n× OVERSAMPLING PRESERVES ANY TIGHT AFFINE FRAME FOR ODD n

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AbstracT. If ψ generates an affine frame \( \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \ j, k \in \mathbb{Z}, \ \) of \( L^2(\mathbb{R}), \) we prove that \( \{n^{-1/2} \psi_{j,k/n}\} \) is also an affine frame of \( L^2(\mathbb{R}) \) with the same frame bounds for any positive odd integer \( n. \) This establishes the result stated as the title of this paper. A counterexample of this statement for \( n = 2 \) is also given.

1. Introduction and results

Let \( L^2 = L^2(\mathbb{R}) \) denote, as usual, the space of all complex-valued square-integrable functions on the real line with inner product \( \langle , \rangle \) and norm \( \| \|. \) For any \( f \in L^2, \) we will use the notation

\[
f_{j,\alpha}(x) = 2^{j/2} f(2^j x - \alpha), \quad j \in \mathbb{Z}, \ \alpha \in \mathbb{R}.
\]

A function \( \psi \in L^2 \) is said to generate an affine frame

\[
\{ \psi_{j,k} : j, k \in \mathbb{Z} \}
\]

of \( L^2, \) with frame bounds \( A \) and \( B, \) where \( 0 < A \leq B < \infty, \) if it satisfies

\[
A\|f\|^2 \leq \sum_{j, k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq B\|f\|^2, \quad f \in L^2.
\]

The frame (1.2) of \( L^2 \) is called a tight frame, if (1.3) holds with \( A = B. \) The importance of a tight frame is that any \( f \in L^2 \) can be recovered from its integral wavelet transform (IWT)

\[
\langle f, \psi_{j,k} \rangle = 2^{j/2} \int_{-\infty}^{\infty} f(x) \psi\left(2^j \left(x - \frac{k}{2^j}\right)\right) \, dx
\]

relative to \( \psi \) at the time-scale locations \( (2^{-j}, k/2^j), \ j, k \in \mathbb{Z}, \) via the formula

\[
f(x) = \frac{1}{A} \sum_{j, k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}(x),
\]
where $A = B$. It should be noted that a frame, such as (1.2)-(1.3), tight or not, may be redundant in the sense that (1.2) does not have to be $l^2$-linearly independent. However, any Riesz (or unconditional) basis is also a frame.

In the above discussion, we only consider, without loss of generality, the sampling period $b = 1$ and scaling parameter $a = 2$ as in (1.2). Details and generality are discussed in the wavelet literature [1, 2, 3, 5, 6], and a general study of frames can be found in the monograph [7] as well as the fundamental paper [4] of Duffin and Schaeffer, where the notion of frames was first introduced.

The objective of this paper is to establish the following.

**Theorem 1.** Let $\psi \in L^2$ generate a frame $\{\psi_j, k\}$ of $L^2$ with frame bounds $A$ and $B$ as given by (1.3). Then for any positive odd integer $n$, the family

$$\{n^{-1/2}\psi_{j, k/n}: j, k \in \mathbb{Z}\}$$

remains a frame of $L^2$ with the same bounds: that is,

$$nA\|f\|^2 \leq \sum_{j, k \in \mathbb{Z}} |\langle f, \psi_{j, k/n}\rangle|^2 \leq nB\|f\|^2, \quad f \in L^2.$$

In particular, if $\{\psi_{j, k}\}$ is a tight frame (with $A = B$) and $n > 0$ is odd, then the family in (1.6) satisfies

$$\sum_{j, k \in \mathbb{Z}} |\langle f, \psi_{j, k/n}\rangle|^2 = nA\|f\|^2, \quad f \in L^2.$$

On the other hand, (1.8) does not necessarily hold for even $n > 0$.

## 2. Preliminary results

A sequence of three lemmas will be needed for the proof of Theorem 1. Let $n$ be any positive odd integer and set

$$\lambda_1(p) = 2p - \frac{n}{2}(1 + \text{sgn}(2p - n)).$$

Then $\lambda_1$ is a permutation of the set $\{0, \ldots, n-1\}$. This permutation gives rise to a rearrangement operator $\tau$ defined on $\mathbb{R}^n$ by

$$\tau(a) := (a_{\lambda_1(0)}, \ldots, a_{\lambda_1(n-1)}), \quad a = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n,$$

where $n > 0$ is odd. As usual, we set

$$\tau^0 = I \quad \text{and} \quad \tau^j = \tau(\tau^{j-1}), \quad j = 1, 2, \ldots,$$

with $I$ denoting the identity operator. Hence, for each $j = 0, 1, \ldots$, we may write

$$\tau^j(a) = (a_{\lambda_j(0)}, \ldots, a_{\lambda_j(n-1)}), \quad a \in \mathbb{R}^n,$$

where $\lambda_j$ is a permutation of $\{0, \ldots, n-1\}$ induced by $\lambda_1$. We have the following result.
Lemma 1. Let $n$ be any positive odd integer and $\lambda_j$ be defined by (2.1)–(2.4). Then for any $j \in \mathbb{Z}_+$,
\begin{equation}
\lambda_j(p) \equiv 2^j p \pmod{n}, \quad p = 0, \ldots, n-1.
\end{equation}

