let $a \in \mathbb{R}$, and let $f \in \mathcal{AP}(\mathbb{R}, X)$. Let $g$ and $\varphi$ be the almost periodic component and the ergodic perturbation of $f$, respectively. If $F(t) = \int_a^t f(u) \, du$ and $G_a(t) = \int_a^t g(u) \, du$ for $t \in \mathbb{R}$, then

$$G_a(\mathbb{R}) \subset 2\overline{conv}(F(\mathbb{R})).$$

**Proof.** Suppose that the conclusion does not hold; we can find a $t_0 \in \mathbb{R}$ such that $G_a(t_0) \notin 2\overline{conv}(F(\mathbb{R}))$. In this case, we point out that $\|\varphi\| \neq 0$; otherwise we will have the conclusion. Also, $t_0 \neq a$ because $G_a(a) = 0 \in 2\overline{conv}F(\mathbb{R})$. By the Hahn-Banach theorem, there is an $x^* \in X^*$ such that

$$\min_{y \in 2\overline{conv}(F(\mathbb{R}))} |x^*(G_a(t_0) - y)| = \epsilon > 0.$$ 

Let

$$0 < \delta = \min \left\{ \frac{\epsilon}{3\|x^*\|(t_0 - a)}, \frac{\epsilon}{3\|x^*\|\|\varphi\|} \right\}$$

and

$$C = \{t \in \mathbb{R} : \|\varphi(t)\| \geq \delta\}.$$ 

By Proposition 4, $C$ is an ergodic zero subset of $\mathbb{R}$. For $\delta > 0$ and $g \in \mathcal{AP}(\mathbb{R}, X)$, let $P(\delta)$ be the relatively dense subset of $\mathbb{R}$. By Lemma 6, there are $(u, v) \subset \mathbb{R}$ and a $\tau \in P(\delta)$ such that $[a, t_0] + \tau \subset (u, v)$ and $m((u, v) \cap C) < \delta$. Now,

$$\epsilon = \min_{y \in 2\overline{conv}(F(\mathbb{R}))} |x^*(G_a(t_0) - y)|$$

$$\leq |x^*\{G_a(t_0) - [F(t_0 + \tau) - F(a + \tau)]\}|$$

$$\leq |x^*\left\{ G_a(t_0) - \int_a^{t_0} g(u + \tau) \, du \right\}|$$

$$+ |x^*\left\{ \int_a^{t_0} g(u + \tau) \, du - \int_a^{t_0+\tau} f(u) \, du \right\}|$$

$$\leq \|x^*\| \int_a^{t_0} |g(u) - g(u + \tau)| \, du + |x^*\left\{ \int_a^{t_0} \varphi(u + \tau) \, du \right\}|$$

$$\leq \|x^*\|((t_0 - a)\delta + \int_a^{t_0} |x^*\{\varphi(u + \tau)\}| \, du$$

$$= \|x^*\|((t_0 - a)\delta + \int_{[a+\tau, t_0+\tau]}C \, |x^*\{\varphi(u)\}| \, du + \int_{[a+\tau, t_0+\tau] \setminus C} \, |x^*\{\varphi(u)\}| \, du$$

$$< \|x^*\|((t_0 - a)\delta + \delta \|\varphi\|\|x^*\| < \epsilon,$$

a contradiction.

**Lemma 11 [5, 4.5].** Let $a \in \mathbb{R}$, and let $g \in \mathcal{AP}(\mathbb{R}, X)$. Put $G_a(t) = \int_a^t g(u) \, du$ for $t \in \mathbb{R}$, and set

$$G(t) = \begin{cases} \int_a^t g(u) \, du, & t \in \mathbb{R}, \\ -\int_t^a g(u) \, du, & t \in \mathbb{R} \setminus \mathbb{R} \setminus \mathbb{R}. \end{cases}$$
Then
\[ G(\mathbb{R}) \subset \text{G}([0,|a|]) + 2c\text{o}(G_a(\mathbb{J}_a)). \]

Taken together, Lemmas 10 and 11 yield the following result.

