PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 1, May 1994

PATH PROPERTIES FOR l^∞-VALUED GAUSSIAN PROCESSES

MIKLÓS CSÖRGŐ, ZHENG-YAN LIN, AND QI-MAN SHAO

(Communicated by George Papanicolaou)

ABSTRACT. We prove moduli of continuity results for l^∞-valued Gaussian processes in general, as well as for l^∞-valued Ornstein-Uhlenbeck processes in particular.

1. INTRODUCTION

Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^\infty$ be a sequence of continuous Gaussian processes with stationary increments. Based on their Fernique type inequalities for Banach space-valued stochastic processes, Csáki and Csörgő [2, 1] and Csáki, Csörgő, and Shao [4, 5] studied the continuity and moduli of continuity sample path properties of $Y(\cdot) \in l^p$, $1 \le p \le 2$, featuring l^p-valued Ornstein-Uhlenbeck (O-U) processes as their main example of interest, as in Schmuland [17] as well for $1 \le p \le 2$. For extensions of results along these lines for $Y(\cdot) \in l^p$, $1 \le p < \infty$, we refer to Csörgő and Shao [7]. The study of infinite-dimensional O-U processes was initiated by Dawson [8], and they have been intensively studied in many papers since. Concerning their path continuity properties, in addition to the just mentioned papers, we refer to Csáki, Csörgő, Lin, and Révész [3], who studied infinite series of independent Ornstein-Uhlenbeck processes, Csörgő and Lin [6] on the l^2-norm squared O-U process, Fernique [9, 10], who gives necessary and sufficient conditions for the continuity of l^p-valued, $2 \le p < \infty$, O-U processes (cf. also Iscoe, Marcus, McDonald, Talagrand and Zinn [11], and Iscoe and McDonald [12] for $p = 2$, as well as Schmuland [13-17]).

In this paper we first investigate moduli of continuity path properties of $Y(\cdot)$ when it is an l^∞-valued process and then establish exact moduli of continuity for l^∞-valued O-U processes (cf. §§2 and 3, respectively).

Received by the editors May 27, 1991.
1991 *Mathematics Subject Classification*. Primary 60G15, 60G10, 60G17, 60F15; Secondary 60G07.

Key words and phrases. l^∞-valued Gaussian processes, l^∞-valued Ornstein-Uhlenbeck processes, moduli of continuity, quasi-increasing functions.

The first author's research was supported by an NSERC Canada grant at Carleton University. The second author's research was supported by the National Natural Science Foundation of China and by an NSERC Canada grant of M. Csörgő at Carleton University. The third author's research was supported by the Fok Yingtung Education Foundation and by an NSERC Canada Scientific Exchange Award at Carleton University.

©1994 American Mathematical Society
0002-9939/94 $1.00 + $.25 per page

2. l^∞-VALUED GAUSSIAN PROCESSES

As in our introduction, let

$$\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$$

be a sequence of continuous Gaussian processes with stationary increments. Throughout this section we assume that $EX_k(t) = 0$ for any t and every k and that $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$ are nondecreasing in h.

Just like in [4], *a function* $f(x)$ *will be called quasi-increasing on* (a, b) *if there exists a constant* $c > 0$ *such that*

$$f(x) \le cf(y) \quad \textit{for } a < x < y < b.$$

Put $\sigma^{*^2}(h) = \max_{k\ge 1} \sigma_k^2(h)$, and suppose that

$$\sigma^{*^2}(h)/h^\alpha \textit{ is quasi-increasing for some } \alpha > 0.$$

We assume, without loss of generality, that for every $k \ge 1$, $\sigma_k(h) > 0$ for $h > 0$. Let y_h be the solution of the equation

$$\sum_{k=1}^{\infty} (hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)} = h. \tag{2.1}$$

We state and prove our first result.

Theorem 1. *Suppose that there exist positive numbers* A *and* h_0 *such that*

$$\sum_{k=1}^{\infty} \sigma_k^A(h_0) < \infty. \tag{2.2}$$

Then

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \max_{k\ge 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \le 1 \quad \textit{a.s.} \tag{2.3}$$

If condition (2.2) *is replaced by conditions for* $0 < h \le h_0$ *so that*

$$\inf_{0<s\le h} \frac{\sigma^*(s)}{\sigma_k(s)} \ge c_1 \frac{\sigma^*(h)}{\sigma_k(h)} \quad \textit{for some } c_1 > 0 \textit{ and every } k \ge 1 \tag{2.4}$$

and

$$\sum_{k=1}^{\infty} \exp\left\{-\frac{\sigma^{*^2}(h)}{\sigma_k^2(h)} \log\frac{1}{h}\right\} < \infty, \tag{2.5}$$

then (2.3) *remains true with* $y_h = 1$. *If, in addition,* $X_k(\cdot)$, $k = 1, 2, \ldots$, *are independent and for* $0 \le t_1 < t_2 \le t_3 < t_4$,

$$E(X_k(t_2) - X_k(t_1))(X_k(t_4) - X_k(t_3)) \le 0, \tag{2.6}$$

then

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \max_{k\ge 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} = 1 \quad \textit{a.s.} \tag{2.7}$$

and

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1} \max_{k\ge 1} \frac{|X_k(t+h)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} = 1 \quad a.s. \tag{2.8}$$