Proof. We will establish (2.5) by induction on $j$. Since $\lambda_0$ is the identity, (2.5) certainly holds for $j = 0$. For $j \geq 1$, we first consider $0 \leq p \leq (n-1)/2$. In this case, it follows from (2.1) that $\lambda_{j+1}(p) = \lambda_j(2p)$. Hence, by the induction hypothesis, we have
\[
\lambda_{j+1}(p) = \lambda_j(2p) \equiv 2^j(2p) \pmod{n} = 2^{j+1}p \pmod{n}.
\]
Similarly, for $(n-1)/2 < p \leq n-1$, it follows from (2.1) and the induction hypothesis that
\[
\lambda_{j+1}(p) = \lambda_j(2p-n) \equiv 2^j(2p-n) \pmod{n} \equiv 2^{j+1}p \pmod{n}.
\]

Since $\lambda_1$ is a one-one map of $\{0, \ldots, n-1\}$ onto itself, the rearrangement operator $\tau$ as defined by (2.2) has an inverse $\tau^{-1}$. Hence, the definition of $\tau^j$ in (2.3) can be extended to all $j \in \mathbb{Z}$. Set
\[
a_0 := (0, 1, \ldots, n-1)
\]
and define $\{e_j, p\}$, $p = 0, \ldots, n-1$, and $j \in \mathbb{Z}$, by
\begin{equation}
(2.6) \quad \tau^j(a_0) := (e_{j,0}, \ldots, e_{j,n-1}), \quad j \in \mathbb{Z}.
\end{equation}
We have the following

Lemma 2. Let $j_0 \in \mathbb{Z}$. Then
\begin{equation}
(2.7) \quad e_{j, p} \equiv 2^{j-j_0} e_{j_0, p} \pmod{n},
\end{equation}
for all $j > j_0$ and $p = 0, \ldots, n-1$.

Proof. We first establish the relation:
\begin{equation}
(2.8) \quad e_{j+1, p} \equiv 2e_{j, p} \pmod{n}, \quad j \leq -1, \; p = 0, \ldots, n-1,
\end{equation}
by induction. For $j = -1$, observe that for even $p$,
\[
2e_{-1, p} = 2e_{0, p/2} \equiv p \pmod{n},
\]
and that for odd $p$,
\[
2e_{-1, p} = 2e_{0, [p/2]+(n+1)/2} = 2[p/2] + n + 1 \equiv p \pmod{n}.
\]
Hence, (2.8) holds for $j = -1$. For $j < -1$, we have, for even $p$,
\[
2e_{j, p} = 2e_{j+1, p/2} \equiv e_{j+2, p/2} \pmod{n}
\]
by applying the induction hypothesis. Since $e_{j+2, p/2} = e_{j+1, p}$, it follows that
\[
2e_{j, p} \equiv e_{j+1, p} \pmod{n},
\]
for even $p$. For odd $p$, we also have
\[
2e_{j, p} = 2e_{j+1, [p/2]+(n+1)/2} \equiv e_{j+2, [p/2]+(n+1)/2} \pmod{n}
\]
\[
\equiv e_{j+1, p} \pmod{n}
\]
again by applying the induction hypothesis. This establishes (2.8).
Of course, (2.7) is an immediate consequence of (2.8) for $0 > j > j_0$. On the other hand, if $j > 0 > j_0$, then we may obtain (2.7) by applying (2.5) in Lemma 1 as well. Finally, for $j > j_0 > 0$, then
\[ e_{j,p} \equiv 2^j e_{0,p} (\mod n) \equiv 2^{j-j_0} 2^{j_0} e_{0,p} (\mod n) \equiv 2^{j-j_0} e_{j_0,p} (\mod n). \]

**Lemma 3.** Let $j, j_0 \in \mathbb{Z}$ with $j \geq j_0$ and $p = 0, \ldots, n-1$. Then the two collections of functions
\begin{align*}
(2.9) & \quad \{2^j x - e_{j,p}/n - k : k \in \mathbb{Z}\} \\
(2.10) & \quad \{2^j (x - 2^{-j_0} e_{j_0,p}/n) - k' : k' \in \mathbb{Z}\}
\end{align*}
are identical.

