PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 2, June 1994

A HOMOLOGICAL CHARACTERIZATION OF ABELIAN B_2-GROUPS

K. M. RANGASWAMY

(Communicated by Ronald Solomon)

ABSTRACT. Assuming the Continuum Hypothesis, we show that a torsion-free abelian group G is a B_2-group if and only if $\mathrm{Bext}^1(G, T) = 0 = \mathrm{Bext}^2(G, T)$, for every torsion group T.

INTRODUCTION

All the groups considered here, unless otherwise stated, are torsion-free abelian groups. For unexplained notation and terminology we refer to Fuchs [F]. A torsion-free abelian group G is called a B_2-group if G is a union of a continuous well-ordered ascending chain of pure subgroups

$$0 = G_0 \subset G_1 \subset \cdots \subset G_\alpha \subset \cdots \subset G_\tau = G$$

such that, for each $\alpha < \tau$, $G_{\alpha+1} = G_\alpha + B_\alpha$, where B_α is a finite rank pure subgroup of a completely decomposable group. Such B_α are also called finite rank Butler groups. Our main result, under CH, is a homological characterization of B_2-groups: A torsion-free abelian group G is a B_2-group if and only if $\mathrm{Bext}^1(G, T) = 0 = \mathrm{Bext}^2(G, T)$, for every torsion group T. (Here Bext^1 denotes the subfunctor of Ext^1 consisting of all balanced extensions.)

We shall briefly discuss the implication of this result. A torsion-free abelian group G is called a B_1-group (or a Butler group) if $\mathrm{Bext}^1(G, T) = 0$ for every torsion group T. Two open problems in the theory of B_1-groups are: (1) whether $\mathrm{Bext}^2(G, T) = 0$ for all torsion-free groups G and all torsion groups T and (2) whether every B_1-group G is a B_2-group.

Affirmative answers to these questions were obtained by Bican and Salce [BS] for countable groups and in [AH, DHR] for groups of cardinality $\aleph_1$. Under continuum hypothesis, the same results were shown to hold for groups of cardinalities $\leq \aleph_\omega$ in [DHR] while in [R] (in [FR]) affirmative answers were obtained, under ZFC, when the groups are (unions of pure subgroups) with at most countable typesets. Recently, Fuchs and Magidor [FM] extended these results to groups of arbitrary cardinalities in the constructible universe

Received by the editors August 7, 1992 and, in revised form, September 30, 1992.
1991 *Mathematics Subject Classification*. Primary 20K20; Secondary 20K35, 20K40.
Key words and phrases. Torsion-free abelian groups, Butler groups, B_2-groups, pure subgroups, axiom-3 family of subgroups.

L. On the other hand, Dugas and Thome [DT] showed that the negation of the continuum hypothesis leads to a negative answer to question (1).

In view of our characterization of B_2-groups, a B_1-group G of arbitrary cardinality will be, under CH, a B_2-group exactly when $\mathrm{Bext}^2(G, T) = 0$. Thus the two problems mentioned above can effectively be reduced to the consideration of the first problem. Actually we prove something more. Suppose C is a B_2-group and H a TEP subgroup of C with $C/H = G$. We show, under ZFC, that if H is a B_2-group, so is G and, under CH, if G is a B_2-group, then H becomes a B_2-group, provided H is balanced in C.

Preliminaries

A subgroup A of a group G is *TEP* in G or is a *TEP subgroup* (has the torsion extension property) if every homomorphism from A to a torsion group T extends to a homomorphism from G to T (see [DR]). A pure subgroup A is said to be *decent* in G (see [AH]), if for every finite rank pure subgroup C/A of G/A, there exists a finite rank Butler group B such that $C = A + B$. A group G is called *finitely Butler* if every finite rank pure subgroup of G is a Butler group. The characteristic of an element a is denoted by $\chi(a)$. A pure subgroup A of a group G is said to be *balanced* if, for any $g \in G$, the coset $g + A$ contains an element $g + a$ $(a \in A)$ such that $\chi(g + x) \leq \chi(g + a)$ for all $x \in A$. Suppose A and B are subgroups of G. Following Hill (see, for example, [AH]), we write $A \| B$ and say A is *compatible* with B in G, if for each pair $(a, b) \in A \times B$ there exists c in $A \cap B$ such that $\chi(a+b) \leq \chi(a+c)$. Observe that if $A \| B$ then $B \| A$.

