SEMIPRIMITIVITY OF GROUP ALGEBRAS
OF INFINITE SIMPLE GROUPS OF LIE TYPE

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Abstract. Let G be a simple group of Lie type over an infinite locally finite
field F. For any field K, we prove that the group algebra K[G] is semiprimitive. The argument here is a mixture of combinatorial and topological methods.
Combined with earlier results, it now follows that any group algebra of an in-
finte locally finite simple group is semiprimitive. Furthermore, if the group is
countably infinite, then the group algebra is primitive. In particular, if G is a
simple group of Lie type over the field F, then K[G] is a primitive ring.

Let G be a simple group of Lie type over an infinite locally finite field F of
characteristic q. Say G ⊆ M_n(F), and use ϕ(x) to denote the characteristic
polynomial of x ∈ G in the variable ζ. Let U(G) denote the set of unipotent
elements of G so that U(G) = G_q is the set of q-elements of G.

Now let K be a field of characteristic p > 0, and let K'[G] be a twisted
group algebra of G over K. We do not assume that p and q are distinct. Let
G_p,q denote the set of (p, q)-elements of G. Our goal is to prove that K'[G]
is semiprimitive. Suppose, by way of contradiction, that this is not the case,
and let

0 ≠ α = 1 + \sum_{i=1}^{s} k_i x_i ∈ J K'[G]

with each 1 ≠ x_i ∈ G.

Lemma 1. There exists a constant c = c(p, q) such that |q^r - 1|_p ≤ cr for all
r > 0.

Proof. This is trivial if p = q. Suppose p ≠ q, and choose e minimal with
p|(q^e - 1) and 4|(q^e - 1) if p = 2. Then, as is well known,

|q^r - 1|_p ≤ |q^{er} - 1|_p = |q^e - 1|_p |r|_p ≤ (q^e - 1)r,

so the result follows with c = q^e - 1. □

The next lemma isolates a small part of the unipotent structure of G. See
[C] or [St] for the information we require.

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Lemma 2. There exists a constant \( l \geq 1 \), matrices \( v_{a,b} \in M_n(F) \) for \( 1 \leq a, b \leq l \), and a field automorphism \( \theta \) of \( F \) such that:

(i) \( u_x = 1 + \sum_{a,b} \lambda^{a+b} v_{a,b} \) for all \( \lambda \in F \). Furthermore, \( u_x \) is not identically equal to 1.

(ii) The restriction of the action of \( \theta \) to any \( GF(q^r) \subseteq F \) is given by \( \gamma \to \gamma^{m(r)} \), where \( 1 \leq m(r) \leq q^{(r+1)/2} \).

Proof. Let \( \theta \) be the field automorphism used to construct \( G \). If \( \theta \) has order \( \leq 2 \), then \( m(r) = 1 \) or \( r \) is even and \( m(r) = q^{r/2} \). Otherwise, \( G \) is a Suzuki or Ree group, \( q = 2 \) or \( 3 \), \( r \) is odd, and \( m(r) = q^{(r+1)/2} \). Finally, \( u_x \) and the matrices \( v_{a,b} \) are easily seen to exist by considering a nontrivial root subgroup of \( G \) for a maximal root.

We now proceed to the combinatorial part of the proof using the above notation.

Lemma 3. Let \( w \in G_{p,q} \) satisfy \( \phi(u_xw) = \phi(w) \) for all \( \lambda \in F \). Then there exists \( 1 \leq i \leq s \) such that \( wx_i \in G_{p,q} \) and \( \phi(u_xwx_i) = \phi(wx_i) \) for all \( \lambda \in F \).

Proof. Choose \( GF(q^r) \subseteq F \) so that \( w \), all \( v_{a,b} \), and all \( x_i \) are contained in \( M_n(q^r) \). There are of course infinitely many such \( r \); note that the constants used below are all independent of \( r \) unless otherwise indicated. Note also that \( \phi(u_xw) = \phi(w) \) implies that \( u_xw \in G_{r,p} \) for all \( \lambda \in GF(q^r) \).

