PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 2, June 1994

FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN SUBSYMMETRIZABLE SPACES

NGUYEN HUY VIET

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Three fixed point theorems for multi-valued mappings in symmetric spaces with closure operator are proved. As corollaries of these theorems we obtain the existence of fixed points of mappings in compact metric spaces. Another corollary of the last theorem extends a condition of Kannan's theorem.

1. INTRODUCTION

The fixed point theorems for convex-valued mappings in convex sets of Banach spaces are well known. These theorems stem from Kakutani's works, and have since been extensively generalized. The analysis of the logical structure of the arguments in their proofs shows that the conditions of convexity and of spaces being Banach are not necessary. In fact the structure used requires only a weak metric (symmetric) and a closure operator with several properties. This idea was formed by Liepins in [1, 2] for single-valued mappings. In this paper these fixed point results are extended to the case of multi-valued mappings. Also, Corollary 1 of Theorem 3 extends Theorem 2 of [3].

2. NOTATION AND DEFINITIONS

Let X be a set and 2^X be the set of all subsets of X. A mapping $S: 2^X \to 2^X$ is called a closure operator iff it satisfies the following conditions: $A, B \in 2^X$ and $A \subset B \Rightarrow S(A) \subset S(B)$; $A \subset S(A)$ and $S(S(A)) = S(A)$. A set A is called S-closed iff $A = S(A)$.

The topological closure in topological spaces and the convex closure in linear spaces provide examples of closure operators.

Any family of subsets $\mathscr{A} \subset 2^X$ invariant with respect to the set intersection gives a closure operator by the formula:

$$S(A) = \bigcap\{B | B \supset A, \ B \in \mathscr{A}\}.$$

Let $F: 2^X \to 2^X$ be a multi-valued mapping and $\mathscr{A}$ be the family of all subsets A of X for which $F(A) \subset A$. Then $\mathscr{A}$ is invariant with respect to the set intersection and so it induces a closure operator which is denoted by S_F.

Received by the editors June 25, 1989 and, in revised form, January 4, 1990.
1991 *Mathematics Subject Classification*. Primary 46N99, 46S30.

Let S_1 and S_2 be closure operators. We define $S' = \inf\{S_1, S_2\}$ by the formula:

$$S'(A) = \bigcap (B | B \supset A, \ B \text{ is } S_1\text{- and } S_2\text{-closed}).$$

It is clear that S' is a closure operator.

A set H is called S-compact iff the condition "H has an empty intersection with a family of S-closed subsets" always implies "H has an empty intersection with some finite subfamily".

The operator S is called algebraic iff for every $A \in 2^X$ and $x \in S(A)$ there exists a finite subset $K \subset A$ such that $x \in S(K)$.

A mapping $t: X \times X \to R_+$ is called a symmetric on X iff $t(x, y) = 0 \iff x = y$ and $t(x, y) = t(y, x)$ for all $x, y \in X$.

A set X equipped with a symmetric t is called a symmetric space and denoted by (X, t).

Denote:

$$D(A, B) = \sup\{t(x, y) | x \in A, \ y \in B\},$$
$$D(A, x) = D(A, \{x\}), \quad \operatorname{diam} A = D(A, A),$$
$$B(A, r) = \{x \in X | D(A, x) \leq r\}, \qquad B(x, r) = B(\{x\}, r).$$

A topological space X is called subsymmetrizable iff there exists a continuous symmetric on X.

Let S be a closure operator on X. A symmetric t is called S-continuous iff $\operatorname{diam} A = \operatorname{diam} S(A)$ for every $A \in 2^X$.

Let $F: X \to 2^X$ be a multi-valued mapping. A sequence x_n, $n = 0, 1, 2, \ldots$, is called an orbit of F at x iff $x_0 = x$ and $x_{n+1} \in F(x_n)$ for $n = 0, 1, 2, \ldots$.

Let X, Y be topological spaces and let $F: X \to 2^Y$ be a multi-valued mapping. F is called lower-semi-continuous (l.s.c.) at x iff for every open set $G \subset Y$ such that $F(x) \cap G \neq \varnothing$ there exists a neighborhood $V(x)$ of x such that $F(y) \cap G \neq \varnothing$ for all $y \in V(x)$. F is called l.s.c. iff it is l.s.c. at every $x \in X$.

3. Main results

Theorem 1. *Let (X, t) be a symmetric space with closure operator S, let $F: X \to 2^X$ be a multi-valued mapping, let $S' = \inf\{S, S_F\}$, and let H be an S'-compact set. Suppose:*

(1) *t is S-continuous,*

(2) $(\forall x, y \in X)\ D(F(x), F(y)) \leq \max\{D(x, F(x)), \ D(y, F(y))\}$.