A collection $\mathbb{C}$ of subgroups of G is called an *axiom-3 family*, if $\mathbb{C}$ satisfies the following conditions: (a) $0, G \in \mathbb{C}$; (b) if $\{S_i : i \in I\}$ is any set of subgroups in $\mathbb{C}$, then their group union $\sum\{S_i : i \in I\}$ also belongs to $\mathbb{C}$; and (c) if $A \in \mathbb{C}$ and X is a countable subset of G, then there is a $B \in \mathbb{C}$ containing both A and X such that B/A is countable. Let κ be an infinite cardinal. In the definition of the axiom-3 family above, if condition (b) is required only for ascending chains $\{S_i : i \in I\}$ and, if in condition (c), countability is replaced by cardinality $\leq \kappa$, then we say that the family $\mathbb{C}$ is a *$G(\kappa)$-family*. It is known (see [AH, R]) that every B_2-group possesses an axiom-3 family of pure decent TEP subgroups.

Suppose $0 \to H \to C \to G \to 0$ is a balanced exact sequence, where C is completely decomposable. Then, for any torsion group T, applying the functor $\mathrm{Bext}(-, T)$ we obtain an exact sequence

$$\mathrm{Bext}^1(C, T) = 0 \to \mathrm{Bext}^1(H, T) \to \mathrm{Bext}^2(G, T) \to \mathrm{Bext}^2(C, T) = 0$$

so that $\mathrm{Bext}^1(H, T) \cong \mathrm{Bext}^2(G, T)$. This fact will be tacitly used in the sequel.

The reader is especially recommended to consult [DHR] from which we shall be using a number of results and concepts.

Main results

We begin with the following useful observations.

Observation 1. Let $G = \bigcup_{\alpha<\tau} G_\alpha$ be a B_2-group where, for each $\alpha < \tau$, $G_{\alpha+1} = G_\alpha + B_\alpha$, with B_α a finite rank Butler group. For later reference, we say $\{G_\alpha : \alpha < \tau\}$ is a B_2-filtration of G and that G_α fits into a B_2-filtration of G. A

subset S of ordinals $< \tau$ is called *closed* if for each $\lambda \in S$, we have $G_\lambda \cap B_\lambda \subset \sum\{B_\alpha : \alpha \in S\}$ (as pointed out in [FM], the additional purity requirement in the definition of closed sets given in [AH] is superfluous). For a closed set S of ordinals $< \tau$, let $G(S)$ denote the subgroup generated by $\{B_\alpha : \alpha \in S\}$. In [AH], it was shown (see [FM] for a corrected proof) that the family $\mathbb{C} = \{G(S) : S \text{ a closed set}\}$ is an axiom-3 family of pure decent subgroups. In [R], it was pointed out that members of the family $\mathbb{C}$ above are also TEP in G. Considering the closed subsets contained in S, it is clear that each $G(S)$ itself has an axiom-3 family of decent subgroups $\{G(S') : S' \subset S, S' \text{ closed}\}$ and hence $G(S)$ is a B_2-group. Thus a B_2-group G has an axiom-3 family $\mathbb{C}$ of pure decent TEP subgroups each of which is a B_2-group and each of which fits into a B_2-filtration of G so that G/A is again a B_2-group for all $A \in \mathbb{C}$.

Observation 2. An examination of the proof of Corollary 6.3 of [DHR] reveals that we could drop the condition that G is a Butler group from its hypothesis. The modified statement is: A torsion-free abelian group G of singular cardinality λ is a B_2-group if it admits a λ-family of pure B_2-subgroups in the sense of Definition 6.1 of [DHR].