Proof. At first, we list the following facts. We have $0 < y_h \le 1$, since

$$h = \sum_{k=1}^{\infty}(hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)} \ge hy_h .$$

Moreover, by elementary calculations, it is easy to see that condition (2.2) implies condition (2.5), and the latter condition guarantees that the solution of equation (2.1) exists and is unique. We have also the following property: There exists a constant $d > 0$, such that

$$\sigma^{*^2}(h) \ge dh^2 . \tag{2.9}$$

In fact, noting the definition of $\sigma_k^2(h)$, we have $\sigma_k^2(2h) \le 4\sigma_k^2(h)$. So, inductively,

$$\sigma_k^2(h) \ge \frac{1}{4}\sigma_k^2(2h) \ge \cdots \ge \frac{1}{4^l}\sigma_k^2(2^l h) \ge h^2\sigma_k^2\left(\frac{1}{2}\right) \quad \text{for } \frac{1}{2} \le 2^l h \le 1, \tag{2.10}$$

which implies (2.9). Furthermore, combining this with condition (2.2), we arrive at

$$\sum_{k=1}^{\infty}\left(\frac{\sigma_k^2(h)}{\sigma^{*^2}(h)}\right)^A \le \frac{c_2}{h^a} \quad \text{for } 0 < h \le h_0, \tag{2.11}$$

where $c_2 = d^{-A}\sum_{k=1}^{\infty}\sigma_k^{2A}(h_0)$, $a = 2A$.

As the first step, we prove that for a given $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \ge 1+\varepsilon\right\} \le Ch^\varepsilon y_h^\varepsilon . \tag{2.12}$$

It is easy to see that for $0 < s \le h$

$$\begin{aligned} &P\left\{\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}} \ge 1+\varepsilon\right\} \\ &\quad\le \sum_{k=1}^{\infty}\exp\left\{-(1+\varepsilon)^2\left(\log\frac{1}{hy_h}\right)\frac{\sigma^{*^2}(h)}{\sigma_k^2(h)}\right\} \\ &\quad= \sum_{k=1}^{\infty}(hy_h)^{(1+\varepsilon)^2\sigma^{*^2}(h)/\sigma_k^2(h)} \le h^{1+2\varepsilon}y_h^{2\varepsilon} \end{aligned} \tag{2.13}$$

by (2.1). For any positive number t and positive integer $r = r(\varepsilon)$, put $r_1 = h/2^r$ and $t_r = [t/r_1]r_1$. We have

$$\begin{aligned} |X_k(t+s)-X_k(t)| \le{}& |X_k((t+s)_r)-X_k(t_r)| \\ &+ \sum_{j=0}^{\infty}|X_k((t+s)_{r+j+1})-X_k((t+s)_{r+j})| \\ &+ \sum_{j=0}^{\infty}|X_k(t_{r+j+1})-X_k(t_{r+j})| . \end{aligned}$$

Then, from (2.13),

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h-r_1}\max_{k\ge 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1+\frac{\varepsilon}{2}\right\} \le \frac{4}{h}2^{2r}h^{1+\varepsilon}y_h^\varepsilon \le Ch^\varepsilon y_h^\varepsilon. \tag{2.14}$$

Since $\sigma^{*^2}(h)/h^\alpha$ is quasi-increasing, there exists a $c_0>0$ such that

$$\frac{\sigma^{*^2}(h)}{\sigma_k^2(2h/2^r)}\ge c_0 2^{\alpha(r-1)}\frac{\sigma^{*^2}(2h/2r^r)}{\sigma_k^2(2h/2r^r)}\ge c_0 2^{\alpha(r-1)}.$$

If condition (2.2) is satisfied, then, from (2.11), for r large enough and h small enough, similarly to (2.13), we get

$$\begin{aligned} p_1 &:= P\left\{\sup_{0\le t\le 1}\sup_{h-r_1\le s\le h}\max_{k\ge 1}\frac{|X_k((t+s)_r)-X_k(t_r)|}{\sigma^*(h)(2\log(1/hy_h)))^{1/2}}\ge\frac{\varepsilon}{4}\right\}\\ &\le \frac{2}{h}2^r\sum_{k=1}^\infty (hy_h)^{\varepsilon^2/16\cdot\sigma^{*^2}(h)/\sigma_k^2(2h/2^r)}\\ &\le 2^{r+1}h^{1+a}y_h\sum_{k=1}^\infty \exp\left\{-\left(\frac{\varepsilon^2}{16}\frac{\sigma^{*^2}(h)}{\sigma_k^2(2h/2^r)}-2-a\right)\log\frac{1}{hy_h}\right\}\\ &\le 2^{r+1}h^{1+a}y_h\sum_{k=1}^\infty \exp\left\{-\left(\frac{\sigma^{*^2}(h)}{\sigma_h^2(2h/2^r)}-1\right)A\right\}\\ &\le 2^{r+1}h^{1+a}y_h\sum_{k=1}^\infty\left(\frac{\sigma_k^2(2h/2^r)}{\sigma^{*^2}(h)}\right)^A\\ &\le c_2 2^{r+1}h^{1+a}y_h\left(\frac{\sigma^{*^2}(2h/2^r)}{\sigma^{*^2}(h)}\right)^A\left(\frac{2h}{2^r}\right)^{-a}\le Chy_h. \end{aligned}\tag{2.15}$$