Let \( x \) be any \((p,q)\)-element of \( G \), and use the notation of [P]. Then \( C_G(x) = C_G(x) \) since \( K \) has characteristic \( p \) and the Schur multiplier of \( G \) is a finite group of order prime to \( q \) (see [G, Theorem 4.235] or [St]). Furthermore, if \( x \in GL_n(q^r) \) and \( \epsilon \) is an eigenvalue of \( x \), then \( \epsilon \) is a \( p \)-element in \( GF(q^{nr}) \). Hence, by Lemma 1, there are at most \( c' \) choices for \( \epsilon \), where \( c' \) is a constant independent of \( r \). In particular, if \( c = (c'p)^{n} \), then there are at most \( cr^n \) choices for the characteristic polynomial \( \phi(x) \). Say these possibilities are \( \phi_1, \phi_2, \ldots, \phi_d \), where \( d = d(r) \leq cr^n \).

For each \( 1 \leq i \leq s \) and \( 1 \leq j \leq d(r) \), let

\[ F_{i,j} = \{ \lambda \in GF(q^r) | \phi(u_xwx_i) = \phi_j \} \]

We claim that \( \bigcup_{i,j} F_{i,j} = GF(q^r) \). To this end, let \( \lambda \in GF(q^r) \). Since \( u_xw \) and all its powers are \((p,q)\)-elements, [P, Lemma 6.2] and the preceding comments on \( C_G(x) \) imply that there exist \( k, i \) with

\[(u_xwx_i)^{\rho^k} \sim (u_xw)^{\rho^k},\]

where \( \sim \) indicates that the elements are \( G \)-conjugate. In particular, \( u_xwx_i \) is a \((p,q)\)-element in \( GL_n(q^r) \), so \( \phi(u_xwx_i) = \phi_j \) for some \( j \) and \( \lambda \) is contained in \( F_{i,j} \).

Suppose that, for all \( i, j \), we have

\[|F_{i,j}| \leq n!q^{(r+1)/2}.\]

Since \( GF(q^r) = \bigcup_{i,j} F_{i,j} \), this yields

\[q^r = |GF(q^r)| \leq sd(r)n!q^{(r+1)/2} \leq scln!q^{(r+1)/2},\]

so \( q^{(r-1)/2} \leq scln!n \), an equation which certainly cannot hold for all such \( r \). Thus, when \( r \) is taken to be sufficiently large, there exist subscripts \( i \) and \( j \) with

\[|F_{i,j}| > n!q^{(r+1)/2}.\]
Now notice that, by Lemma 2, the coefficients of $\phi(u_xw_ix)$ are all polynomials in $g$ of degree $\leq nq^{(r+1)/2}$. In particular, if one of these polynomials is not identically constant, then the number of $g \in GF(q^r)$ with $\phi(u_xw_ix) = \phi_j$ is bounded by this $g$-degree, a contradiction. Thus these polynomials must all be identically constant, and therefore, for all $g \in F$, we have

$$\phi(u_xw_ix) = \phi_j = \phi(u_0w_ix) = \phi(w_ix).$$

Since $\phi(w_ix) = \phi_j$, it follows that $w_ix \in G_{p,q}$, and the lemma is proved. \[\square\]

Now let $G \subseteq GL_n(F)$ be endowed with the Zariski topology. As will be apparent, the remainder of the argument is topological in nature. Define

$$W = \{w \in G_{p,q}|\phi(g^{-1}u_xgw) = \phi(w) \text{ for all } g \in G, \lambda \in F\},$$

and set

$$W' = \{w \in G|\phi(g^{-1}u_xgw) = \phi(w) \text{ for all } g \in G, \lambda \in F\}.$$

Notice that $W \subseteq W'$ and $W'$ is Zariski closed in $G$. Thus, if $W$ denotes the closure of $W$ in $G$, then $W \subseteq W \subseteq W'$. Furthermore, $1 \in W \subseteq W$ since $u_x$ is unipotent. Recall that a topological space is irreducible if it is not the union of two proper closed subspaces.

