PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 1, May 1994

# THE COMPACT NEIGHBORHOOD EXTENSION PROPERTY AND LOCAL EQUI-CONNECTEDNESS

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(Communicated by James West)

ABSTRACT. It is shown that any $\sigma$-compact metrizable space is an AR (ANR) if and only if it is (locally) equi-connected and has the compact (neighborhood) extension property.

## INTRODUCTION

In this paper, *all spaces are metrizable* unless otherwise stated. An AR (or ANR) means an absolute retract (or absolute neighborhood retract). A space $X$ is an AR or ANR if and only if $X$ is an AE (= absolute extensor) or ANE (= absolute neighborhood extensor) for metrizable spaces, respectively. It is said that a space $X$ has the *compact extension property* (CEP) [K] provided for any space $Y$ and any compactum $A$ in $Y$, each map $f: A \to X$ can be extended over $Y$. If $f: A \to X$ can be extended over a neighborhood of $A$ in $Y$, we say that $X$ has the *compact neighborhood extension property* (CENP). As is easily observed, $X$ has the CEP if $X$ is contractible and has the CNEP. Since any compactum can be embedded in a compact AR (e.g., the Hilbert cube), it follows that $X$ has the CNEP (or the CEP) if and only if $X$ is an ANE (or AE) for compacta. Hence every compactum with the CNEP (or the CEP) is an ANR (or an AR). J. van Mill [vM] constructed a separable metrizable space which has the CEP but is not an ANR. In [CM], this example was improved to be completely metrizable by using an example of Edwards, Walsh, and Dranishnikov [E, Wa, Dr] of dimension-raising cell-like maps. In the light of the example of van Mill, the following problem arises:

**Problem** [We, Problem ANR 13]. Is every $\sigma$-compact space $X$ with the CEP an ANR?

Received by the editors March 24, 1992.
1991 *Mathematics Subject Classification*. Primary 54C20, 54C55; Secondary 54D45, 54H11.
*Key words and phrases*. The compact (neighborhood) extension property, AR, ANR, (locally) equi-connected.
The first author was supported by the Royal Society. This paper was written while the first author visited the University College of North Wales as a Royal Society Fellow. He would like to thank the Royal Society for financial support, and he is very grateful to Professor R. Brown for his kind support and encouragement and to all members of the staff of the School of Mathematics for their hospitality.

In [Do], Dobrowolski showed that this problem is affirmative in case $X$ is a convex set in a linear topological space (cf. [BM]). A space $X$ is *locally equi-connected* (LEC) if there exists a neighborhood $U$ of the diagonal $\Delta X$ in $X^2$ and a map $\lambda : U \times \mathbf{I} \to X$ such that $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ for any $(x, y) \in U$ and $\lambda(x, x, t) = x$ for any $x \in X$ and $t \in \mathbf{I}$. In case $U = X^2$, $X$ is *equi-connected* (EC). Note that a space is EC if and only if it is contractible and LEC. Any linear topological space is EC. More generally, any contractible topological group $X$ is EC (e.g., see [Do, Example 2]). Furthermore a topological group $X$ is LEC if the unit of $X$ has a neighborhood which is contractible in $X$. We generalize their result as follows:

**Main Theorem.** *Any $\sigma$-compact space is an AR (ANR) if and only if it is EC (LEC) and has the CEP (CNEP).*

Note that an ANR is an AR if it is $C^\infty$ [Hu, Chapter III, Proposition 7.3] and that both the CEP and equi-connectedness imply $C^\infty$. Thus any $\sigma$-compact space is an AR if and only if it is LEC and has the CEP if and only if it is EC and has the CNEP.

We have the following corollary (cf. [Do, Proposition and Note 5]):

**Corollary.** *Any $\sigma$-compact topological group $X$ is an ANR if the unit of $X$ has a neighborhood which is contractible in $X$ and $X$ has the CNEP. Hence any contractible $\sigma$-compact topological group $X$ is an AR if $X$ has the CNEP.*

We introduce in §2 a weaker property than the local equi-connectedness and establish in §3 a more general result than the main theorem by using this property. This property gives a necessary and sufficient condition in order that a strongly countable dimensional $LC^\infty$ space is an ANR.

