A HOMOLOGICAL CHARACTERIZATION OF ABELIAN $B_2$-GROUPS

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Abstract. Assuming the Continuum Hypothesis, we show that a torsion-free abelian group $G$ is a $B_2$-group if and only if $\text{Bext}^1(G, T) = 0 = \text{Bext}^2(G, T)$, for every torsion group $T$.

Introduction

All the groups considered here, unless otherwise stated, are torsion-free abelian groups. For unexplained notation and terminology we refer to Fuchs [F]. A torsion-free abelian group $G$ is called a $B_2$-group if $G$ is a union of a continuous well-ordered ascending chain of pure subgroups

$$0 = G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset \cdots \subset G_\tau = G$$

such that, for each $\alpha < \tau$, $G_{\alpha+1} = G_\alpha + B_\alpha$, where $B_\alpha$ is a finite rank pure subgroup of a completely decomposable group. Such $B_\alpha$ are also called finite rank Butler groups. Our main result, under CH, is a homological characterization of $B_2$-groups: A torsion-free abelian group $G$ is a $B_2$-group if and only if $\text{Bext}^1(G, T) = 0 = \text{Bext}^2(G, T)$, for every torsion group $T$. (Here $\text{Bext}^1$ denotes the subfunctor of Ext$^1$ consisting of all balanced extensions.)

We shall briefly discuss the implication of this result. A torsion-free abelian group $G$ is called a $B_1$-group (or a Butler group) if $\text{Bext}^1(G, T) = 0$ for every torsion group $T$. Two open problems in the theory of $B_1$-groups are: (1) whether $\text{Bext}^2(G, T) = 0$ for all torsion-free groups $G$ and all torsion groups $T$ and (2) whether every $B_1$-group $G$ is a $B_2$-group.

Affirmative answers to these questions were obtained by Bican and Salce [BS] for countable groups and in [AH, DHR] for groups of cardinality $\aleph_1$. Under continuum hypothesis, the same results were shown to hold for groups of cardinalities $\leq \aleph_\omega$ in [DHR] while in [R] (in [FR]) affirmative answers were obtained, under ZFC, when the groups are (unions of pure subgroups) with at most countable typesets. Recently, Fuchs and Magidor [FM] extended these results to groups of arbitrary cardinalities in the constructible universe.
On the other hand, Dugas and Thome [DT] showed that the negation of the continuum hypothesis leads to a negative answer to question (1).

In view of our characterization of $B_2$-groups, a $B_1$-group $G$ of arbitrary cardinality will be, under CH, a $B_2$-group exactly when $\text{Bext}^2(G,T) = 0$. Thus the two problems mentioned above can effectively be reduced to the consideration of the first problem. Actually we prove something more. Suppose $C$ is a $B_2$-group and $H$ a TEP subgroup of $C$ with $C/H = G$. We show, under ZFC, that if $H$ is a $B_2$-group, so is $G$ and, under CH, if $G$ is a $B_2$-group, then $H$ becomes a $B_2$-group, provided $H$ is balanced in $C$.

**Preliminaries**

A subgroup $A$ of a group $G$ is *TEP* in $G$ or is a *TEP subgroup* (has the torsion extension property) if every homomorphism from $A$ to a torsion group $T$ extends to a homomorphism from $G$ to $T$ (see [DR]). A pure subgroup $A$ is said to be *decent* in $G$ (see [AH]), if for every finite rank pure subgroup $C/A$ of $G/A$, there exists a finite rank Butler group $B$ such that $C = A + B$. A group $G$ is called *finitely Butler* if every finite rank pure subgroup of $G$ is a Butler group. The characteristic of an element $a$ is denoted by $\chi(a)$. A pure subgroup $A$ of a group $G$ is said to be *balanced* if, for any $g \in G$, the coset $g + A$ contains an element $g + a$ ($a \in A$) such that $\chi(g + x) \leq \chi(g + a)$ for all $x \in A$. Suppose $A$ and $B$ are subgroups of $G$. Following Hill (see, for example, [AH]), we write $A \| B$ and say $A$ is *compatible with* $B$ in $G$, if for each pair $(a,b) \in A \times B$ there exists $c \in A \cap B$ such that $\chi(a + b) \leq \chi(a + c)$. Observe that if $A \| B$ then $B \| A$.