Further, suppose that for every $x \in X$ there exists an orbit $O(x)$ of F at x which satisfies the following conditions:

(3) $S'(O(x)) \cap H \neq \varnothing$,

(4) $x \in F(x) \Rightarrow \sup_{y \in S'(O(x))} D(y, F(y)) < \operatorname{diam} S'(O(x))$.

Then there exists $\overline{x} \in H$ such that $\overline{x} \in F(\overline{x})$.

Proof. Consider the family of all nonempty S'-closed subsets of X. This family is nonempty because it contains X. It is partially ordered by set inclusion. Let $\mathscr{C}$ be a chain in the family. We prove that $\mathscr{C}$ has a minimal element. Let A be a nonempty S'-closed subset and $x \in A$. We have $S'(O(x)) \subset A$. Using condition (3) we have $A \cap H \neq \varnothing$. So, every finite intersection of elements of $\mathscr{C}$

meets H. By the S'-compactness of H we conclude that $\bigcap(A|A \in \mathscr{C}) \cap H \neq \varnothing$. Therefore $\bigcap(A|A \in \mathscr{C})$ is a nonempty minimal element for $\mathscr{C}$ in the family. By Zorn's lemma there exists a minimal element M in the family of all nonempty S'-closed subsets under consideration and $M \cap H \neq \varnothing$.

Let $a \in M$. We shall prove that $a \in F(a)$. By the minimality of M we have $S'(O(a)) = M$. Suppose to the contrary that $a \notin F(a)$. Then by condition (4) we have

$$\sup_{y \in M} D(y, F(y)) = r < \operatorname{diam} M. \tag{1}$$

We show that $S(F(M)) = M$. Indeed, by the S'-closedness of M we have $S(F(M)) \subset M$. Hence $F(S(F(M))) \subset F(M) \subset S(F(M))$. This means that $S(F(M))$ is SF-closed. Therefore $S(F(M))$ is S'-closed. By the minimality of M we conclude that $S(F(M)) = M$. Using condition (1) we have $\operatorname{diam} M = \operatorname{diam} S(F(M)) = \operatorname{diam} F(M)$. Further, we have

$$\operatorname{diam} M = \operatorname{diam} F(M) = \sup_{x, y \in F(M)} t(x, y) \leq \sup_{x, y \in M} D(F(x), F(y)).$$

Using condition (2) we get

$$\operatorname{diam} M \leq \sup_{x, y \in M} \max\{D(x, F(x)), D(y, F(y))\} = r.$$

This inequality contradicts inequality (1). Thus, every $a \in M$ is a fixed point for F and in particular so is every point from the nonempty subset $M \cap H$. The theorem is proved.

Theorem 2. *Let (X, t) be a symmetric space with closure operator S, let $F: X \to 2^X$ be a multi-valued mapping, let $S' = \inf\{S, S_F\}$, and let H be an S'-compact set. Suppose*:

(1) *$B(x, r)$ and $B(F(x), r)$ are S-closed for all $r \in R_+$ and $x \in X$,*
(2) $(\forall x, y \in X)$ $D(F(x), F(y)) \leq \max\{D(x, F(x)), D(y, F(y))\}$.

Further, suppose that for every $x \in X$ there exists an orbit $O(x)$ which satisfies the following conditions:

(3) $S'(O(x)) \cap H \neq \varnothing$,
(4) $x \notin F(x) \Rightarrow \sup_{y \in S'(O(x))} D(y, F(y)) < \operatorname{diam} S'(O(x))$.

Then there exists $\overline{x} \in H$ such that $\overline{x} \in F(\overline{x})$.

Proof. Repeat the proof of Theorem 1 up to inequality (1). Further, let $x \in M$. Consider the set $A_1 = B(F(x), r) \cap M$. It is nonempty because $x \in A_1$. By condition (1), A_1 is S-closed. Let $y \in A_1$. By condition (2) we have

$$D(F(x), F(y)) \leq \max\{D(x, F(x)), D(y, F(y))\} \leq r.$$

Thus $F(y) \subset A_1$ for any $y \in A_1$. This means that $FA_1 \subset A_1$ and A_1 is S_F-closed. So A_1 is S'-closed and by the minimality of M we have $A_1 = B(F(x), r) \cap M = M$. From there $M \subset B(F(x), r)$, i.e., $D(F(x), y) \leq r$ for all $y \in M$. Setting $A_2 = \bigcap(B(y, r)|y \in M) \cap M$ we have $F(x) \subset A_2$. Thus, taking $x \in A_1 = M$ we have $F(x) \subset A_2$. This means that $F(M) \subset A_2$. But $A_2 \subset M$, and we have $F(A_2) \subset A_2$. So A_2 is S_F-closed. By condition (1), A_2 is also S-closed as an intersection of S-closed sets; so it is S'-closed. By the minimality of M we have $A_2 = M$. Then $\operatorname{diam} M = \operatorname{diam} A_2 \leq r$. This contradicts inequality (1). Thus $a \in F(a)$ for all $a \in M$ and in particular so is every point from the nonempty subset $M \cap H$. The theorem is proved.

