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A MATRICIAL IDENTITY INVOLVING THE SELF-COMMUTATOR OF A COMMUTING n-TUPLE

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ABSTRACT. For a commuting n-tuple $a = (a_1, \dots, a_n)$ of elements of a unital C^*-algebra $\mathscr{A}$, we establish a matricial identity linking the self-commutator of a to the $2^{n-1} \times 2^{n-1}$ matrix $\hat{a}$ that detects the Taylor invertibility of a. As a consequence, we obtain a simple proof of a result of D. Xia (Oper. Theory: Adv. Appl. **48** (1990), 423–448), which states that for commuting t-hyponormal n-tuples, $\sigma_T(a) = \sigma_r(a)$.

Let $a = (a_1, \dots, a_n)$ be a commuting n-tuple of elements of a unital C^*-algebra $\mathscr{A}$. The main result in [Cu1] states that a is invertible on $\mathscr{A}$ (in the sense of Taylor) if and only if a certain $2^{n-1} \times 2^{n-1}$ matrix over $\mathscr{A}$, denoted $\hat{a}$, is invertible. Doubly commuting n-tuples (those where $a_i a_j = a_j a_i$ and $a_i a_j^* = a_j^* a_i$ for all $i \neq j$) admit a much simpler criterion, namely, a is invertible if and only if all 2^n positive elements $a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n$, $a_1^* a_1 + a_2^* a_2 + \cdots + a_n a_n^*, \dots, a_1 a_1^* + a_2 a_2^* + \cdots + a_n^* a_n$, $a_1 a_1^* + a_2 a_2^* + \cdots + a_n a_n^*$ are invertible. In this note we establish a simple matricial formula linking $\hat{a}$ to the self-commutator of a, the $n \times n$ matrix $[a^*, a] := ([a_j^*, a_i])_{i,j=1}^n$, where $[a_j^*, a_i] := a_j^* a_i - a_i a_j^*$. As a corollary, we obtain a new proof of a result of D. Xia which states that for t-hyponormal n-tuples a, the Taylor spectrum agrees with the right spectrum.

To explain our results, we need to recall the construction of $\hat{a}$. Rather than repeating the details in [Cu1], we give here an alternate inductive definition. First, for M and N $k \times k$ matrices over $\mathscr{A}$ we let

$$(M, N)^\frown := \begin{pmatrix} M & N \\ -N^* & M^* \end{pmatrix}.$$

Now, if $n = 1$ we let $\hat{a} := (a_1)$, and if $n > 1$ we let $\hat{a} := ((a')^\frown, a_n I)^\frown$, where $a' := (a_1, \dots, a_{n-1})$ and I is the $2^{n-2} \times 2^{n-2}$ identity matrix. This definition of $\hat{a}$, slightly different from the one used in [Cu1], is more convenient for our purposes; the following result, nevertheless, still holds.

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Lemma 1 [Cu1, Theorem 1]. *The commuting n-tuple a is invertible on $\mathscr{A}$ if and only if $\hat{a}$ is invertible (as a $2^{n-1} \times 2^{n-1}$ matrix over $\mathscr{A}$).*

For the next definition we need some notation. If $1 \le k \le n$, we let $\mathbf{k} := \{1, \dots, k\}$, $\mathbf{n} := \{1, \dots, n\}$, and we let $\mathrm{Inc}(\mathbf{k}, \mathbf{n})$ denote the set of increasing maps from $\mathbf{k}$ to $\mathbf{n}$. We also let $\mathscr{S}_n$ denote the set of $n \times n$ diagonal matrices with diagonal entries equal to 1 or -1.