If conditions (2.4) and (2.5) are satisfied, then on noting that $\sigma^{*^2}(h)/h^\alpha$ is quasi-increasing and taking r to be large enough and h to be small enough, we obtain

$$\begin{aligned} p_1 &\le \frac{2}{h}2^r\sum_{k=1}^\infty (hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)\cdot\varepsilon^2/16\cdot\sigma_k^2(h)/\sigma_k^2(2h/2^r)}\\ &\le \frac{2}{h}2^r\sum_{k=1}^\infty (hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)\cdot\varepsilon^2/16\cdot c_1\sigma^{*^2}(h)/\sigma^{*^2}(2h/2^r)}\\ &\le \frac{2}{h}2^r\sum_{k=1}^\infty (hy_h)^{2\sigma^{*^2}(h)/\sigma_k^2(h)}\le Chy_h. \end{aligned}\tag{2.16}$$

Furthermore, let $x_j^2=2B\log\frac{1}{hy_h}+2(2+a)j$, where $B=2/c_1$. If condition

(2.2) is satisfied, then similarly to (2.15),

$$p_2 := P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\sum_{j=0}^{\infty}|X_k((t+s)_{r+j+1}) - X_k((t+s)_{r+j})| \ge \sum_{j=0}^{\infty} x_j\sigma^*(h/2^{r+j+1})\right\}$$

$$\le \frac{2}{h}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^{2(r+j+1)}\exp\left\{-\frac{x_j^2}{2}\frac{\sigma^{*^2}(h/2^{r+j+1})}{\sigma_k^2(h/2^{r+j+1})}\right\}$$

$$\le 2\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^{2(r+j+1)}e^{-(2+a)j}h^{1+a}y_h \times \exp\left\{-\left(\frac{B\sigma^{*^2}(h/2^{r+j+1})}{\sigma_k^2(h/2^{r+j+1})} - 2 - a\right)\log\frac{1}{hy_h}\right\} \tag{2.17}$$

$$\le 8\, 2^{2r}h^{1+a}y_h\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^{2j}e^{-(2+a)j}\left(\frac{\sigma_k^2(h/2^{r+j+1})}{\sigma^{*^2}(h/2^{r+j+1})}\right)^A$$

$$\le 8c_2 2^{r(2+a)+a}hy_h\sum_{j=0}^{\infty} 2^{(2+a)j}e^{-(2+a)j} \le Chy_h.$$

Also, if conditions (2.4) and (2.5) are satisfied, then, similarly to (2.16) and (2.17),

$$p_2 \le \frac{2}{h}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^{2(r+j+1)}e^{-(2+a)j}\exp\left\{-B\left(\log\frac{1}{hy_h}\right)c_1\frac{\sigma^{*^2}(h)}{\sigma_k^2(h)}\right\} \le Chy_h. \tag{2.18}$$

Similarly, in both cases, we have

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\sum_{j=0}^{\infty}|X_k(t_{r+j+1}) - X_k(t_{r+j})| \ge \sum_{j=0}^{\infty} x_j\sigma^*(h/2^{r+j+1})\right\} \le Chy_h. \tag{2.19}$$

Moreover, since $\sigma^{*^2}(h/2^{r+j+1})/\sigma^{*^2}(h) \le c_0^{-1}2^{-\alpha(r+j+1)}$, we have

$$\sum_{j=0}^{\infty} x_j\sigma^*\left(\frac{h}{2^{r+j+1}}\right) = \sigma^*(h)\left\{(2Bc_0^{-1})^{1/2}\left(\log\frac{1}{hy_h}\right)^{1/2}\sum_{j=0}^{\infty}2^{-\alpha(r+j+1)/2} + c_0^{-1/2}\sum_{j=0}^{\infty}\frac{(2(2+a)j)^{1/2}}{2^{\alpha(r+j+1)/2}}\right\} \le \frac{\varepsilon}{8}\sigma^*(h)\left(\log\frac{1}{hy_h}\right)^{1/2}, \tag{2.20}$$

provided r is large enough. Combining these estimations, we get (2.12).

In order to prove (2.3), we use (2.9) again and obtain from (2.12)

$$P\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1+\varepsilon\right\} \le C(hy_h)^{\varepsilon/2}\left(\log\sigma^{*^{-1}}(h)\right)^{-2}, \tag{2.21}$$

provided h is small enough.