**Lemma 4.** If $w \in W$, then $w_x \in W$ for some $1 \leq i \leq s$.

**Proof.** First let $w \in W$ and, for each $i$ with $w_x \in G_{p,q}$, let

$$B_i = \{g \in G|\phi(g^{-1}u_xgw) = \phi(w_x) \text{ for all } g \in G, \lambda \in F\}.$$

Since $w$ is fixed, it is clear that each $B_i$ is a Zariski closed subset of $G$. If $g \in G$, then $g^{-1}u_xg = 1 + \sum_{a,b} \lambda^a g^{-1}v_{a,b}g \in U(G)$, and therefore Lemma 3, with $u_x$ replaced by $g^{-1}u_xg$, implies that there exists $i$ with $w_x \in G_{p,q}$ and

$$\phi(g^{-1}u_xgw) = \phi(w_x)$$

for all $g \in F$. In other words, $g \in B_i$, and we have shown that $G = \bigcup_i B_i$. Furthermore, $G$ is connected by [W, Lemma 5.2], so we conclude from [W, Lemma 14.3] that $G$ is irreducible. Thus $G = B_j$ for some $j$, and hence $w_x \in W$.

Now let $\overline{w} \in \overline{W}$, and suppose, by way of contradiction, that no $i$ exists with $\overline{w}_x \in \overline{W}$. Then, for each $i$, there exists an open subset $\mathcal{G}_i$ of $G$ with $\overline{w}_x \in \mathcal{G}_i$ and $\mathcal{G}_i \cap W = \emptyset$. Now each $\mathcal{G}_i^{-1}$ is open and contains $\overline{w}$, so $\mathcal{G} = \bigcap_i \mathcal{G}_i^{-1}$ is an open neighborhood of $\overline{w} \in \overline{W}$. Therefore, there exists $w' \in \mathcal{G} \cap W$. Finally, by the result of the previous paragraph, $w'x_k \in W$ for some $k$ and hence $w'x_k \in \mathcal{G}_k \cap W$, a contradiction \[\square\]

Since the closed subsets of $\overline{W}$ satisfy the descending chain condition, it follows from the proof of [W, Lemma 14.3] that $\overline{W} = C_1 \cup C_2 \cup \cdots \cup C_i$ is uniquely a finite irredundant union of closed irreducible subspaces $C_i$. We call these $C_i$ the irreducible components of $\overline{W}$.

**Lemma 5.** $\overline{W}$ and each $C_j$ are stable under the conjugation action of $G$. Furthermore, for each irreducible component of $\overline{W}$, there exist $x_i$ and an irreducible component $C'$ with $C_ix_i \subseteq C'$.

**Proof.** If $w \in W$ and $y \in G$, then

$$\phi(y^{-1}wy) = \phi(w) = \phi(\phi(y^{-1}u_xgy^{-1}w) = \phi(g^{-1}u_xgy^{-1}wy)$$

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for all \( \lambda \in F \) and \( g \in G \). Since \( y^{-1}wy \in G_{p,q} \), it follows that \( y^{-1}wy \in W \).

In other words, \( W \) is stable under \( G \)-conjugation, and hence so is \( \bar{W} \). This implies that \( G \) permutes the finitely many irreducible components \( C_j \) and, since \( G \) has no proper subgroup of finite index, the first part is proved.

Next, notice that

\[
\bar{W}_i = \{ \bar{w} \in \bar{W} | \bar{w}x_i \in \bar{W} \} = \bar{W} \cap \bar{W} x_i^{-1}
\]

is a closed subset of \( \bar{W} \) and that \( \bar{W} = \bigcup_i \bar{W}_i \) by the previous lemma. Thus \( C = \bigcup_i (\bar{W}_i \cap C) \) and, since \( C \) is irreducible, it follows that \( \bar{W}_i \supseteq C \) for some \( i \).

In other words, \( Cx_j \subseteq \bar{W} \) and, since \( Cx_j \) is irreducible and \( Cx_i \subseteq \bigcap_j C_j \), we have \( Cx_i \subseteq C' \) for some appropriate \( C' \).

We can now complete the proof of

**Theorem 6.** If \( G \) is a simple group of Lie type over an infinite locally finite field \( F \), then any twisted group algebra \( K'[G] \) is semiprimitive.

**Proof.** If \( K \) has characteristic 0, then \( K'[G] \) is semiprimitive for any locally finite group \( G \). Thus it suffices to assume that \( \text{char} K = p > 0 \). We continue with the preceding notation. In addition, let \( \mathcal{P} \) denote the set \( \{x_1, x_2, \ldots, x_t\} \), and write \( C = \{C_1, C_2, \ldots, C_t\} \).