Definition 2. Let $1 \le k \le n$ be given, and let $\varphi \in \mathrm{Inc}(\mathbf{k}, \mathbf{n})$. We define $\varphi^*\colon \mathscr{M}_n(\mathscr{A}) \to \mathscr{M}_k(\mathscr{A})$ by $\varphi^*(M)_{i,j} := m_{\varphi(i),\varphi(j)}$, where $M = (m_{ij})_{i,j=1}^n \in \mathscr{M}_n(\mathscr{A})$, and $\varphi_*\colon \mathscr{M}_k(\mathscr{A}) \to \mathscr{M}_n(\mathscr{A})$ by

$$\varphi_*(M)_{i,j} := \begin{cases} m_{\varphi^{-1}(i),\varphi^{-1}(j)} & \text{if } i, j \in \varphi(\mathbf{k}) \\ 0 & \text{otherwise} \end{cases} \qquad (M \in \mathscr{M}_k(\mathscr{A})).$$

Definition 3. Let $1 \le k \le n$ and let $N \in \mathscr{M}_k(\mathscr{A})$ and $M \in \mathscr{M}_n(\mathscr{A})$. We say that N is a k-generalized principal submatrix of M (abbreviated k-GPSM) if there exist φ, $\psi \in \mathrm{Inc}(\mathbf{k}, \mathbf{n})$ and a diagonal matrix $D \in \mathscr{S}_n$ such that $N = \psi_*(\varphi^*(DMD))$.

Theorem 4. *Let $a = (a_1, \dots, a_n)$ be a commuting n-tuple of elements of a unital C^*-algebra $\mathscr{A}$, let $\hat{a}$ be as in Definition 1, and let $M(a) := [a^*, a]^t$. Then*

(i) $\hat{a}\hat{a}^* = \mathrm{diag}(r) + s$ *and*
(ii) $\hat{a}^*\hat{a} = \mathrm{diag}(r) + t$,

where $r = r_n = r(a) := a_1a_1^* + \cdots + a_na_n^*$, and s and t are sums of GPSM's of $M(a)$.

Before we state the following consequence, we recall that a commuting n-tuple $a = (a_1, \dots, a_n)$ is said to be hyponormal if $[a^*, a] \ge 0$ [At, Cu2, CMX, McCP] and t-hyponormal if $M(a) \ge 0$ [Xia].

Corollary 5 [Xia, Theorem 5]. *Assume that a is t-hyponormal. Then a is Taylor invertible on $\mathscr{A}$ if and only if $a_1a_1^* + \cdots + a_na_n^*$ is invertible.*

Proof. Observe that the maps φ^* and ψ_* in Definition 2 preserve positivity, so s and t are both positive. Thus, if $a_1a_1^* + \cdots + a_na_n^*$ is invertible, we obtain that $\hat{a}\hat{a}^*$ and $\hat{a}^*\hat{a}$ are invertible, which implies that a is Taylor invertible. □

Corollary 6. *Let a be a t-hyponormal commuting n-tuple of elements of $\mathscr{A}$. Then $\sigma_r(a) = \sigma_T(a)$.*

Proof. Let $\lambda \in \mathbb{C}^n$. Observe that $a - \lambda$ is again t-hyponormal, so Corollary 5 applies. □

Proof of Theorem 4

Our ploy is to use mathematical induction on the number of coordinates. For $n = 1$ the result is obvious, as $a_1^*a_1 = r + [a_1^*, a_1]$. Let us consider the case $n = 2$. Here

$$\hat{a}\hat{a}^* = \begin{pmatrix} r & 0 \\ 0 & a_1^*a_1 + a_2^*a_2 \end{pmatrix} = \mathrm{diag}(r) + \begin{pmatrix} 0 & 0 \\ 0 & [a_1^*, a_1] \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & [a_2^*, a_2] \end{pmatrix}$$

and

$$\hat{a}^*\hat{a} = \begin{pmatrix} a_1^*a_1 + a_2a_2^* & [a_1^*, a_2] \\ [a_2^*, a_1] & a_2^*a_2 + a_1a_1^* \end{pmatrix} = \operatorname{diag}(r) + M(a),$$