## 1. A characterization of ANR's

Let $X = (X, d)$ be a metric space with no isolated point. The nerve of an open cover $\mathscr{U}$ of $X$ is denoted by $N(\mathscr{U})$. The polyhedron $|N(\mathscr{U})|$ is considered with *the Whitehead topology*, not with the metric topology. Let $\mathscr{U} = (\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of $X$. It is said that $\mathscr{U}$ is a *zero* sequence if $\operatorname{mesh}\mathscr{U}_n \to 0$ as $n \to \infty$. We denote

$$TN(\mathscr{U}) = \bigcup_{n\in\mathbb{N}} N(\mathscr{U}_n \cup \mathscr{U}_{n+1}).$$

For each $\sigma \in TN(\mathscr{U})$, we define

$$n(\sigma) = \sup\{n \in \mathbb{N} | \sigma \in N(\mathscr{U}_n \cup \mathscr{U}_{n+1})\}.$$

Since $\operatorname{mesh}\mathscr{U}_n \to 0$ as $n \to \infty$, $\bigcap_{n\in\mathbb{N}}\mathscr{U}_n = \varnothing$, so $n(\sigma) \in \mathbb{N}$ is well defined. In case $(\mathscr{U}_n)_{n\in\mathbb{N}}$ is nested, i.e., $\mathscr{U}_1 \supset \mathscr{U}_2 \supset \cdots$, we have $TN(\mathscr{U}) = N(\mathscr{U}_1)$ and

$$m(\sigma) = \sup\{n \in \mathbb{N} \mid \sigma \in N(\mathscr{U}_n)\} = n(\sigma) + 1 \quad \forall \sigma \in \mathscr{U}_2.$$

The following is an alteration of the characterization of ANR's of [N, Theorem 1-1]:

**Theorem 1.** *A metric space $X$ with no isolated point is an ANR if and only if $X$ has a zero sequence $\mathscr{U} = (\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers with a map $g\colon |TN(\mathscr{U})| \to X$*

*such that* $g(U) \in U$ *for each* $U \in \mathscr{U}$ *and* $\operatorname{diam} g(\sigma_i) \to 0$ *if* $n(\sigma_i) \to \infty$ *for any sequence* $(\sigma_i)_{i \in \mathbb{N}}$ *in* $TN(\mathscr{U})$.

*Proof.* First note that the notation $K \prec \{\mathscr{U}_n\}$ in [N] means that $K$ is a subcomplex of $TN(\mathscr{U})$. Obviously the condition (ii) of [N, Theorem 1-1] implies the above condition. Let $\mathscr{U}$ and $g$ be as in the condition, $K$ a subcomplex of $TN(\mathscr{U})$, and $f\colon K^{(0)} \to X$ a selection (i.e., $f(U) \in U$ for each $U \in K^{(0)}$). For any sequence $(\sigma_i)_{i \in \mathbb{N}}$ in $K$ with $n(\sigma_i) \to \infty$,

$$\begin{aligned}\delta(\sigma_i) &= \sup\{d(g(x), f(V)) \mid x \in \sigma_i,\ V \in \sigma_i^{(0)}\} \\ &\leqslant \operatorname{diam} g(\sigma_i) + \max\{\operatorname{mesh} \mathscr{U}_{n(\sigma_i)}, \operatorname{mesh} \mathscr{U}_{n(\sigma_i)+1}\} \to 0.\end{aligned}$$

Hence the above condition implies the condition (iii) of [N, Theorem 1-1]. Thus the theorem follows from [N, Theorem 1-1]. □