**Lemma 12.** Let \(a, f, g, \varphi, F\), and \(G_a\) be as in Lemma 10. If \(F(\mathbb{J}_a)\) is bounded [weakly relatively compact] \{relatively compact\} in \(X\), then the same is true for \(G(\mathbb{R})\).

Let \(\mathcal{B}(\mathbb{J}_a, X)\) denote the set of all bounded functions from \(\mathbb{J}_a\) to \(X\). Now we have the following theorem.

**Theorem 13.** Let \(a, f, g, \varphi, \) and \(F\) be as in Lemma 10. Suppose either

(i) \(F \in \mathcal{B}(\mathbb{J}_a, X)\) and \(X\) does not contain an isomorphic copy of \(c_0\), or

(ii) \(F(\mathbb{J}_a)\) is weakly relatively compact in \(X\).

Then \(F \in \mathcal{WAP}(\mathbb{J}_a, X)\) if and only if there is an \(A \in X\) such that \(\Phi\), defined by

\[ \Phi(x) = \int_0^x \varphi(u) \, du - A, \]

is in \(\mathcal{WAP}_0(\mathbb{J}_a, X)\).

**Proof.** Necessity. We define the indefinite integrals \(G_a : \mathbb{J}_a \to X\) and \(G : \mathbb{R} \to X\) of \(g\) as in Lemma 11.

Since (i) or (ii) is satisfied, Lemma 12 shows that at least one of the following holds: \(c_0 \not\subset X\), \(G(\mathbb{R})\) is bounded, and \(G(\mathbb{R})\) is weakly relatively compact in \(X\). By [4, Theorem 1] in case (i), or by [4, Theorem 2] in case (ii), \(G\) is almost periodic. Therefore, \(G_a = G|_{\mathbb{J}_a} - G(a) \in \mathcal{WAP}(\mathbb{J}_a, X)\), as is \(\psi\), where

\[ \psi(t) = \int_a^t \varphi(u) \, du = F(t) - \int_a^t g(u) \, du. \]

Now the necessity follows from Theorem 7.

The sufficiency is easy to prove; we omit the proof.

The following theorem was shown in [5, Theorem 4.11].

**Theorem 14.** Let \(f = g + \varphi\), where \(g \in \mathcal{A}(\mathbb{R}, X)\) and \(\varphi \in \mathcal{C}_0(\mathbb{R}, X)\). Define \(F : \mathbb{R} \to X\) by \(F(t) = \int_0^t f(u) \, du\). Then \(F\) is in \(\mathcal{WAP}(\mathbb{R}, X)\) if and only if either

(i) \(F(\mathbb{R})\) is weakly relatively compact in \(X\) or

(ii) \(c_0 \not\subset X\) and \(F \in \mathcal{B}(\mathbb{R}, X)\),

and the following limits exist and satisfy

\[ \lim_{t \to -\infty} \int_0^t \varphi(u) \, du = \lim_{t \to -\infty} \int_0^t \varphi(u) \, du. \]

Since \(\mathcal{C}_0(\mathbb{R}, X) \subset \mathcal{WAP}_0(\mathbb{R}, X)\), Russ and Summers pointed out in [5, p. 33] that Theorem 14 does not answer the question: when is the integral of an \(f \in \mathcal{WAP}(\mathbb{R}, X)\) again in \(\mathcal{WAP}(\mathbb{R}, X)\)?