verifying (i) and (ii). For $n \geq 2$,

$$(a, a_{n+1})\hat{} = \begin{pmatrix} \hat{a} & \operatorname{diag}(a_{n+1}) \\ -\operatorname{diag}(a_{n+1}^*) & \hat{a}^* \end{pmatrix},$$

so

$$\begin{aligned}(1)\qquad &(a, a_{n+1})\hat{}[(a, a_{n+1})\hat{}]^* \\ &= \begin{pmatrix} \hat{a}\hat{a}^* + \operatorname{diag}(a_{n+1}a_{n+1}^*) & [\operatorname{diag}(a_{n+1}), \hat{a}] \\ [\hat{a}^*, \operatorname{diag}(a_{n+1}^*)] & \hat{a}^*\hat{a} + \operatorname{diag}(a_{n+1}^*a_{n+1}) \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}(2)\qquad &[(a, a_{n+1})\hat{}]^*(a, a_{n+1})\hat{} \\ &= \begin{pmatrix} \hat{a}^*\hat{a} + \operatorname{diag}(a_{n+1}a_{n+1}^*) & [\hat{a}^*, \operatorname{diag}(a_{n+1})] \\ [\operatorname{diag}(a_{n+1}^*), \hat{a}] & \hat{a}\hat{a}^* + \operatorname{diag}(a_{n+1}^*a_{n+1}) \end{pmatrix}.\end{aligned}$$

A moment's thought reveals that a successful inductive argument can be completed once the structure of the commutators $[\hat{a}, \operatorname{diag}(a_{n+1})]$ and $[\hat{a}, \operatorname{diag}(a_{n+1}^*)]$ is fully determined. (Observe that $r(a, a_{n+1}) = r(a) + a_{n+1}a_{n+1}^*$.) To reveal this structure, we first need a definition.

Definition 7. Let $1 \leq k \leq n$ and $M \in \mathscr{M}_n(\mathscr{A})$. A vector $\mathbf{v} \in \mathscr{A} \otimes \mathbb{C}^n$ is said to be a k-generalized column vector of M (abbreviated k-GCV) if the nonzero entries of $\mathbf{v}$ are part of a column of a $(k+1)$-GPSM of M.

Lemma 8. *Let $a = (a_1, \ldots, a_n)$ be a commuting n-tuple in $\mathscr{A}$, and assume that $a_{n+1} \in \mathscr{A}$ commutes with $a_1, \ldots, a_n$. Then*

(i) *If n is odd, $[\operatorname{diag}(a_{n+1}), \hat{a}]$ consists of $\binom{n}{n-1}$ $(n-1)$-GCV's, $\binom{n}{n-3}$ $(n-3)$-GCV's, …, $\binom{n}{2}$ 2-GCV's of $M(a, a_{n+1})$, and a zero column vector (the last column), and $[\hat{a}^*, \operatorname{diag}(a_{n+1})]$ consists of $\binom{n}{n}$ n-GCV's, $\binom{n}{n-2}$ $(n-2)$-GCV's, …, $\binom{n}{1}$ 1-GCV's of $M(a, a_{n+1})$.*

(ii) *If n is even, $[\operatorname{diag}(a_{n+1}), \hat{a}]$ consists of $\binom{n}{n-1}$ $(n-1)$-GCV's, $\binom{n}{n-3}$ $(n-3)$-GCV's, …, $\binom{n}{1}$ 1-GCV's of $M(a, a_{n+1})$, and $[\hat{a}^*, \operatorname{diag}(a_{n+1})]$ consists of $\binom{n}{n}$ n-GCV's, $\binom{n}{n-2}$ $(n-2)$-GCV's, …, $\binom{n}{2}$ 2-GCV's of $M(a, a_{n+1})$, and a zero column vector (the last column).*