## 2. Property $(S)$

Here we introduce some property of LEC metric spaces. A sequence $\mathscr{U} = (\mathscr{U}_n)_{n \in \mathbb{N}}$ of families of open sets in a metric space $X$ is called a *covering sequence* in $X$ if $\operatorname{mesh} \mathscr{U}_n \to 0$ as $n \to \infty$ and each $\mathscr{U}_n^\infty = \bigcup_{i \geqslant n} \mathscr{U}_i$ covers $X$, namely, $\mathscr{U}^\infty = (\mathscr{U}_n^\infty)_{n \in \mathbb{N}}$ is a zero sequence of open covers of $X$. A *refinement* of a covering sequence $(\mathscr{U}_n)_{n \in \mathbb{N}}$ of $X$ is a covering sequence $(\mathscr{V}_n)_{n \in \mathbb{N}}$ of $X$ such that each $\mathscr{V}_n$ refines $\mathscr{U}_n$.

**Lemma 1.** *Each covering sequence* $(\mathscr{U}_n)_{n \in \mathbb{N}}$ *in* $X$ *has a refinement* $(\mathscr{V}_n)_{n \in \mathbb{N}}$ *such that each* $\mathscr{V}_n$ *is locally finite in* $X$.

*Proof.* Let $\mathscr{W}_n$ be locally finite open covers of $X$ $(n \in \mathbb{N})$ such that each $\mathscr{W}_n$ is a star-refinement of $\mathscr{U}_n^\infty$ and also a refinement of $\mathscr{W}_i$ for all $i < n$. For each $n \in \mathbb{N}$, we define

$$\mathscr{V}_n = \{W \in \mathscr{W}_n \mid \exists U \in \mathscr{U}_n \text{ s.t. } W \subset U\} \subset \mathscr{W}_n.$$

Then each $\mathscr{V}_n^\infty$ covers $X$. In fact, each $x \in X$ is contained in some $W \in \mathscr{W}_n$, and the star $\operatorname{st}(W, \mathscr{W}_n)$ is contained in some $U \in \mathscr{U}_m$ $(m \geqslant n)$. Choose $V \in \mathscr{W}_m$ so that $x \in V$. Since $\mathscr{W}_m$ refines $\mathscr{W}_n$, $V \subset \operatorname{st}(W, \mathscr{W}_n) \subset U$, which implies $V \in \mathscr{V}_m \subset \mathscr{V}_n^\infty$. Thus $(\mathscr{V}_n)_{n \in \mathbb{N}}$ is a covering sequence in $X$, which is the desired refinement of $(\mathscr{U}_n)_{n \in \mathbb{N}}$. □

**Definition of Property** $(S)$. A metric space $X$ with no isolated point has *Property* $(S)$ provided the following conditions hold:

(1) for any sequence $\{G_n \mid n \in \mathbb{N}\}$ of open sets in $X$ such that $\bigcup_{i \geqslant n} G_i = X$, there exists a covering sequence $(\mathscr{V}_n)_{n \in \mathbb{N}}$ such that $\bigcup \mathscr{V}_n \subset G_n$ for each $n \in \mathbb{N}$; and

(2) for any refinement $(\mathscr{W}_n)_{n \in \mathbb{N}}$ of $(\mathscr{V}_n)_{n \in \mathbb{N}}$ such that $\mathscr{W}_n \cap \mathscr{W}_m = \varnothing$ if $n \neq m$ and for any selection $f_0\colon \mathscr{W}_1^\infty \to X$ (i.e., $f(W) \in W$ for each $W \in \mathscr{W}_1^\infty$), if $f_0$ extends to a map $f\colon |\bigoplus_{n \in \mathbb{N}} N(\mathscr{W}_n)| \to X$ so that $f(\sigma) \cup W_1 \cup \cdots \cup W_k$ is contained in some $V \in \mathscr{V}_n$ for each $\sigma = \langle W_1, \ldots, W_k \rangle \in N(\mathscr{W}_n)$, then $f_0$ extends to a map $\tilde{f}\colon |N(\mathscr{W}_1^\infty)| \to X$ so that $\operatorname{diam} \tilde{f}(\sigma_i) \to 0$ as $n(\sigma_i) \to \infty$ for any sequence $(\sigma_i)_{i \in \mathbb{N}}$ of simplexes of $N(\mathscr{W}_1^\infty)$.