Corollary. *Let (X, t) be a nonempty compact metric space, let $F: X \to 2^X$ be a multi-valued mapping, and let $S' = \inf\{S_t, S_F\}$, where S_t is the topological closure operator. Suppose:*

(1) $D(F(x), F(y)) \leq \max\{D(x, F(x)), D(y, F(y))\}$ $(\forall x, y \in X)$,
(2) *if* $x \notin F(x)$ *there exists an orbit* $O(x)$ *such that* $\sup_{y \in S'(O(x))} D(y, F(y)) < \operatorname{diam} S'(O(x))$.

Then F has a fixed point.

Proof. Use Theorem 1 or 2 with $H = X$ and $S = S_t$.

Lemma 1. *Let X be a subsymmetrizable topological space, t be a continuous symmetric on X, and $F: X \to 2^X$ be an l.s.c. mapping. Then $\{x \in X | D(x, F(x)) \leq r\}$ is a closed set for every positive number $r \in R_+$.*

Proof. Let $y \notin \{x \in X | D(x, F(x)) \leq r\}$, i.e., $D(y, F(y)) = r + \varepsilon$, where $\varepsilon \in R_+$. By the definition of D there exists $z_1 \in F(y)$ for which $t(y, z_1) > r + \frac{\varepsilon}{2}$. By the continuity of t there exist open neighborhoods $U(y)$ and $U_1(z_1)$ such that $t(w, z) \geq r + \frac{\varepsilon}{4}$ for all $w \in U(y)$ and $z \in U_1(z_1)$. The set $F(y) \cap U_1(z_1)$ is nonempty because it contains z_1. By the lower-continuity of F there exists an open neighborhood $U_2(y)$ for which $F(w) \cap U_1(z_1) \neq \varnothing$ for all $w \in U_2(y)$. Setting $V(y) = U(y) \cap U_2(y)$ we have $t(w, F(w)) \geq r + \frac{\varepsilon}{4}$ for all $w \in V(y)$. Thus $V(y) \bigcap \{x \in X | D(x, F(x)) \leq r\} = \varnothing$ and so $\{x \in X | D(x, F(x)) \leq r\}$ is closed.

Lemma 2. *Let X and Y be topological spaces, $F: X \to 2^X$ be an l.s.c. mapping, and A be a subset in X. Then $S_t F(A) \supset F(S_t A)$, where S_t denotes the topological closure operator.*

Proof. Let $a \in S_t A$. We have to prove that $F(a) \subset S_t F(A)$. Suppose the contrary, that $F(a) \cap (X \backslash S_t F(A)) \neq \varnothing$. Then by the lower-continuity of F, there exists a neighborhood $V(a)$ of a for which $F(v) \cap (X \backslash S_t F(A)) \neq \varnothing$ for every $v \in V(a)$. Take $w \in V(a) \cap A$. Then $F(w) \cap (X \backslash S_t F(A)) \neq \varnothing$ and so $F(A) \cap (X \backslash S_t F(A)) \neq \varnothing$. But this is impossible. The lemma is proved.

Theorem 3. *Let X be a subsymmetrizable topological space, t be a continuous symmetric on X, and $F: X \to 2^X$ be l.s.c. Let S be an algebraic closure operator on X, let S_t be the topological closure operator, let $S' = \inf\{S, S_t, S_F\}$, and let $H \in 2^X$ be a nonempty S'-compact subset. Suppose:*

(1) $(\forall x \in X)$ $(\forall r \in R_+)$ $B(F(x), r) \in S(2^X)$,
(2) $(\forall A \in 2^X)$ $S(S_t(S(A))) = S_t(S(A))$,
(3) $(\forall x, y \in X)$ $(\exists \alpha(x, y) \in (0, 1))$: $D(F(x), F(y)) \leq \alpha D(x, F(x)) + (1 - \alpha) D(y, F(y))$.

Further, suppose that for every $x \in X$ there exists an orbit $O(x)$ of F which satisfies the following conditions:

(4) $S'(O(x)) \cap H \neq \varnothing$,
(5) $x \notin F(x) \Rightarrow (\exists y \in S'(O(x)))$: $D(y, F(y)) < \sup\{D(z, F(z)) | z \in S'(O(x))\}$.

Then F has a fixed point in H.

Proof. Following the proofs of Theorems 1 and 2 we construct the S'-closed minimal subset M for which $M \cap H \neq \varnothing$. Let $a \in M$. We prove that $a \in F(a)$. Suppose to the contrary that $a \notin F(a)$. According to condition (5) there exists $a_1 \in S'(O(x))$ such that $D(a_1, F(a_1)) = r < \sup\{D(x, F(x)) | x \in M\}$ because $S'(O(a)) = M$ by the minimality of M. Consider the subsets

$$A = \{x \in M | D(x, F(x)) \leq r\}, \qquad A_1 = S(F(A)).$$

They are nonempty because $a_1 \in A$.