Let $\theta>1$. Define $A_i=\{h\colon \theta^{-i-1}\le\sigma^*(h)<\theta^{-i}\}$, $A_{ij}=\{h\colon \theta^{-j-1}\le hy_h<\theta^{-j},\ h\in A_i\}$, and $h_{ij}=\sup\{h\colon h\in A_{ij}\}$. Then

$$\begin{aligned}&\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\\
&\quad\le\limsup_{i\to\infty}\sup_{j\ge 0}\sup_{0\le t\le 1}\sup_{0\le s\le h_{ij}}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{-i-1}(2\log\theta^{j})^{1/2}}\\
&\quad\le\limsup_{i\to\infty}\sup_{j\ge 0}\sup_{0\le t\le 1}\sup_{0\le s\le h_{ij}}\max_{k\ge 1}\frac{\theta^2|X_k(t+s)-X_k(t)|}{\sigma^*(h_{ij})(2\log(1/(h_{ij}y_{h_{ij}})))^{1/2}}.\end{aligned}$$

Using (2.21), we have

$$\begin{aligned}&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\{\sup_{0\le t\le 1}\sup_{0\le s\le h_{ij}}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h_{ij})(2\log(1/(h_{ij}y_{h_{ij}})))^{1/2}}\ge 1+\varepsilon\right\}\\
&\quad\le C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(h_{ij}y_{h_{ij}})^{\varepsilon/2}\left(\log\sigma^{*^{-1}}(h_{ij})\right)^{-2}\le C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta^{-j\varepsilon/2}(i\log\theta)^{-2}<\infty.\end{aligned}$$

Hence (2.3) is proved by the Borel-Cantelli lemma if only we can show that $y_h=1$ under conditions (2.4) and (2.5). It suffices to prove that $\log\frac{1}{y_h}=o(\log\frac{1}{h})$ in this case. Consider the equation

$$\sum_{k=1}^{\infty}x^{c_1(\log 2)\sigma^{*^2}(1/2)/\sigma_k^2(1/2)}=1.$$

From condition (2.5), its solution $x=x_0>0$ exists. Then for any $0<h\le\frac{1}{2}$,

$$1=\frac{1}{h}\sum_{k=1}^{\infty}(hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)}\le\sum_{k=1}^{\infty}y^{c_1\sigma^{*^2}(1/2)/\sigma_k^2(1/2)}.$$

Consequently, $y_h\ge x_0^{\log 2}$ for $0<h\le\frac{1}{2}$, as required.

Next, we prove (2.8) under the assumption of independence and condition (2.6). Having (2.3), it is enough to show

$$\limsup_{h\downarrow 0}\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\ge 1\quad\text{a.s.} \tag{2.22}$$

To this end, it suffices to prove that for any $h_n\downarrow 0$, $0<\varepsilon<1$,

$$\lim_{n\to\infty}P\left\{\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h_n)-X_k(t)|}{\sigma^*(h_n)(2\log(1/(h_ny_{h_n})))^{1/2}}\ge 1-\varepsilon\right\}=1. \tag{2.23}$$

In fact, for n large enough, i.e. h_n small enough, we have

$$\begin{aligned}
&P\left\{\sup_{0\le t\le 1}\max_{k\ge 1}\frac{|X_k(t+h_n)-X_k(t)|}{\sigma^*(h_n)(2\log(1/h_n y_{h_n})))^{1/2}}<1-\varepsilon\right\}\\
&\le P\left\{\max_{0\le j\le 1/h_n}\max_{k\ge 1}\frac{X_k((j+1)h_n)-X_k(jh_n)}{\sigma^*(h_n)(2\log(1/(h_n y_{h_n})))^{1/2}}<1-\varepsilon\right\}\\
&\le \prod_{j=0}^{[1/h_n]}\prod_{k=1}^{\infty}P\left\{\frac{X_k((j+1)h_n)-X_k(jh_n)}{\sigma^*(h_n)(2\log(1/(h_n y_{h_n})))^{1/2}}<1-\varepsilon\right\}\\
&\le \prod_{j=0}^{[1/h_n]}\prod_{k=1}^{\infty}\left\{1-\exp\left\{-\frac{(1-\varepsilon)\sigma^{*^2}(h_n)}{\sigma_k^2(h_n)}\log\frac{1}{h_n y_{h_n}}\right\}\right\}\\
&= \prod_{j=0}^{[1/h_n]}\prod_{k=1}^{\infty}\left\{1-(h_n y_{h_n})^{(1-\varepsilon)\sigma^{*^2}(h_n)/\sigma_k^2(h_n)}\right\}\\
&\le \prod_{j=0}^{[1/h_n]}\exp\left\{-\sum_{k=1}^{\infty}(h_n y_{h_n})^{(1-\varepsilon)\sigma^{*^2}(h_n)/\sigma_k^2(h_n)}\right\}\\
&\le \exp(-h_n^{-\varepsilon})\to 0
\end{aligned}$$

as $n\to\infty$, where in our second inequality we used independence and Slepian's lemma. Hence (2.8) is proved.