*Remark.* In the above definition, it is necessary to assume that $X$ has no isolated point. In fact, as is easily observed, if $X$ has an isolated point, then $X$

does not have a covering sequence $(\mathscr{W}_n)_{n\in\mathbb{N}}$ as described above. In case $X$ has no isolated point, we have the following:

**Lemma 2.** *If $X$ has no isolated point, then any covering sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of $X$ has a refinement $(\mathscr{W}_n)_{n\in\mathbb{N}}$ such that $\mathscr{W}_n \cap \mathscr{W}_m = \varnothing$ if $n \neq m$.*

*Proof.* By Lemma 1, we may assume that each $\mathscr{V}_n$ is locally finite in $X$. Let $\mathscr{W}_1 = \mathscr{V}_1$, and assume that $\mathscr{W}_1, \dots, \mathscr{W}_{n-1}$ are defined so that each $\mathscr{W}_i$ is locally finite in $X$ and refines $\mathscr{V}_i$, $\bigcup \mathscr{W}_i = \bigcup \mathscr{V}_i$, and $\mathscr{W}_i \cap \mathscr{W}_j = \varnothing$ if $i \neq j$. Since $X$ has no isolated point and $\bigcup_{i<n} \mathscr{W}_i$ is locally finite in $X$, each $V \in \mathscr{V}_n \cap \bigcup_{i<n} \mathscr{W}_i$ contains infinite many points $x$ such that $V \setminus \{x\} \notin \bigcup_{i<n} \mathscr{W}_i$. Let $x_V$ and $y_V$ be such distinct points of $V$. We define

$$\mathscr{W}_n = \left(\mathscr{V}_n \setminus \textstyle\bigcup_{i<n} \mathscr{W}_i\right) \cup \left\{V \setminus \{x_V\},\, V \setminus \{y_V\} \;\middle|\; V \in \mathscr{V}_n \cap \textstyle\bigcup_{i<n} \mathscr{W}_i\right\}.$$

Then $\mathscr{W}_1, \dots, \mathscr{W}_n$ satisfy the above inductive assumption. By induction, we can obtain the desired refinement $(\mathscr{W}_n)_{n\in\mathbb{N}}$ of $(\mathscr{V}_n)_{n\in\mathbb{N}}$. □

**Theorem 2.** *If a metric space $X$ is LEC and has no isolated point, then $X$ has Property $(S)$.*

*Proof.* Since $X$ is LEC, $X$ has a local equi-connecting map $\lambda\colon U \times \mathbf{I} \to X$, where $U$ is an open neighborhood of the diagonal $\Delta X$ in $X^2$. Then $X$ has a sequence of open covers $(\mathscr{V}_n)_{n\in\mathbb{N}\cup\{0\}}$ such that $\operatorname{mesh} \mathscr{V}_n < 2^{-n}$, $\operatorname{st}(x, \mathscr{V}_n)^2 \subset U$ for each $x \in X$, and

$$\left\{\lambda(\operatorname{st}(x, \mathscr{V}_n)^2 \times \mathbf{I}) \;\middle|\; x \in X\right\} \prec \mathscr{V}_{n-1}.$$

For any sequence $\{G_n \mid n \in \mathbb{N}\}$ of open sets in $X$ such that $\bigcup_{i\geqslant n} G_i = X$, let $\mathscr{V}_n' = \{V \cap G_n \mid V \in \mathscr{V}_n\}$. Then it is easy to see that $(\mathscr{V}_n')_{n\in\mathbb{N}}$ is a covering sequence of $X$.