A collection $C$ of subgroups of $G$ is called an *axiom-3 family*, if $C$ satisfies the following conditions: (a) $0, G \in C$; (b) if $\{S_i : i \in I\}$ is any set of subgroups in $C$, then their group union $\sum \{S_i : i \in I\}$ also belongs to $C$; and (c) if $A \in C$ and $X$ is a countable subset of $G$, then there is a $B \in C$ containing both $A$ and $X$ such that $B/A$ is countable. Let $\kappa$ be an infinite cardinal. In the definition of the axiom-3 family above, if condition (b) is required only for ascending chains $\{S_i : i \in I\}$ and, if in condition (c), countability is replaced by cardinality $\leq \kappa$, then we say that the family $C$ is a *$G(\kappa)$-family*. It is known (see [AH, R]) that every $B_2$-group possesses an axiom-3 family of pure decent TEP subgroups.

Suppose $0 \to H \to C \to G \to 0$ is a balanced exact sequence, where $C$ is completely decomposable. Then, for any torsion group $T$, applying the functor $\text{Bext}(\cdot, T)$ we obtain an exact sequence

$$\text{Bext}^1(C,T) = 0 \to \text{Bext}^1(H,T) \to \text{Bext}^2(G,T) \to \text{Bext}^2(C,T) = 0$$

so that $\text{Bext}^1(H,T) \cong \text{Bext}^2(G,T)$. This fact will be tacitly used in the sequel.

The reader is especially recommended to consult [DHR] from which we shall be using a number of results and concepts.

**Main results**

We begin with the following useful observations.

*Observation 1.* Let $G = \bigcup_{\alpha < \tau} G_\alpha$ be a $B_2$-group where, for each $\alpha < \tau$, $G_{\alpha + 1} = G_\alpha + B_\alpha$, with $B_\alpha$ a finite rank Butler group. For later reference, we say $\{G_\alpha : \alpha < \tau\}$ is a $B_2$-filtration of $G$ and that $G_\alpha$ fits into a $B_2$-filtration of $G$. A
subset $S$ of ordinals $< \tau$ is called closed if for each $\lambda \in S$, we have $G_\lambda \cap B_\lambda \subset \sum\{B_\alpha : \alpha \in S\}$ (as pointed out in [FM], the additional purity requirement in the definition of closed sets given in [AH] is superfluous). For a closed set $S$ of ordinals $< \tau$, let $G(S)$ denote the subgroup generated by $\{B_\alpha : \alpha \in S\}$. In [AH], it was shown (see [FM] for a corrected proof) that the family $\mathbb{C} = \{G(S) : S \text{ a closed set}\}$ is an axiom-3 family of pure decent subgroups. In [R], it was pointed out that members of the family $\mathbb{C}$ above are also TEP in $G$. Considering the closed subsets contained in $S$, it is clear that each $G(S)$ itself has an axiom-3 family of decent subgroups $\{G(S') : S' \subset S, S' \text{ closed}\}$ and hence $G(S)$ is a $B_2$-group. Thus a $B_2$-group $G$ has an axiom-3 family $\mathbb{C}$ of pure decent TEP subgroups each of which is a $B_2$-group and each of which fits into a $B_2$-filtration of $G$ so that $G/A$ is again a $B_2$-group for all $A \in \mathbb{C}$.

Observation 2. An examination of the proof of Corollary 6.3 of [DHR] reveals that we could drop the condition that $G$ is a Butler group from its hypothesis. The modified statement is: A torsion-free abelian group $G$ of singular cardinality $\lambda$ is a $B_2$-group if it admits a $\lambda$-family of pure $B_2$-subgroups in the sense of Definition 6.1 of [DHR].