Finally we prove (2.7). With the help of (2.3), it suffices to show

$$\liminf_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/hy_h)))^{1/2}}\ge 1\quad\text{a.s.}\tag{2.24}$$

Define A_{ij} as above, and put $h'_{ij}=\inf\{h\colon h\in A_{ij}\}$. Then

$$\begin{aligned}
&\liminf_{h\downarrow 0}\sup_{0\le t\le 1}\sup_{0\le s\le h}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\sigma^*(h)(2\log(1/(hy_h)))^{1/2}}\\
&\ge \liminf_{i\to\infty}\inf_{j\ge 0}\sup_{0\le t\le 1}\sup_{0\le s\le h'_{ij}}\max_{k\ge 1}\frac{|X_k(t+s)-X_k(t)|}{\theta^{-i}(2\log\theta^{j+1})^{1/2}}\\
&\ge \liminf_{i\to\infty}\inf_{j\ge 0}\max_{0\le l\le 1/h'_{ij}}\max_{k\ge 1}\frac{|X_k((l+1)h'_{ij})-X_k(lh'_{ij})|}{\theta^2\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}}.
\end{aligned}\tag{2.25}$$

Using Slepian's lemma again, we obtain

$$\begin{aligned}
&P\left\{\max_{0\le l\le 1/h'_{ij}}\max_{k\ge 1}\frac{(X_k((l+1)h'_{ij})-X_k(lh'_{ij}))}{\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}}\le 1-\varepsilon\right\}\\
&\le \prod_{l=0}^{[1/h'_{ij}]}\prod_{k=1}^{\infty}\left\{1-\exp\left\{-\frac{(1-\varepsilon)\sigma^{*^2}(h'_{ij})}{\sigma_k^2(h'_{ij})}\log\frac{1}{h'_{ij}y_{h'_{ij}}}\right\}\right\}\\
&\le \exp\left\{-(h'_{ij}y_{h'_{ij}})^{-\varepsilon}\right\}\le \exp\left\{-(h'_{ij}y_{h'_{ij}})^{-\varepsilon/2}\log\sigma^{*^{-1}}(h'_{ij})\right\}\\
&\le \exp\left\{-\theta^{j\varepsilon/2}(i\log\theta)\right\}.
\end{aligned}$$

The last but one inequality is due to (2.9). So we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\left\{\max_{0\le l\le 1/h'_{ij}} \max_{k\ge 1} \frac{|X_k((l+1)h'_{ij}) - X_k(lh'_{ij})|}{\sigma^*(h'_{ij})(2\log(1/(h'_{ij}y_{h'_{ij}})))^{1/2}} \le 1-\varepsilon\right\} < \infty. \tag{2.26}$$

Now (2.25) and (2.26) together imply (2.24). This completes the proof of Theorem 1.

Corollary 1. *Suppose that there exist constants* $0 < c_1 \le c_2 < \infty$, *a sequence of positive numbers* $\{a_k,\ k \ge 1\}$, *and a nondecreasing function* $\sigma(h)$ *such that*

$$c_1 a_k \sigma(h) \le \sigma_k(h) \le c_2 a_k \sigma(h) \tag{2.27}$$

for any $h > 0$ *and every* $k \ge 1$. *Moreover, suppose that* $\sigma^2(h)/h^\alpha$ *is quasi-increasing for some* $\alpha > 0$ *and*

$$\sum_{k=1}^{\infty} \exp\{-Aa^{*^2}/a_k^2\} < \infty \tag{2.28}$$

for some $A > 0$, *where* $a^* = \max_{k\ge 1} a_k$. *Then we have*

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \max_{k\ge 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log\frac{1}{h})^{1/2}} \le 1 \quad a.s. \tag{2.29}$$

If, in addition, $\{X_k(\cdot)\}_{k=1}^{\infty}$ *are independent and* (2.6) *is satisfied, then*

$$\lim_{h\downarrow 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \max_{k\ge 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2\log\frac{1}{h})^{1/2}} = 1 \quad a.s. \tag{2.30}$$

and

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1} \max_{k\ge 1} \frac{|X_k(t+h) - X_k(t)|}{\sigma^*(h)(2\log\frac{1}{h})^{1/2}} = 1 \quad a.s. \tag{2.31}$$

Proof. It is clear that (2.27) implies (2.4). Now the conclusion follows from Theorem 1 immediately.

The solution y_h of the equation (2.1) seems very difficult to find when condition (2.2) is satisfied but neither (2.4) nor (2.5) is satisfied. Below we give upper and lower estimators for y_h.

Lemma 1. *Suppose that* (2.2) *is satisfied. Then we have*

$$\log\frac{1}{y_h} \le \log\sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A \tag{2.32}$$

for any $0 < h \le e^{-A}$.