Let $(\mathscr{W}_n)_{n\in\mathbb{N}}$ be a refinement of $(\mathscr{V}_n')_{n\in\mathbb{N}}$ such that $\mathscr{W}_n \cap \mathscr{W}_m = \varnothing$ if $n \neq m$, and let $f\colon |\bigoplus_{n\in\mathbb{N}} N(\mathscr{W}_n)| \to X$ be a map such that $f|\mathscr{W}_1^\infty$ is a selection and

(0) $\forall n \in \mathbb{N}, \forall \sigma = \langle W_1, \dots, W_k\rangle \in N(\mathscr{W}_n), \exists V \in \mathscr{V}_n$ s.t.

$$f(\sigma) \cup W_1 \cup \dots \cup W_k \subset V.$$

Let $L_0 = \bigoplus_{n\in\mathbb{N}} N(\mathscr{W}_n)$, and for each $k \in \mathbb{N}$, let

$$L_k = L_{k-1} \cup \left\{\sigma'\sigma'' \in N(\mathscr{W}_1^\infty) \;\middle|\; \sigma' \in L_0,\, \sigma'' \in L_{k-1}\right\}.$$

Then clearly $N(\mathscr{W}_1^\infty) = \bigcup_{k\in\mathbb{N}\cup\{0\}} L_k$. Recall $m(\sigma) = \sup\{n \in \mathbb{N} \mid \sigma \in N(\mathscr{W}_n^\infty)\}$ for each $\sigma \in N(\mathscr{W}_1^\infty)$. Then each $\sigma \in L_k \setminus L_{k-1}$ can be uniquely written as the join:

$$(*) \qquad \sigma = \sigma'\sigma'', \quad \sigma' \in L_0, \quad \sigma'' \in L_{k-1}, \quad m(\sigma') < m(\sigma'').$$

Then note $m(\sigma) = m(\sigma')$. By induction, we can construct maps $f_k\colon |L_k| \to X$, $k \in \mathbb{N}$, such that

$(k)_1$ $\forall \sigma = \langle W_1, \dots, W_n\rangle \in L_k, \exists V \in \mathscr{V}_{m(\sigma)-1}$ s.t.

$$f(\sigma) \cup W_1 \cup \dots \cup W_n \subset V,$$

$(k)_2$ if $\sigma'\sigma'' \in L_k$ and $m(\sigma') < m(\sigma'')$, then $\forall y \in \sigma' \in L_0, \forall z \in \sigma'' \in L_{k-1}, \forall t \in \mathbf{I}$,

$$f_k((1-t)y + tz) = \lambda(f(y), f_{k-1}(z), t),$$

where $f_0 = f$. In fact, write $\sigma = \langle W_1, \dots, W_n \rangle \in L_k \setminus L_{k-1}$ as $(*)$, where $\sigma' = \langle W_1, \dots, W_m \rangle$ and $\sigma'' = \langle W_{m+1}, \dots, W_n \rangle$. Since $\sigma \in N(\mathscr{W}_1^\infty)$, there is $x \in W_1 \cap \dots \cap W_n$. Since $f$ and $f_{k-1}$ satisfy (0) and $(k-1)_1$, respectively, we have $V', V'' \in \mathscr{V}_{m(\sigma)}$ such that

$$f(\sigma') \cup W_1 \cup \dots \cup W_{n'} \subset V',$$
$$f_{k-1}(\sigma'') \cup W_{n'+1} \cup \dots \cup W_n \subset V''.$$

Then $x \in V' \cap V''$, so

$$f(\sigma') \times f_{k-1}(\sigma'') \subset V' \times V'' \subset \mathrm{st}(x, \mathscr{V}_{m(\sigma)})^2 \subset U.$$

Hence $f_{k-1}|\sigma \cap |N(L_{k-1})|$ can be extended over $\sigma$ by $(k)_2$ because it satisfies $(k-1)_2$. Moreover, in the above,

$$W_1 \cup \dots \cup W_n \subset V' \cup V'' \subset \mathrm{st}(x, \mathscr{V}_{m(\sigma)}).$$