Remark. It is worth recording the following interesting observation by Professor L. Fuchs: A TEP subgroup $H$ of a torsion-free group $C$ is always pure. Because, otherwise, for some prime $p$, there is an element $a \in (H \cap pC) \setminus pH$ and a homomorphism $\varphi : H \to \mathbb{Z}(p)$, the cyclic group of order $p$, with $\varphi(a) \neq 0$. This $\varphi$ cannot be extended to a homomorphism from $C$ to $\mathbb{Z}(p)$.

Consider a TEP exact sequence

$$0 \to H \to C \to G \to 0$$

of torsion-free abelian groups.

Question. In the above sequence, if any two of three groups are $B_2$-groups, is the third group also a $B_2$-group?

The first case, when $H$ and $G$ are $B_2$-groups, has already been considered in [DHR] and we rephrase it in the next lemma and shall be using it several times in the sequel.

Lemma 3 [DHR, Proposition 3.9]. In the TEP sequence (1), if both $H$ and $G$ are $B_2$-groups, then so is $C$, provided $C$ is finitely Butler. In this case $H$, considered as a subgroup of $C$, fits into a $B_2$-filtration of $C$.

The next theorem considers the first of the two remaining cases.

Theorem 4. In the TEP exact sequence (1) above, if $H$ and $C$ are $B_2$-groups, then so is $G$.

Proof. For convenience in writing, consider $H$ as a subgroup of $C$ with $C/H = G$. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be axiom-3 families of pure decent TEP $B_2$-subgroups of $H$ and $C$ respectively. We claim that $\mathbb{C}_3' = \{X \in \mathbb{C}_2 : X + H \text{ pure in } C\}$ is a $G(\aleph_0)$-family. We need only to verify the condition (c) of a $G(\aleph_0)$-family. Suppose $A \in \mathbb{C}_3'$ so that $A + H$ is pure in $C$. Let $S$ be a countable subset of $C$. Construct a countable ascending chain of subgroups $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ such that $A \cup S \subset A_1$, for each $i \geq 1$, $A_i \in \mathbb{C}_2$, $|A_i/A| \leq \aleph_0$, and $(A_{i+1} + H)/(A + H)$ contains the purification of $(A_i + H)/(A + H)$ in
$C/(A + H)$. Then $B = \bigcup_{i<\omega} A_i \subseteq C_2$ and $B + H$ is pure in $C$ so that $B \subseteq C_3$. Since $A \cup S \subseteq B$ and $B/A$ is countable, condition (c) is satisfied. A similar back-and-forth argument implies that $C_3 = \{X \in C_3 : X \cap H \subseteq C_1\}$ is a $G(\aleph_0)$-family. Since $H$ is TEP in $C$, by Corollary 4.3 of [DHR], $G$ is a $B_1$-group. Also for all $X \in C_3$, $X \cap H$ is TEP in $H$ and hence in $X$. Then $C = \{(X + H)/H : X \subseteq C_3\}$ is a $G(\aleph_0)$-family of pure $B_1$-subgroups of $G$. We apply induction on the cardinality $\kappa$ of $G$. If $\kappa \leq \aleph_0$, we could assume $C$ is countable and then Theorem 7 of [DR] implies that $H$ is decent $C$. By [BS], we conclude that $G$ is a $B_2$-group. Suppose $\kappa > \aleph_0$ and assume the theorem holds for groups $G$ of cardinality $< \kappa$.

Suppose $\kappa$ is a regular cardinal. Using the family $C$, build a smooth $\kappa$-filtration $G = \bigcup_{\alpha<\kappa} G_\alpha$ where, for each $\alpha < \kappa$, $G_\alpha \subseteq C$, and $|G_\alpha| < \kappa$. By Theorem 7.1 of [DHR], we may assume each $G_\alpha$ is TEP in $G$ which implies that $G_{\alpha+1}/G_\alpha$ is a $B_1$-group. Since $G_\alpha \subseteq C$, write, for each $\alpha < \kappa$, $G_\alpha = (X_\alpha + H)/H$ where $X_\alpha \subseteq C_3$ and let $H_\alpha = H \cap X_\alpha$. By induction, each $G_\alpha$ is a $B_2$-group so that $G$ is finitely Butler. Clearly

$$G_{\alpha+1}/G_\alpha \cong (X_{\alpha+1} + H)/(X_\alpha + H) \cong X_{\alpha+1}/(X_{\alpha+1} \cap (X_\alpha + H)) = X_{\alpha+1}/(X_\alpha + H_{\alpha+1}).$$

