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# HOMOGENEOUS PARTIAL DERIVATIVES OF RADIAL FUNCTIONS

RUSSELL LYONS AND KEVIN ZUMBRUN

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The following surprising identity for differentiation of radial functions by homogeneous partial differential operators appears to be new. For a polynomial $P(x_1, \ldots, x_n)$, write, as usual, $P(D) := P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Write $r := (x_1^2 + \cdots + x_n^2)^{1/2}$.

**Theorem.** *Let $P$ be a polynomial of $n$ variables homogeneous of degree $h$. Let $f$ be a function of one variable. Then*

$$P(D)f(r) = \sum_{k=0}^{\lfloor h/2 \rfloor} \frac{1}{2^k k!} \Delta^k P(x) \cdot \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k} f(r).$$

*Proof.* It suffices to prove this when $P$ is a monomial. The assertion for monomials is established by induction on the degree $h$ as follows. Note that for all $i$ and all $f$,

$$\frac{1}{x_i}\frac{\partial}{\partial x_i} f(r) = \frac{1}{r}\frac{\partial}{\partial r} f(r).$$

Also, by induction on $k \geq 0$, we have that

$$\Delta^k (x_i P(x)) = x_i \Delta^k P(x) + 2k \frac{\partial}{\partial x_i} \Delta^{k-1} P(x).$$

Therefore, if

$$P(D)f(r) = \sum_{k=0}^{\infty} \frac{1}{2^k k!} \Delta^k P(x) \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k} f(r),$$

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then for any $i$,

$$\begin{aligned}
(x_iP)(D)f(r) &= \frac{\partial}{\partial x_i}P(D)f(r) \\
&= \sum_{k=0}^{\infty}\left[\frac{1}{2^k k!}\frac{\partial}{\partial x_i}\left[\Delta^k P(x)\right]\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k} f(r)\right. \\
&\qquad \left. +\frac{1}{2^k k!}\Delta^k P(x)\cdot x_i\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k+1} f(r)\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{1}{2^k k!}\frac{1}{2(k+1)}\left\{\Delta^{k+1}(x_iP(x))-x_i\Delta^{k+1}P(x)\right\}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(h+1)-(k+1)} f(r)\right. \\
&\qquad \left. +\frac{1}{2^k k!}\Delta^k P(x)\cdot x_i\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(h+1)-k} f(r)\right] \\
&= \sum_{k=0}^{\infty}\frac{1}{2^k k!}\Delta^k(x_iP(x))\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(h+1)-k} f(r),
\end{aligned}$$

which is the desired induction step. □

Let $K[g](x_1, \dots, x_n) := r^{2-n}g(x_1/r^2, \dots, x_n/r^2)$ be the Kelvin transform of $g$. We were motivated to discover the above identity in order to understand the following result [1, p. 90], which is crucial to the fast algorithm of [2] for a symbolic solution of the Dirichlet problem on balls with polynomial data. Another derivation of the following corollary with more conceptual understanding comes from the representation theory of $\mathrm{SO}(n)$, the special orthogonal group acting on $\mathbf{R}^n$. It was known to Axler, Bourdon, and Ramey and to Lenard [3].

**Corollary.** *Let $P$ be a harmonic polynomial of $n$ variables homogeneous of degree $h$. Then*

$$P = c_h K[P(D)f_n(r)],$$

*where $f_n(r) := r^{2-n}$ for $n > 2$, $f_n(r) := \log r$ if $n = 2$, $c_h := \prod_{i=1}^{h}(4-n-2i)^{-1}$ for $n > 2$, and $c_h := (-1)^{h-1}/[2^{h-1}(h-1)!]$ if $n = 2$.*

*Proof.* Since $K$ is an involution, it suffices to show that

$$c_h^{-1}K[P] = P(D)f_n(r).$$

This is now an immediate consequence of the above theorem and the definition of $K$. □

## References


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2. S. Axler and W. Ramey, *Dirichlet problems with polynomial data* (to appear).
3. A. Lenard, *A remark on harmonic polynomials*, unpublished manuscript, 1992.


Department of Mathematics, Indiana University, Bloomington, Indiana 47405-5701