Therefore, such extensions give $f_k$ which satisfies $(k)_1$. Since $f_k||L_{k-1}| = f_{k-1}$ for each $n \in \mathbb{N}$, the desired extension $\tilde{f}\colon |N(\mathscr{W}_1^\infty)| \to X$ can be defined by $\tilde{f}||L_k| = f_k$ for each $k \in \mathbb{N}$. In fact, $\operatorname{diam} \tilde{f}(\sigma) \leqslant \operatorname{mesh} \mathscr{V}_{m(\sigma)-1} < 2^{-m(\sigma)+1}$. □

## 3. Proof of the main theorem

We extend the concept of CNEP as follows: Let $\mathscr{C}$ be a class of spaces. A space $X$ has the *$\mathscr{C}$-neighborhood extension property* ($\mathscr{C}$-NEP) provided for any space $Y$ and any closed set $A$ in $Y$ such that $A \in \mathscr{C}$, each map $f\colon A \to X$ can be extended over a neighborhood of $A$ in $Y$. Similarly the *$\mathscr{C}$-extension property* ($\mathscr{C}$-EP) is defined. It is said that $\mathscr{C}$ is *topological* if any space homeomorphic to a space in $\mathscr{C}$ is also contained in $\mathscr{C}$. And $\mathscr{C}$ is *closed additive* provided if $X, Y \in \mathscr{C}$ and both $X$ and $Y$ are closed in $X \cup Y$, then $X \cup Y \in \mathscr{C}$. We denote

$$\mathscr{C}_\sigma = \{X = \textstyle\bigcup_{n \in \mathbb{N}} X_n \mid \forall X_n \in \mathscr{C} \text{ is closed in } X\}.$$

The main theorem easily follows from Theorem 2 and the following:

**Theorem 3.** *Let $\mathscr{C}$ be a closed additive topological class of metrizable spaces and $X \in \mathscr{C}_\sigma$ with no isolated point. Then $X$ is an ANR if and only if $X$ has the $\mathscr{C}$-NEP and Property $(S)$ with respect to some/any metric compatible with the topology.*

*Proof*. Since any ANR is LEC and has the $\mathscr{C}$-NEP, the "only if" part follows from Theorem 2. We show the "if" part. We can assume that $X$ is a closed subset of a normed linear space $E$ by Arens-Eells's Embedding Theorem [AE] (cf. [T]). Since $X \in \mathscr{C}_\sigma$, we can write $X = \bigcup_{n \in \mathbb{N}} X_n$, where $X_1 \subset X_2 \subset \cdots$ is a tower of closed sets in $X$ and $X_n \in \mathscr{C}$ for each $n \in \mathbb{N}$. By the $\mathscr{C}$-NEP, we have open neighborhoods $G_n$ of $X_n$ in $E$ and maps $g_n\colon G_n \to X$ such that $g_n|X_n = \mathrm{id}$. By using Property $(S)$, we have a covering sequence $(\mathscr{V}_n)_{n \in \mathbb{N}}$ of $X$ which satisfies the condition for $(G_n \cap X)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, let $\mathscr{W}_n$ be an open cover of $\bigcup \mathscr{V}_n$ such that

(i) $\forall x \in \bigcup \mathscr{V}_n$, $\exists V \in \mathscr{V}_n$ s.t.
$\operatorname{conv} \mathrm{st}(x, \mathscr{W}_n) \subset G_n \cap g_n^{-1}(V)$ and $X \cap \operatorname{conv} \mathrm{st}(x, \mathscr{W}_n) \subset V$;