Let $x \in A_1$. Because S is algebraic there exists a finite subset $K \subset F(A)$ such that $x \in S(K)$. Setting $q = \max\{D(y, F(x)) | y \in K\}$ we have $K \subset B(F(x), q)$. By condition (1), $S(K) \subset B(F(x), q)$. Therefore $x \in B(F(x), q)$, i.e., $D(x, F(x)) \leq q$. So we have $D(x, F(x)) \leq D(F(x), y)$ for some $y \in F(z)$, $z \in A$. Therefore $D(x, F(x)) \leq D(F(x), F(z))$. Using condition (3) we get

$$D(x, F(x)) \leq \alpha D(x, F(x)) + (1 - \alpha) D(z, F(z)),$$

where $\alpha \in (0, 1)$. It follows that $D(x, F(x)) \leq D(z, F(z)) \leq r$, i.e., $x \in A$. So we have $A_1 \subset A$.

Denote $A_2 = S_t(A_1)$. We have $A_2 = S_t(A_1) \subset S_t(A)$. By Lemma 1, $S_t(A) = A$ and therefore $A_2 \subset A$.

A_2 is S-closed by condition (2). Using Lemma 2 we have

$$\begin{aligned} F(A_2) &= F(S_t(A_1)) \subset S_t(F(A_1)) \subset S_t(F(A)) \subset S_t(S(F(A))) \\ &= S_t(A_1) = A_2. \end{aligned}$$

So A_2 is also S_F-closed and then S'-closed.

Using the minimality of M and the fact that $A_2 \subset M$, we conclude that $A_2 = M$. At the same time, using the fact that $A_2 \subset A$ we have

$$\sup\{D(x, F(x)) | x \in A_2\} \leq r < \sup\{D(x, F(x)) | x \in M\},$$

a contradiction.

Thus, every $a \in M$ is a fixed point of F and in particular so is every point from the nonempty subset $H \cap M$. The theorem is proved.

Since every convex closed subset in a normed space is also weakly closed, we infer the existence of fixed points for multi-valued l.s.c. mappings which satisfy a condition of Kannan's type.

Corollary 1. *Let X be a convex weakly compact subset in a normed space, and let $F: X \to 2^X$ be a convex-valued l.s.c. mapping with nonempty images and satisfying the condition*

$$D(F(x), F(y)) \leq \tfrac{1}{2}(D(x, F(x)) + D(y, F(y))) \quad \textit{for all } x, y \in X,$$

where D is defined by $D(A, B) = \sup\{\|a - b\| \,|a \in A,\ b \in B\}$.

Suppose that for every convex subset $A \in 2^X$ containing more that one element and invariant under F, there exists $x \in A$ such that

$$D(x, F(x)) < \sup\{D(y, F(y)) | y \in A\}.$$

Then F has a fixed point in X.

Proof. Use Theorem 3 with $t(x, y) = \|x - y\|$, $H = X$, $\alpha = \frac{1}{2}$, and S the convex closure operator.

Let us note that in reflexive Banach spaces all bounded closed convex subsets are weakly compact and by our definition every continuous single-valued mapping is l.s.c. Hence this corollary extends Theorem 2 of [3].

Corollary 2. *Let* (X, t) *be a metric compact space and let* $F: X \to 2^X$ *be an l.s.c. mapping with nonempty images and satisfying the condition*

$$D(F(x), F(y)) \leq \tfrac{1}{2}(D(x, F(x)) + D(y, F(y))) \quad \textit{for all } x, y \in X,$$

where D *is defined by the metric* t.

Suppose that for every closed subset $A \in 2^X$ *containing more than one element and invariant under* F, *there exists* $x \in A$ *such that*

$$D(x, F(x)) < \sup\{D(y, F(y)) | y \in A\}.$$

Then F *has a fixed point.*

Proof. Use Theorem 3 with $H = X$, $\alpha = \frac{1}{2}$, and $S(A) = A$ for all $A \in 2^X$.

Acknowledgment

The author wishes to thank Professor Tan D. H. for his lecture and correction of the manuscript.

References

1. A. Liepins, *Topological spaces and their mappings*, Riga, 1983, pp. 61–69.
2. ——, *Fixed point theorems in subsymmetrizable spaces*, Fifth Prague Topological Symposium, Theses (Prague, 1981).
3. R. Kannan, *Fixed point theorems in reflexive Banach spaces*, Proc. Amer. Math. Soc. **38** (1973), 111–118.

Institute of Mathematics, 1000 Hanoi, Vietnam