Proof. We use mathematical induction. The conclusions certainly hold for $n = 2$ and $n = 3$; assume first that n is odd, and that the assertions are true for all $k < n$. From the inductive definition of $\hat{a}$, it follows at once that

$$[\operatorname{diag}(a_{n+1}), \hat{a}] = \begin{pmatrix} [\operatorname{diag}(a_{n+1}), (a')\hat{}] & 0 \\ \operatorname{diag}([a_n^*, a_{n+1}]) & [\operatorname{diag}(a_{n+1}), [(a')\hat{}]^*] \end{pmatrix}.$$

By the inductive hypothesis, the first 2^{n-2} columns of $[\operatorname{diag}(a_{n+1}), \hat{a}]$ contain $\binom{n-1}{n-2}$ $(n-1)$-GCV's, …, $\binom{n-1}{2}$ 3-GCV's, and one 1-GCV of $M(a, a_{n+1})$. Similarly, the last 2^{n-2} columns of $[\operatorname{diag}(a_{n+1}), \hat{a}]$ contain $\binom{n-1}{n-1}$ $(n-1)$-GCV's, …, $\binom{n-1}{1}$ 1-GCV's of $M(a, a_{n+1})$. The conclusion then follows by recalling that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. The cases of n odd, and the two cases corresponding to $[\hat{a}^*, \operatorname{diag}(a_{n+1})]$ are entirely similar. □

We can now conclude the proof of Theorem 4. Assume that n is even. By inductive hypothesis and formula (2), we see that

$$[(a, a_{n+1})^\frown]^*(a, a_{n+1})^\frown$$
$$= \operatorname{diag}(r_{n+1}) + \begin{pmatrix} t_n & [\hat{a}^*, \operatorname{diag}(a_{n+1})] \\ [\operatorname{diag}(a_{n+1}^*), \hat{a}] & s_n + \operatorname{diag}([a_{n+1}^*, a_{n+1}]) \end{pmatrix}.$$

Now observe that each GPSM of $M(a, a_{n+1})$ is either a GPSM of $M(a)$ or it involves a_{n+1} in both a column and a column. Since t_n as well as s_n are sums of GPSM's of $M(a)$, an application of Lemma 8 shows that

$$\begin{pmatrix} t_n & [\hat{a}^*, \operatorname{diag}(a_{n+1})] \\ [\operatorname{diag}(a_{n+1}^*), \hat{a}] & s_n + \operatorname{diag}([a_{n+1}^*, a_{n+1}]) \end{pmatrix}$$

can also be written as a sum of GPSM's of $M(a, a_{n+1})$. Thus,

$$[(a, a_{n+1})^\frown]^*(a, a_{n+1})^\frown - \operatorname{diag}(r_{n+1})$$

is a sum of GPSM's of $M(a, a_{n+1})$. A similar argument holds in the remaining cases.

Remark 9. Formulas (1) and (2), and the proofs of Theorem 4 and Lemma 8 show that s_n and t_n can be written recursively in terms of $a_1, \ldots, a_n$ starting from $[(a_1, a_2)^\frown]^*[(a_1, a_2)^\frown]$ and $[(a_1, a_2)^\frown][a_1, a_2)^\frown]^*$.

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References

[At] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[Cu1] R. Curto, *Spectral permanence for joint spectra*, Trans. Amer. Math. Soc. **270** (1982), 659–665.

[Cu2] ——, *Joint hyponormality: a bridge between hyponormality and subnormality*, Proc. Sympos. Pure Math., vol. 51, part 2, Amer. Math. Soc., Providence, RI, 1990, pp. 69–91.

[CMX] R. Curto, P. Muhly, and J. Xia, *Hyponormal pairs of commuting operators*, Oper. Theory: Adv. Appl., vol. 35, Birkhäuser Verlag, Basel, 1988, pp. 1–22.

[McCP] S. McCullough and V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107** (1989), 187–195.

[Xia] D. Xia, *On some classes of hyponormal tuples of commuting operators*, Oper. Theory: Adv. Appl., vol. 48, Birkhäuser Verlag, Basel, 1990, pp. 423–448.

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