Proof. Note that

$$\begin{aligned} 1 &\le y_h \sum_{k=1}^{\infty} h^{\sigma^{*^2}(h)/\sigma_k^2(h)-1} = y_h \sum_{k=1}^{\infty} \exp\left\{-\left(\log\frac{1}{h}\right)\left(\frac{\sigma^{*^2}(h)}{\sigma_k^2(h)} - 1\right)\right\} \\ &\le y_h \sum_{k=1}^{\infty} \exp\left\{-\frac{A}{2}\left(\frac{\sigma^{*^2}(h)}{\sigma_k^2(h)} - 1\right)\right\} \le y_h \sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A, \end{aligned}$$

that is, we have (2.32).

Lemma 2. *Suppose that* (2.2) *is satisfied. Then we have*

$$\log \frac{1}{hy_h} \geq \frac{1}{\theta} \log \frac{\# A_\theta(h)}{h} \tag{2.33}$$

for any $h > 0$ *and* $\theta > 1$, *where* $A_\theta(h) = \{k\colon \sigma_k^2(h)/\sigma^{*^2}(h) \geq 1/\theta\}$.

Proof. Note that

$$h = \sum_{k=1}^{\infty} (hy_h)^{\sigma^{*^2}(h)/\sigma_k^2(h)} \geq \sum_{k \in A_\theta(h)} (hy_h)^\theta = \# A_\theta(h)(hy_h)^\theta,$$

which implies $1/hy_h \geq (\# A_\theta(h)/h)^{1/\theta}$, as desired.

Theorem 1 in combination with Lemmas 1 and 2 yields the next result.

Corollary 2. *Assuming that* (2.2) *is satisfied, we have*

$$\limsup_{h\downarrow 0} \sup_{0\leq t\leq 1} \sup_{0\leq s\leq h} \max_{k\geq 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)\{2\log(\frac{1}{h}\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A)\}^{1/2}} \leq 1 \quad a.s. \tag{2.34}$$

If, in addition, $\{X_k(\cdot)\}_{k=1}^{\infty}$ *are independent,* (2.6) *is satisfied, and*

$$\lim_{h\downarrow 0} \left(\log \frac{\# A_\theta(h)}{h}\right) \bigg/ \left(\log \frac{1}{h} \sum_{k=1}^{\infty} \left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A\right) = 1 \tag{2.35}$$

for any $\theta < 1$, *then*

$$\lim_{h\downarrow 0} \sup_{0\leq t\leq 1} \sup_{0\leq s\leq h} \max_{k\geq 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)\{2\log(\frac{1}{h}\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A)\}^{1/2}} = 1 \quad a.s. \tag{2.36}$$

and

$$\limsup_{h\downarrow 0} \sup_{0\leq t\leq 1} \max_{k\geq 1} \frac{|X_k(t+h) - X_k(t)|}{\sigma^*(h)\{2\log(\frac{1}{h}\sum_{k=1}^{\infty}(\sigma_k(h)/\sigma^*(h))^A)\}} = 1 \quad a.s. \tag{2.37}$$

In particular, if

$$\sum_{k=1}^{\infty} (\sigma_k(h)/\sigma^*(h))^A = o\left(\frac{1}{h}\right) \quad as\ h \to 0, \tag{2.38}$$

then, (2.30) *and* (2.31) *hold true.*

As an application of Corollary 2, we deal with the problem of moduli of continuity for l^∞-valued Ornstein-Uhlenbeck processes.

3. l^∞-VALUED O-U PROCESSES

Let $\{Y(t),\ -\infty < t < \infty\} = \{X_k(t),\ -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k, i.e., $X_k(\cdot)$ is a stationary, mean zero Gaussian process with

$$EX_k(s)X_k(t) = \frac{\gamma_k}{\lambda_k}\exp(-\lambda_k|t-s|), \qquad k = 1, 2, \ldots,$$

where $\gamma_k \geq 0$, $\lambda_k > 0$. Thus, we have

$$\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2 = \frac{2\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}).$$

It is clear that (2.6) is satisfied for $\{X_k(\cdot)\}$ (cf. [3, (4.2)]).

Theorem 2. *Suppose that* $\sigma^{*^2}(h)/h^\alpha$ *is quasi-increasing and that*

$$\sum_{k=1}^{\infty} \gamma_k^A < \infty \tag{3.1}$$

for some $A \geq 2$. *Then* (2.30) *and* (2.31) *hold true.*

Proof. It follows from (3.1) that

$$\sum_{k=1}^{\infty} \sigma_k^{2A}(h) \leq (2h)^A \sum_{k=1}^{\infty} \gamma_k^A .$$

On the other hand, it is easy to see that

$$\liminf_{h\downarrow 0} \sigma^{*^2}(h)/h > 0 ,$$

on recalling the proof of (2.9) and noting that $E(X_k(t+2h) - X_k(t+h)) \cdot (X_k(t+h) - X_k(t)) \leq 0$. So we have

$$\limsup_{h\downarrow 0} \sum_{k=1}^{\infty} (\sigma_k(h)/\sigma^*(h))^{2A} < \infty ,$$

that is, (2.38) is satisfied. Hence, (2.30) and (2.31) hold by Corollary 2.