We claim that $X_\alpha + H_{\alpha+1}$ is a $B_2$-group which is TEP in $X_{\alpha+1}$. The TEP property follows if one observes that $H_{\alpha+1}$ is TEP in $X_{\alpha+1}$ and that $(X_\alpha + H_{\alpha+1})/H_{\alpha+1} \cong G_\alpha$ is TEP in $X_{\alpha+1}/H_{\alpha+1} \cong G_{\alpha+1}$. Moreover, since both $H_{\alpha+1}$ and $(X_\alpha + H_{\alpha+1})/H_{\alpha+1} \cong G_{\alpha+1}$ are $B_2$-groups and $X_\alpha + H_{\alpha+1}$ is finitely Butler, Lemma 3 implies that $X_\alpha + H_{\alpha+1}$ is a $B_2$-group. Thus we get a TEP exact sequence

$$0 \to X_\alpha + H_{\alpha+1} \to X_{\alpha+1} \to G_{\alpha+1}/G_\alpha \to 0$$

where the first two groups are $B_2$-groups and all the groups have cardinality $< \kappa$. By induction hypothesis, $G_{\alpha+1}/G_\alpha$ is therefore a $B_2$-group. Lemma 3 then implies that $G_\alpha$ fits into a $B_2$-filtration of $G_{\alpha+1}$. Consequently, $G = \bigcup_{\alpha<\kappa} G_\alpha$ is a $B_2$-group.

Suppose $\kappa$ is a singular cardinal. Now the family $C^* = \{Y \subseteq C : |Y| < \kappa\}$ is easily seen to be a $\kappa$-family in the sense of Definition 6.1 of [DHR]. Since each member of $C^*$ is, by induction, a $B_2$-group, Observation 2 implies that $G$ is a $B_2$-group.

**Corollary 5.** Let $G = C/H$ be a $B_1$-group, where $C$ is completely decomposable and $H$ is balanced in $C$. Then $G$ is a $B_2$-group if $H$ is.

**Proof.** Just observe that the balanced subgroup $H$ is TEP in $C$ since $G$ is a $B_1$-group. Then Theorem 4 applies.

By [AH], the group $H$ above will always be a $B_2$-group if $|H| \leq \aleph_1$. Then Corollary 5 yields the following

**Corollary 6** [DHR]. A $B_1$-group of cardinality $\leq \aleph_1$ is a $B_2$-group.

To consider the next case in our question, we begin with the following generalization of Theorem 7.5 of [DHR]. We first state a useful lemma.

**Lemma 7** [DHR]. Let $A$, $B$, and $H$ be pure subgroups of a torsion-free group $G$.

(a) If $A$ is balanced in $G$ and $A\Vert H$, then $A \cap H$ is balanced in $H$.

(b) If $A\Vert H$ and $H + A\Vert B$ then $H\Vert A + B$.
Proof. See the proof of Lemma 7.2 of [DHR].

**Proposition 8.** Let $C = \bigcup_{\alpha<\tau} \mathcal{C}_{\alpha}$ be a fixed $B_2$-filtration of a $B_2$-group $C$ and let $H$ be a pure subgroup of $C$. Then there exists a $G(2^{\aleph_0})$-family of pure decent subgroups of the form $C(S)$, where $S$ is a closed set of ordinals $< \tau$ such that $C(S)||H$.

