HOMOGENEOUS PARTIAL DERIVATIVES OF RADIAL FUNCTIONS

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The following surprising identity for differentiation of radial functions by homogeneous partial differential operators appears to be new. For a polynomial $P(x_1, \ldots, x_n)$, write, as usual, $P(D) := P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Write $r := (x_1^2 + \cdots + x_n^2)^{1/2}$.

**Theorem.** Let $P$ be a polynomial of $n$ variables homogeneous of degree $h$. Let $f$ be a function of one variable. Then

$$P(D)f(r) = \sum_{k=0}^{[h/2]} \frac{1}{2^k k!} \Delta^k P(x) \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{h-k} f(r).$$

**Proof.** It suffices to prove this when $P$ is a monomial. The assertion for monomials is established by induction on the degree $h$ as follows. Note that for all $i$ and all $f$,

$$\frac{1}{x_i} \frac{\partial}{\partial x_i} f(r) = \frac{1}{r} \frac{\partial}{\partial r} f(r).$$

Also, by induction on $k \geq 0$, we have that

$$\Delta^k (x_i P(x)) = x_i \Delta^k P(x) + 2k \frac{\partial}{\partial x_i} \Delta^{k-1} P(x).$$

Therefore, if

$$P(D)f(r) = \sum_{k=0}^{\infty} \frac{1}{2^k k!} \Delta^k P(x) \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{h-k} f(r),$$

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then for any \( i \),

\[
(x_i P)(D)f(r) = \frac{\partial}{\partial x_i} P(D)f(r)
\]

\[
= \sum_{k=0}^{\infty} \left[ \frac{1}{2^k k!} \frac{\partial}{\partial x_i} \left[ \Delta^k P(x) \right] \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{h-k} f(r) + \frac{1}{2^k k!} \Delta^k P(x) \cdot x_i \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{h-k+1} f(r) \right]
\]

\[
= \sum_{k=0}^{\infty} \left[ \frac{1}{2^k k!} \frac{1}{2(k+1)} \left\{ \Delta^{k+1}(x_i P(x)) - x_i \Delta^{k+1} P(x) \right\} \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{(h+1)-(k+1)} f(r) + \frac{1}{2^k k!} \Delta^k P(x) \cdot x_i \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{(h+1)-k} f(r) \right]
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{2^k k!} \Delta^k (x_i P(x)) \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^{(h+1)-k} f(r),
\]

which is the desired induction step. \( \square \)

Let \( K[g](x_1, \ldots, x_n) := r^{2-n} g(x_1/r^2, \ldots, x_n/r^2) \) be the Kelvin transform of \( g \). We were motivated to discover the above identity in order to understand the following result [1, p. 90], which is crucial to the fast algorithm of [2] for a symbolic solution of the Dirichlet problem on balls with polynomial data. Another derivation of the following corollary with more conceptual understanding comes from the representation theory of \( \text{SO}(n) \), the special orthogonal group acting on \( \mathbb{R}^n \). It was known to Axler, Bourdon, and Ramey and to Lenard [3].

**Corollary.** Let \( P \) be a harmonic polynomial of \( n \) variables homogeneous of degree \( h \). Then

\[
P = c_h K[P(D)f_n(r)],
\]

where \( f_n(r) := r^{2-n} \) for \( n > 2 \), \( f_n(r) := \log r \) if \( n = 2 \), \( c_h := \prod_{i=1}^{h} (4-n-2i)^{-1} \) for \( n > 2 \), and \( c_h := (-1)^{h-1}/[2^{h-1}(h-1)!] \) if \( n = 2 \).

**Proof.** Since \( K \) is an involution, it suffices to show that

\[
c_h^{-1} K[P] = P(D)f_n(r).
\]

This is now an immediate consequence of the above theorem and the definition of \( K \). \( \square \)

**References**


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