The following sufficiency condition for $\sigma^{*^2}(h)/h^\alpha$ being quasi-increasing is due to Csáki, Csörgő, and Shao (cf. [4, Lemma 4.2]).

Lemma 3. *Assuming that* $\gamma_i/(1+\lambda_i)^{1-\alpha}$ *is quasi-decreasing for some* $0 < \alpha < 1$ *and that* $1+\lambda_{i+1} \leq c(1+\lambda_i)$ *for some* c *and every* $i \geq 1$, *we have that* $\sigma^{*^2}(h)/h^\alpha$ *is quasi-increasing on* $(0, \frac{1}{2})$.

Theorem 3. *Assuming* $\gamma_k = k^\alpha$ *and* $\lambda_k = k^\beta$ *with* $\beta > \alpha \geq 0$, *we have*

$$\lim_{h\downarrow 0} \sup_{0\leq t\leq 1} \sup_{0\leq s\leq h} \max_{k\geq 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(2(1+\frac{1}{\beta})\log\frac{1}{h})^{1/2}} = 1 \quad a.s. \tag{3.2}$$

and

$$\limsup_{h\downarrow 0} \sup_{0\leq t\leq 1} \max_{k\geq 1} \frac{|X_k(t+h) - X_k(t)|}{\sigma^*(h)(2(1+\frac{1}{\beta})\log\frac{1}{h})^{1/2}} = 1 \quad a.s. \tag{3.3}$$

Proof. At first, we consider the case of $\alpha > 0$. Let k_h be the integer such that $\sigma^*(h) = \sigma_{k_h}(h)$. Note that

$$\sigma_k^2(h) = 2k^{\alpha-\beta}(1 - e^{-k^\beta h}) .$$

Then there exist $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 h^{-1/\beta} \leq k_h \leq c_2 h^{-1/\beta} , \tag{3.4}$$

by observing the maximum value of the function $f(x) = x^{\alpha-\beta}(1 - e^{-x^\beta h})$. Hence

$$c_3 h^{(\beta-\alpha)A/\beta} \leq \sigma^{*^A}(h) \leq c_4 h^{(\beta-\alpha)A/\beta}$$

for some $0 < c_3 \le c_4 < \infty$. Moreover, taking A such that $(\beta - \alpha)A > 1$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \sigma_k^A(h) &\le \sum_{\lambda_k \le 1/h} k^{\alpha A} h^A + \sum_{\lambda_k > 1/h} k^{-(\beta-\alpha)A} \\ &\le \frac{1}{\alpha A + 1}\left(\frac{1}{h}\right)^{(\alpha A+1)/\beta} h^A + \frac{1}{(\beta-\alpha)A - 1}\left(\frac{1}{h}\right)^{(-(\beta-\alpha)A+1)/\beta} \\ &\le c_5 h^{((\beta-\alpha)A-1)/\beta}\end{aligned}$$

for some $c_5 > 0$. Similarly, there exists $c_6 > 0$ such that

$$\sum_{k=1}^{\infty} \sigma_k^A(h) \ge c_6 h^{((\beta-\alpha)A-1)/\beta}.$$

So we get

$$\log \sum_{k=1}^{\infty} \left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A \sim \frac{1}{\beta}\log\frac{1}{h}. \tag{3.5}$$

On the other hand, for $\theta > 1$ we have

$$\begin{aligned}\#A_\theta(h) &= \#\left\{k \colon \frac{\sigma_k^2(h)}{\sigma^{*2}(h)} \ge \frac{1}{\theta}\right\} \\ &\ge \#\left\{k \colon k \ge k_h,\ \frac{k^{\alpha-\beta}(1-e^{-k^\beta h})}{k_h^{\alpha-\beta}(1-e^{-k_h^\beta h})} \ge \frac{1}{\theta}\right\} \\ &\ge \#\left\{k \colon k \ge k_h,\ (k_h/k)^{\beta-\alpha} \ge \frac{1}{\theta}\right\} \\ &\ge \#\{k \colon k_h \le k \le \theta^{1/(\beta-\alpha)} k_h\} \\ &= (\theta^{1/(\beta-\alpha)} - 1)k_h \ge c_7 h^{-1/\beta}\end{aligned} \tag{3.6}$$

for some $c_7 > 0$.