**Proof.** We claim $C = \{C(S) : S \text{ a closed set and } C(S)||H\}$ is a $G(2^{\aleph_0})$-family. The same proof of Theorem 7.5 of [DHR] with easy modifications works here. We shall indicate the proof for the sake of completeness. Since, by [AH], arbitrary union of closed sets are closed, conditions (a) and (b) of a $G(2^{\aleph_0})$-family hold (see the preliminaries above). To verify the condition (c), let $C(J) \in C$ and let $A$ be a subset of $C$ with cardinality $\leq 2^{\aleph_0}$. Observe that every infinite subset of $C$ can be embedded in a subgroup of the form $C(S)$ having the same cardinality. By induction on $n$, we define closed subsets $Z_n$ of cardinality $\leq 2^{\aleph_0}$. Let $Z_1$ be minimal with $A \subset C(Z_1)$. Clearly $|Z_1| \leq 2^{\aleph_0}$. Suppose $Z_n$ has already been defined. Let $g \in C(Z_n)$. Consider the coset $g + (H + C(J))$. Since there are only $2^{\aleph_0}$ height sequences, we can find a subset of at most $2^{\aleph_0}$ elements $\{h_{\alpha,g}, \alpha < 2^{\aleph_0}\} \subset H + C(J)$ so that for each $y \in H + C(J)$ there is an $h_{\alpha,g}$ such that $\chi(g+y) \leq \chi(g+h_{\alpha,g})$. Let $Z'$ be a closed set minimal with respect to $\{h_{\alpha,g}, \alpha < 2^{\aleph_0}, g \in C(Z_n)\} \subset C(Z')$. Then define $Z_{n+1} = Z_n \cup Z'$. If we let $Z = \bigcup_{n<\alpha} Z_n$, then $|C(Z)| \leq 2^{\aleph_0}$ and $H + C(J)||C(Z)$. Also $H||C(J)$, since $C(J) \in C$. Then, by Lemma 7(b), $H||C(J) + C(Z)$. If $J^* = J \cup Z$, then $C(J^*) = C(J \cup Z) = C(J) + C(Z)$ contains $C(J)$ and $A$, $|C(J^*)/C(J)| \leq 2^{\aleph_0}$ and $C(J^*)||H$. Thus $C$ is a $G(2^{\aleph_0})$-family.

We are now ready to investigate whether, in the exact sequence (1), the group $H$ will be a $B_2$-group if $C$ and $G$ are both $B_2$-groups. The answer is in the affirmative if we assume Continuum Hypothesis (CH) and that the sequence (1) is balanced.

**Theorem 9 (CH).** Suppose $0 \rightarrow H \rightarrow C \rightarrow G \rightarrow 0$ is a balanced exact sequence. If $C$ and $G$ are $B_2$-groups, so is $H$.

**Proof.** We shall consider $H$ a subgroup of $C$ with $C/H = G$. By [AH] and [R], $G$ has an axiom-3 family $\mathbb{R}$ or pure decent TEP subgroups each of which is a $B_2$-group. Proposition 8 above yields a $G(\aleph_1)$-family $C^*$ of pure decent TEP subgroups of $C$ of the form $C(S)$, where $S$ a closed set such that $C(S)||H$. Then $C^* = \{X \in C^* \setminus (X + H)/H \in \mathbb{R}\}$ is a $G(\aleph_1)$-family. Let $C = \{H \cap X : X \in C^*\}$. By Lemma 7(a), $H \cap X$ is balanced in $X$ for all $X \in C^*$.