**Proof.** Let $k \in \mathbb{Z}$. Since $e_{j,k} \equiv 2^{-j_0} e_{j_0,p} (\mod n)$, we have
\[ 2^j x - e_{j,p}/n - k = 2^j x - 2^{-j_0} e_{j_0,p}/n - k' = 2^j \left( x - \frac{2^{-j_0} e_{j_0,p}}{n} \right) - k' \]
for some $k' \in \mathbb{Z}$. In addition, it is quite easy to see that the mapping $k \rightarrow k'$ is one-to-one. Hence, the two collections (2.9) and (2.10) are identical. \qed

### 3. Proof of Theorem 1

Let $n$ be a positive odd integer. We first decompose the collection of functions $\psi_{j,k,n}$, $j, k \in \mathbb{Z}$, into $n$ disjoint subcollections $S_0, \ldots, S_{n-1}$, where
\[ S_p := \{2^{j/2}\psi(2^j x - e_{j,p}/n - k) : j, k \in \mathbb{Z}\}. \]
Since $S_0$ is the set $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$, the assumption (1.3) can be expressed as
\[ A\|f\|^2 \leq \sum_{g \in S_0} |\langle f, g \rangle|^2 \leq B\|f\|^2, \quad f \in L^2. \]

Let $j_0 \in \mathbb{Z}$ and consider
\[ \sigma_{j_0,p}(f) := \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} \left| \langle 2^{j/2} \psi \left( 2^j \left( \cdot - \frac{2^{-j_0} e_{j_0,p}}{n} \right) - k \right), f \rangle \right|^2, \]
where $p = 0, \ldots, n-1$, and $f \in L^2$. By Lemma 3, we see that
\[
\sigma_{j_0,p}(f) = \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} \left| \langle 2^{j/2} \psi(2^j \cdot - k), f \left( \cdot + \frac{2^{-j_0} e_{j_0,p}}{n} \right) \rangle \right|^2 \\
\leq B \left\| f \left( \cdot + \frac{2^{-j_0} e_{j_0,p}}{n} \right) \right\|_2^2 = B\|f\|^2.
\]
Hence, for each $p = 1, \ldots, n-1$, we have
\[ \sum_{g \in S_p} |\langle f, g \rangle|^2 = \lim_{j_0 \to -\infty} \sigma_{j_0,p}(f) \leq B\|f\|^2. \]
Combining this with (3.3) yields

\[
\sum_{j, k \in \mathbb{Z}} |\langle f, \psi_{j, k/n} \rangle|^2 = \sum_{p=0}^{n-1} \sum_{g \in S_p} |\langle f, g \rangle|^2 \leq nB\|f\|^2.
\]

To establish the lower bound in (1.7) we consider the class $L_c^\infty$ of all a.e. bounded functions with compact support in $\mathbb{R}$. Since $L_c^\infty$ is dense in $L^2$, it is sufficient to prove that the lower bound in (3.3) holds for all $f \in L_c^\infty$. Let $f \in L_c^\infty$ and suppose that

\[
supp f \subset [-L, L], \quad L > 0.
\]

Set

\[
\Theta_{j_0, p}(f) := \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} |\langle 2^{j/2} \psi(2^{j} \cdot -k), f_{j_0}^p \rangle|^2
\]

and

\[
\Lambda_{j_0, p}(f) := \sum_{j < j_0} \sum_{k \in \mathbb{Z}} \left| \langle 2^{j/2} \psi(2^j \cdot -\frac{e_{j, p}}{n} - k), f \rangle \right|^2,
\]

where

\[
f_{j_0}^p := f(x + 2^{-j_0} e_{j_0, p}/n).
\]

By (3.2), we have, for each $p = 1, \ldots, n - 1$,

\[
\sigma_{j_0, p}(f) + \Theta_{j_0, p}(f) = \sum_{j, k \in \mathbb{Z}} |\langle \psi_{j, k}, f_{j_0}^p \rangle|^2 \geq A\|f_{j_0}^p\|^2 = A\|f\|^2.
\]

This yields

\[
\sum_{g \in S_p} |\langle f, g \rangle|^2 = \sum_{j, k \in \mathbb{Z}} \left| \langle 2^{j/2} \psi\left(2^j \cdot -\frac{e_{j, p}}{n} - k\right), f \rangle \right|^2
\]

\[
= \sigma_{j_0, p}(f) + \Theta_{j_0, p}(f) - \Theta_{j_0, p}(f) + \Lambda_{j_0, p}(f)
\]

\[
\geq A\|f\|^2 - \Theta_{j_0, p}(f).
\]