If $\alpha = 0$, we have $\sigma^{*^2}(h) = \sigma_1^2(h) = 2(1 - e^{-h})$ and

$$\begin{aligned}\#A_\theta(h) &= \#\left\{k \colon \frac{k^{-\beta}(1-e^{-k^\beta h})}{(1-e^{-h})} \ge \frac{1}{\theta}\right\} \\ &\ge \#\left\{k \colon e^{-k^\beta h} k^{-\beta}(e^{k^\beta h} - 1) \ge \frac{h}{\theta}\right\} \\ &\ge \#\left\{k \colon e^{-k^\beta h} \ge \frac{1}{\theta}\right\} \ge \left(\frac{\log\theta}{h}\right)^{1/\beta}.\end{aligned} \tag{3.7}$$

From (3.5), (3.6), and (3.7) we conclude

$$\lim_{h\downarrow 0}(\log \# A_\theta(h)) \Big/ \left(\log \sum_{k=1}^{\infty}\left(\frac{\sigma_k(h)}{\sigma^*(h)}\right)^A\right) = 1$$

for any $\theta > 1$, and hence (2.35) is satisfied. Now, (3.2) and (3.3) follow from (3.5), (2.36), and (2.37).

Remark 1. If $\gamma_k = a^k$ and $\lambda_k = b^k$ with $b > a \geq 1$, then (2.31) and (2.32) hold true.

Remark 2. If $\gamma_k = (\log k)^\alpha$ and $\lambda_k = (\log k)^\beta$ with $\beta \geq \alpha + 1 \geq 1$, then there exist constants $0 < c_1 \leq c_2 < \infty$, such that

$$c_1 \leq \lim_{h \downarrow 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \max_{k \geq 1} \frac{|X_k(t+s) - X_k(t)|}{\sigma^*(h)(1/h)^{1/\beta}} \leq c_2 \quad \text{a.s.}$$

along the lines of the proof of Theorem 1.

References

1. E. Csáki, and M. Csörgő, *Fernique type inequalities for not necessarily Gaussian processes*, C. R. Math. Rep. Acad. Sci. Canada **12** (1990), 149- 154.
2. ——, *Inequalities for increments of stochastic processes and moduli of continuity*, Ann. Probab. **20** (1992), 1031–1052.
3. E. Csáki, M. Csörgő, Z. Y. Lin, and P. Révész, *On infinite series of independent Ornstein-Uhlenbeck processes*, Stochastic Process. Appl. **39** (1991), 25–44.
4. E. Csáki, M. Csörgő, and Q. M. Shao, *Fernique type inequalities and moduli of continuity for l^2-valued Ornstein-Uhlenbeck processes*, Ann. Inst. H. Poincaré Probab. Statist. **28** (1992), 479–517.
5. ——, *Moduli of continuity for l^p-valued Gaussian processes*, Tech. Rep. Ser. Lab. Res. Stat. Probab. No. 160, Carleton University-University of Ottawa.
6. M. Csörgő and Z. Y. Lin, *On moduli of continuity for Gaussian and l^2-norm squared processes generated by Ornstein-Uhlenbeck processes*, Canad. J. Math. **42** (1990), 141–158.
7. M. Csörgő and Q. M. Shao, *Strong limit theorems for large and small increments of l^p-valued Gaussian processes*, Ann. Probab. **21** (1993).
8. D. A. Dawson, *Stochastic evolution equations*, Math. Biosciences **15** (1972), 287–316.
9. X. Fernique, *La régularité des fonctions aléatoires d'Ornstein-Uhlenbeck á valeurs dans l^2*; le cas diagonal, C. R. Acad. Sci. Paris Sér I. Math. **309** (1989), 59–62.
10. ——, *Sur la régularité de certaines fonctions aléatoires d'Ornstein-Uhlenbeck*, Ann. Inst. H. Poincaré Probab. Statist. **26** (1990), 399–417.
11. I. Iscoe, M. Marcus, D. McDonald, M. Talagrand, and J. Zinn, *Continuity of l^2-valued Ornstein-Uhlenbeck processes*, Ann. Probab. **18** (1990), 68–91.
12. I. Iscoe, and D. McDonald, *Large deviations for l^2-valued Ornstein-Uhlenbeck processes*, Ann. Probab. **17** (1989), 58–73.
13. B. Schmuland, *Dirichlet forms and infinite dimensional Ornstein-Uhlenbeck processes*, Ph.D. Dissertation, Carleton University, Ottawa, 1987.
14. ——, *Some regularity results on infinite dimensional diffusions via Dirichlet forms*, Stochastic Anal. Appl. **6** (1988), 327–348.
15. ——, *Regularity of l^2-valued Ornstein-Uhlenbeck processes*, C. R. Math. Rep. Acad. Sci. Canada **10** (1988), 119–124.
16. ——, *Moduli of continuity for some Hilbert space valued Ornstein-Uhlenbeck processes*, C. R. Math. Rep. Acad. Sci. Canada **10** (1988), 197–202.
17. ——, *Sample path properties of l^p-valued Ornstein-Uhlenbeck processes*, Canad. Math. Bull. **33** (1990), 358–366.

(M. Csörgő) Department of Mathematics and statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

(Z.-Y. Lin and Q.-M. Shao) Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, People's Republic of China

(Q.-M. Shao) Department of Mathematics, National University of Singapore, Singapore 0511