We apply induction on the cardinality $\kappa$ of $H$. If $\kappa \leq \aleph_1$, then replacing $C$, if necessary, by a suitable $C(S) \in C^*$, we may assume that $|C| = \kappa$ and then $H$ is a $B_2$-group by [DHR, Proposition 3.11]. Assume $\kappa > \aleph_1$ and that the theorem holds for cardinalities $< \kappa$. Suppose $\kappa$ is regular. By using the $G(\aleph_1)$-family $C^*$, build a smooth $\kappa$-filtration $H = \bigcup_{\alpha<\kappa} H_{\alpha}$, where, for each $\alpha < \kappa$, $H_{\alpha} \in C^*$, say $H_{\alpha} = H \cap X_{\alpha}$ for some $X_{\alpha} \in C^*$, $|H_{\alpha}| < \kappa$ and $|H_{\alpha+1}/H_{\alpha}| \leq \aleph_1$. Since $H_{\alpha}$ is balanced in $X_{\alpha}$ and $X_{\alpha}/H_{\alpha}$ is a $B_2$-group, $H_{\alpha}$ is a TEP subgroup of $X_{\alpha}$. Since $X_{\alpha}$ is TEP in $C$, $H_{\alpha}$ is TEP in $C$ and hence in $H_{\alpha+1}$. Then $H_{\alpha+1}/H_{\alpha}$ is a Butler group of cardinality $\leq \aleph_1$ and so, by [DHR], a $B_2$-group. Then, by Lemma 3, $\bigcup H_{\alpha}$ refines to a $B_2$-filtration of $H$.

Suppose $\kappa$ is a singular cardinal. Since $C^*$ is a $G(\aleph_1)$-family, $C^{**} = \{Y :
$Y \in C$ and $|Y| < \kappa$ is readily seen to a $\kappa$-family in the sense of Definition 6.1 of [DHR] and, by induction hypothesis, members of $C^{**}$ are all $B_2$-groups. By observation 2 above, $H$ is then a $B_2$-group. Hence the theorem.

**Corollary 10 (CH).** $\text{Bext}^2(G, T) = 0$ for any $B_2$-group $G$ and any torsion group $T$.

Theorem 4 and Corollary 10 enable us to answer when a $B_1$-group is $B_2$.

**Theorem 11 (CH).** Let $G$ be a $B_1$-group. Then $G$ is $B_2$-group if and only if $\text{Bext}^2(G, T) = 0$ for every torsion group $T$.

**Proof.** Suppose $\text{Bext}^2(G, T) = 0$ for all torsion $T$. Consider a balanced exact sequence

$$0 \to H \xrightarrow{\varphi} C \to G \to 0$$

where $C$ is completely decomposable. Then for any torsion group $T$, we obtain induced exact sequences

$$\text{Hom}(C, T) \xrightarrow{\varphi^*} \text{Hom}(H, T) \to \text{Bext}^1(G, T) = 0$$

and

$$\text{Bext}^1(C, T) = 0 \to \text{Bext}^1(H, T) \to \text{Bext}^2(G, T) = 0.$$  

Since $\varphi^*$ is an epimorphism, (2) is a TEP exact sequence and since $\text{Bext}^1(H, T) = 0$, $H$ is a $B_1$-group. Now there are at most $2^{\aleph_0} = \aleph_1$ types in the typeset of $C$ and so $C$ can be written as the union of a smooth increasing chain direct summands each of which has its typeset at most countable. Intersecting $H$ with these direct summands, $H$ becomes a union of a smooth increasing chain of pure subgroups each with an at most countable typeset. Since $H$ is also a $B_1$-group, Theorem 3.1 of [FR] implies that $H$ is a $B_2$-group. Then Theorem 4 shows that $G$ is a $B_2$-group. The converse follows from Corollary 10.

**Remark.** Fuchs and Magidor [FM] show, under $(V = L)$, that $\text{Bext}^2(G, T) = 0$ for any torsion-free group $G$ and any torsion group $T$. This, together with Theorem 11, immediately gives their following main theorem (Theorem 10.1) (thus making most of §§5 through 10 (except §7) in [FM], in some sense, redundant): $(V = L)$. Every $B_1$-group is a $B_2$-group.

In [BS] it was shown that a $B_2$-group is always a $B_1$-group. From Theorem 11 and Corollary 10, we then obtain the following homological characterization of $B_2$ groups:

**Theorem 12 (CH).** A torsion-free abelian group $G$ is a $B_2$-group if and only if $\text{Bext}^1(G, T) = 0 = \text{Bext}^2(G, T)$ for every torsion group $T$.

**References**


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