By introducing the notation

\[
I_{j_0, p} := [-L - 2^{-j_0} e_{j_0, p}/n, L - 2^{-j_0} e_{j_0, p}/n],
\]

it follows from (3.5) and (3.8) that

\[
supp f_{j_0}^p \subset I_{j_0, p}, \quad p = 1, \ldots, n - 1.
\]

Hence, by the Cauchy inequality, we have

\[
|\langle 2^{j/2} \psi(2^j \cdot -k), f_{j_0}^p \rangle|^2 \leq 2^{j+1} L\|f\|_\infty \int_{I_{j_0, p}} |\psi(2^j x - k)|^2 \, dx
\]

\[
= 2L\|f\|_\infty \int_{2^j I_{j_0, p}} |\psi(x - k)|^2 \, dx.
\]
Let $j_0, J \in \mathbb{Z}$ be so chosen that $j_0 < -\log_2 nL$ and $J > \log_2 (L + 1)$. Then $I_{j_0, p} \subset (-\infty, 0)$, and hence

\[
\Theta_{j_0, p}(f) \leq 2L\|f\|_\infty \sum_{j_0 - J \leq j < j_0} \sum_{k \in \mathbb{Z}} \int_{k-2^J+2^{-j_0}}^{k+2^J} \int_{k-2^j+2^{-j_0}}^{k+2^j} |\psi(x)|^2 \, dx + 2L\|f\|_\infty \sum_{k \in \mathbb{Z}} \int_{k-2^{-j}}^{k+2^{-J}} |\psi(x)|^2 \, dx.
\]

Let $\eta > 0$ be arbitrarily given. Since $\psi \in L^2$, there is some $\beta > 0$ such that $\int_{|x| \geq \beta} |\psi(x)|^2 \, dx < \eta$. So, by setting $\psi_\beta := \psi \chi_{[-\beta, \beta]}$, $\beta > 0$, we have

\[
\Theta_{j_0, p}(f) \leq 2L\|f\|_\infty \left( \eta + \sum_{j_0 - J \leq j < j_0} \sum_{k \in \mathbb{Z}} \int_{k-2^J+2^{-j_0}}^{k+2^J} \int_{k-2^j+2^{-j_0}}^{k+2^j} |\psi(x)|^2 \, dx + \sum_{k \in \mathbb{Z}} \int_{k-2^{-j}}^{k+2^{-J}} |\psi_\beta(x)|^2 \, dx \right),
\]

where the last term on the right-hand side is smaller than $\eta$ for any sufficiently large $J$. For such a fixed $J$, a $\gamma > 0$ can be chosen to yield

\[
\int_{|x| \geq \gamma} |\psi(x)|^2 \, dx \leq \eta J^{-1}.
\]

Hence, it follows from (3.10) that

\[
\Theta_{j_0, p}(f) \leq 2L\|f\|_\infty \left( 3\eta + \sum_{j_0 - J \leq j < j_0} \sum_{k \in \mathbb{Z}} \int_{k-2^J+2^{-j_0}}^{k+2^J} \int_{k-2^j+2^{-j_0}}^{k+2^j} |\psi_\gamma(x)|^2 \, dx \right),
\]

where (3.11) has been used to take care of $\psi - \psi_\gamma$. Since $\psi_\gamma$ has compact support, the last term on the right-hand side of (3.12) tends to zero as $j_0 \to -\infty$, so that

\[
\limsup_{j_0 \to -\infty} \Theta_{j_0, p}(f) \leq 6L\|f\|_\infty \eta, \quad p = 1, \ldots, n - 1.
\]

In view of (3.9) and (3.2), we have established the lower bound in (1.7).

Finally, to show that (1.8) does not necessarily hold for even $n > 0$, we consider two functions

\[
f_1(x) = \psi_H(x + \frac{1}{2}) \quad \text{and} \quad f_2(x) = \chi_{[-1/2, 1/2]}(x),
\]

where $\psi_H(x) = \chi_{[0, 1]}(x) \sgn(\frac{1}{2} - x)$ is the Haar function. Then for $\psi = \psi_H$ in (1.8), we have

\[
\sum_{j, k \in \mathbb{Z}} |\langle f_1, \psi_H; j, k/2 \rangle|^2 = 3 \sum_{j, k \in \mathbb{Z}} |\langle f_2, \psi_H; j, k/2 \rangle|^2 = \frac{9}{2},
\]

while $\|f_1\| = \|f_2\| = 1$. Hence, $\{\psi_H; j, k/2\}$ cannot be a tight frame. \qed
References


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