Let \( C(1) \) be any irreducible component of \( \bar{W} \). Then, by applying the previous lemma repeatedly, we can construct a sequence \( C(1), C(2), C(3), \ldots \) of elements of \( \mathcal{P} \) and a sequence \( y_1, y_2, y_3, \ldots \) of elements of \( \mathcal{P} \) such that \( C(k)y_k \subseteq C(k + 1) \) for all \( k \geq 1 \). Since \( |\mathcal{P}| = t < \infty \), it follows that \( C = C(i) = C(j) \) for some \( i < j \). Set \( D = C(i + 1) \), and note that \( C y_i \subseteq D \) and \( Dz \subseteq C \) where \( z = y_{i+1} \cdots y_{j-1} \in G \). But \( C \) and \( D \) are \( G \)-stable, so \( Cg^{-1}y_ig \subseteq D \), and hence \( Cg^{-1}y_igz \subseteq C \) for all \( g \in G \). In particular, since \( y_i \neq 1 \), it follows that \( y_i \) is not central, and therefore we can find \( 1 \neq h \in G \) with \( Ch \subseteq C \).

Let \( H = \{g \in G | Cg \subseteq C \} \). Then \( H \) is surely closed under multiplication, so \( H \) is a subgroup of the periodic group \( G \). Furthermore, since \( C \) is \( G \)-stable, it follows that \( H \triangleleft G \). Thus, since \( H \neq 1 \) by the above, we conclude that \( H = G \) and therefore that \( C \supseteq CG = G \). In other words, \( C = G \), so \( \bar{W} = G \), and hence \( W' = G \).

Finally, write \( U = U(G) \), and let \( L = \{g \in G | gU \subseteq U \} \). Again it is clear that \( L \triangleleft G \). Furthermore, as we will see, \( u_\lambda \in L \) for all \( \lambda \in F \). Indeed, if \( w \in U \), then \( w \in G = W' \), so \( \phi(u_\lambda w) = \phi(w) \) for all \( \lambda \in F \). Since \( w \) is unipotent, this implies that \( u_\lambda w \) is also unipotent, and hence \( u_\lambda w \in U \), as required. Furthermore, by Lemma 2, \( u_\lambda \) is not identically equal to 1, so it follows that \( L \neq 1 \) and therefore that \( L = G \). But then \( GU \subseteq U \), so \( G = U \) is unipotent, a contradiction. \( \square \)

A number of corollaries now follow quite quickly. To start with, if \( G \) is an infinite locally finite simple group which is also a linear group, then by [HS, Theorem B] or [T, Theorem 2] we know that \( G \) must be a simple group of Lie type over an infinite locally finite field. Thus the preceding result, along with [PZ, Theorem 1.1] and the remark at the end of that paper, yields

**Corollary 7.** Let \( G \) be an infinite locally finite simple group. Then any twisted group algebra \( K'[G] \) is semiprimitive.

Next, we conclude from [FS, Theorem 2.2] and the above that
Corollary 8. Let $G$ be a countably infinite locally finite simple group. Then any twisted group algebra $K'[G]$ is primitive. In particular, this applies to the simple groups of Lie type over infinite locally finite fields.

Finally, by Theorem 6 and the argument of [P, Theorem 6.1], we obtain

Corollary 9. Let $K$ be a field of characteristic $p > 0$, and let $G$ be a locally finite group having a finite subnormal series

$$1 = G_0 < G_1 < \cdots < G_n = G$$

with each quotient $G_i/G_{i-1}$ either (i) a $p'$-group, or (ii) a finite simple group, or (iii) a simple group of Lie type over an infinite locally finite field. Then $JK'[G]$ is nilpotent, so

$$JK'[G] = JK'[\Delta^p(G)]K'[G]$$

with $\Delta^p(G)$ finite. Furthermore, $K'[G]$ is semiprimitive if $G$ has no finite normal subgroup of order divisible by $p$.

We remark in closing that [Z] contains a direct proof that the ordinary group algebra $K[G]$ is primitive if $G$ belongs to one of the classical families $\text{PSL}_n$, $\text{PSp}_n$, $\text{PO}_n$, or $\text{PSU}_n$, all with $p \neq q$. The argument uses known bounds on the composition lengths of certain permutation modules associated with the finite simple groups in these families.

References


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