The next theorem answers this question. Before stating the theorem, we show that if \(f \in \mathcal{WAP}(\mathbb{R}, X)\) and \(x^* \in X^*\), then \(x^* f|_{\mathbb{J}_0} \in \mathcal{WAP}(\mathbb{J}_0)\), where \(\mathbb{J}_0 = [0, \infty)\). In fact, \(f = g + \varphi\), where \(g \in \mathcal{A}(\mathbb{R}, X)\) and \(\varphi \in \mathcal{WAP}_0(\mathbb{R}, X)\) [3, Theorem 4.11]. \(x^* g \in \mathcal{A}(\mathbb{R})\), and \(x^* \varphi \in \mathcal{WAP}_0(\mathbb{R})\). Since \(\mathcal{WAP}(\mathbb{R}) \subset \mathcal{WAP}_0(\mathbb{R})\) [1, 4.3.13], \(x^* \varphi \in \mathcal{WAP}_0(\mathbb{R})\). So \(x^* f|_{\mathbb{J}_0} \in \mathcal{WAP}_0(\mathbb{J}_0)\) and \(x^* f|_{\mathbb{J}_0} \in \mathcal{WAP}(\mathbb{J}_0)\).
Theorem 15. Let \( f \in \mathcal{WAP}(\mathbb{R}, X) \). Define \( F: \mathbb{R} \to X \) by \( F(t) = \int_0^t f(u) \, du \). Then \( F \) is in \( \mathcal{WAP}(\mathbb{R}, X) \) if and only if either

(i) \( F(\mathbb{R}) \) is weakly relatively compact in \( X \), or

(ii) \( c_0 \not\subset X \) and \( F \in \mathcal{B}(\mathbb{R}, X) \),

and there is a vector \( A \in X \) such that

\[
\psi - A \in \mathcal{WAP}_0(\mathbb{R}, X),
\]

where \( \psi : \mathbb{R} \to X \) is defined by \( \psi(t) = \int_0^t \varphi(u) \, du \).

Proof. Set \( G(t) = \int_0^t g(u) \, du \) for \( t \in \mathbb{R} \). With a proof similar to that for Lemma 12, we can show that if \( F(\mathbb{R}) \) is bounded (weakly relatively compact) in \( X \), then the same is true for \( G(\mathbb{R}) \).

Necessity. Since \( F \in \mathcal{WAP}(\mathbb{R}, X) \), \( F(\mathbb{R}) \) is weakly relatively compact, as is \( G(\mathbb{R}) \). So \( G \in \mathcal{A}(\mathbb{R}, X) \) [4, Theorem 2].

Now we show that (4) holds for some \( A \in X \). Since \( \psi \in \mathcal{WAP}(\mathbb{R}, X) \),

\[
\psi = G_1 + \Phi_1,
\]

where \( G_1 \in \mathcal{A}(\mathbb{R}, X) \) and \( \Phi_1 \in \mathcal{WAP}_0(\mathbb{R}, X) \). We claim that \( G_1 \) is a constant function. For, suppose that there are \( t_1, t_2 \in \mathbb{R} \) such that \( G_1(t_1) \neq G_1(t_2) \). The almost periodicity of \( G_1 \) makes it possible to assume that \( t_1 \) and \( t_2 \) are in \( J_0 \). Then we can find a \( x^* \in X^* \) such that \( x^*G_1(t_1) \neq x^*G_1(t_2) \).

Since \( \psi \in \mathcal{WAP}(\mathbb{R}, X) \), \( x^*\psi|_{J_0} \in \mathcal{AP}(J_0) \). By Theorem 7 there are an \( A_{x^*} \in \mathbb{C} \) and a \( \Phi_{x^*} \in \mathcal{AP}_0(J_0) \) such that

\[
x^*\psi(t) = \int_0^t x^*\varphi(u) \, du = A_{x^*} + \Phi_{x^*}(t) \quad (t \in J_0).
\]

Comparing (6) with (5) and using the uniqueness of the decomposition, we conclude that \( x^*G_1 \) is a constant function, a contradiction.

Sufficiency. It is obvious that \( \psi \in \mathcal{WAP}(\mathbb{R}, X) \) if (4) holds for some \( A \in X \). By the first paragraph in the proof and [4, Theorems 1 and 2], either (i) or (ii) is a sufficient condition for \( G \) to be in \( \mathcal{A}(\mathbb{R}, X) \), so \( F \in \mathcal{WAP}(\mathbb{R}, X) \).

Remark 16. Theorem 14 is a corollary of Theorem 15. In fact, in Theorem 14, \( A = \lim_{t \to \infty} \int_0^t \varphi(u) \, du \).

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Bibliography


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