(ii) $n \neq m \Rightarrow \mathscr{W}_n \cap \mathscr{W}_m = \varnothing$, where $\operatorname{conv} A$ denotes the convex hull of $A$ in $E$. Then $\mathscr{W} = (\mathscr{W}_n)_{n\in\mathbb{N}}$ is a covering sequence of $X$, namely, $\mathscr{W}^\infty = (\mathscr{W}_n^\infty)_{n\in\mathbb{N}}$ is a zero sequence of open covers of $X$. Let $f_0\colon \mathscr{W}_1^\infty \to X$ be a selection. By (i), each $f_0|\mathscr{W}_n$ can be extended to the map $f_n\colon |N(\mathscr{W}_n)| \to G_n$ which is linear on each simplex of $N(\mathscr{W}_n)$. We define a map $f\colon |\bigoplus_{n\in\mathbb{N}} N(\mathscr{W}_n)| \to X$ by $f||N(\mathscr{W}_n)| = g_n f_n$ for each $n \in \mathbb{N}$. For each $\sigma = \langle W_1, \dots, W_k\rangle \in N(\mathscr{W}_n)$, choose $x \in W_1 \cap \dots \cap W_k$. Then we have $V \in \mathscr{V}_n$ by (i) such that

$$f(\sigma) \cup W_1 \cup \dots \cup W_k \subset g_n(\operatorname{conv} \operatorname{st}(x, \mathscr{W}_n)) \cup \operatorname{conv} \operatorname{st}(x, \mathscr{W}_n) \subset V.$$

Hence $f$ extends to a map $\tilde{f}\colon |N(\mathscr{W}_1^\infty)| \to X$ such that $\operatorname{diam} \tilde{f}(\sigma_i) \to 0$ as $n(\sigma_i) \to \infty$. Since $TN(\mathscr{W}^\infty) = N(\mathscr{W}_1^\infty)$, $X$ is an ANR by Theorem 1. □

*Remark.* In the above theorem, the $\mathscr{C}$-NEP of $X$ can be replaced by the condition that $X = \bigcup_{n\in\mathbb{N}} X'_n$, where $X'_1 \subset X'_2 \subset \cdots$ are closed in $X$ and have the $\mathscr{C}$-NEP. In fact, we can take the tower $(X_n)_{n\in\mathbb{N}}$ so that $X_n \subset X'_n$ for each $n \in \mathbb{N}$. Then the above proof goes through.

In the above remark, when $\mathscr{C}$ is the class of all metrizable spaces, we have the following:

**Corollary 1.** *Let $X$ be a metric space with no isolated point such that $X = \bigcup_{n\in\mathbb{N}} X_n$, where $X_1 \subset X_2 \subset \cdots$ are ANR's and closed in $X$. Then $X$ is an ANR if $X$ has Property $(S)$.*

**Corollary 1′.** *Let $X = \bigcup_{n\in\mathbb{N}} X_n$, where $X_1 \subset X_2 \subset \cdots$ are ANR's and closed in $X$. Then $X$ is an ANR if $X$ is LEC.*

Since a space $X$ is $LC^\infty$, $X$ has the $\mathscr{FD}$-NEP [Hu, Chapter V, §2], where $\mathscr{FD}$ is the class of finite-dimensional spaces. A space is *strongly countable dimensional* if it is a countable union of finite-dimensional closed sets. By Theorem 3, we have the following:

**Corollary 2.** *Any strongly countable dimensional metric space $X$ with no isolated point is an ANR if and only if $X$ is $LC^\infty$ and has Property $(S)$.*

*Remarks.* In the above corollary, Property $(S)$ is necessary. In fact, there exists an $LC^\infty$ continuum which is a countable union of finite-dimensional compacta but not locally contractible and so not an ANR (cf. [Hu, Chapter V, §9]).

It is known that a countable dimensional locally contractible space is an ANR (cf. [Ha] and [G]). Then it is natural to ask whether a locally contractible metric space with no isolated point has Property $(S)$ or not. However the classical Borsuk example is a locally contractible continuum which is the union of an increasing sequence of compact ANR's (cf. [Hu, Chapter V, §8]) but it does not have Property $(S)$ by Corollary 1. This is pointed out by the referee. The authors would like to thank him for this remark.

Finally we mention that the main theorem can also be proved by using the result of J. van der Bijl and J. van Mill [BM, Theorem 2.1]. This is rather easy, but the approach in this paper yields such interesting corollaries as those above.

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