Remark. It is worth recording the following interesting observation by Professor L. Fuchs: *A TEP subgroup H of a torsion-free group C is always pure.* Because, otherwise, for some prime p, there is an element $a \in (H \cap pC)\backslash pH$ and a homomorphism $\varphi : H \to Z(p)$, the cyclic group of order p, with $\varphi(a) \neq 0$. This φ cannot be extended to a homomorphism from C to $Z(p)$.

Consider a TEP exact sequence

$$0 \to H \to C \to G \to 0 \tag{1}$$

of torsion-free abelian groups.

Question. In the above sequence, if any two of three groups are B_2-groups, is the third group also a B_2-group?

The first case, when H and G are B_2-groups, has already been considered in [DHR] and we rephrase it in the next lemma and shall be using it several times in the sequel.

Lemma 3 [DHR, Proposition 3.9]. *In the TEP sequence* (1), *if both H and G are B_2-groups, then so is C, provided C is finitely Butler. In this case H, considered as a subgroup of C, fits into a B_2-filtration of C.*

The next theorem considers the first of the two remaining cases.

Theorem 4. *In the TEP exact sequence* (1) *above, if H and C are B_2-groups, then so is G.*

Proof. For convenience in writing, consider H as a subgroup of C with $C/H = G$. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be axiom-3 families of pure decent TEP B_2-subgroups of H and C respectively. We claim that $\mathbb{C}_3' = \{X \in \mathbb{C}_2 : X + H \text{ pure in } C\}$ is a $G(\aleph_0)$-family. We need only to verify the condition (c) of a $G(\aleph_0)$-family. Suppose $A \in \mathbb{C}_3'$ so that $A + H$ is pure in C. Let S be a countable subset of C. Construct a countable ascending chain of subgroups $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ such that $A \cup S \subset A_1$, for each $i \geq 1$, $A_i \in \mathbb{C}_2$, $|A_i/A| \leq \aleph_0$, and $(A_{i+1} + H)/(A + H)$ contains the purification of $(A_i + H)/(A + H)$ in

$C/(A+H)$. Then $B = \bigcup_{i<\omega} A_i \in \mathbb{C}_2$ and $B+H$ is pure in C so that $B \in \mathbb{C}_3'$. Since $A \cup S \subset B$ and B/A is countable, condition (c) is satisfied. A similar back-and-forth argument implies that $\mathbb{C}_3 = \{X \in \mathbb{C}_3' : X \cap H \in \mathbb{C}_1\}$ is a $G(\aleph_0)$-family. Since H is TEP in C, by Corollary 4.3 of [DHR], G is a B_1-group. Also for all $X \in \mathbb{C}_3$, $X \cap H$ is TEP in H and hence in X. Then $\mathbb{C} = \{(X+H)/H : X \in \mathbb{C}_3\}$ is a $G(\aleph_0)$-family of pure B_1-subgroups of G. We apply induction on the cardinality κ of G. If $\kappa \leq \aleph_0$, we could assume C is countable and then Theorem 7 of [DR] implies that H is decent C. By [BS], we conclude that G is a B_2-group. Suppose $\kappa > \aleph_0$ and assume the theorem holds for groups G of cardinality $< \kappa$.

Suppose κ is a regular cardinal. Using the family $\mathbb{C}$, build a smooth κ-filtration $G = \bigcup_{\alpha<\kappa} G_\alpha$ where, for each $\alpha < \kappa$, $G_\alpha \in \mathbb{C}$, and $|G_\alpha| < \kappa$. By Theorem 7.1 of [DHR], we may assume each G_α is TEP in G which implies that $G_{\alpha+1}/G_\alpha$ is a B_1-group. Since $G_\alpha \in \mathbb{C}$, write, for each $\alpha < \kappa$, $G_\alpha = (X_\alpha + H)/H$ where $X_\alpha \in \mathbb{C}_3$ and let $H_\alpha = H \cap X_\alpha$. By induction, each G_α is a B_2-group so that G is finitely Butler. Clearly

$$G_{\alpha+1}/G_\alpha \cong (X_{\alpha+1}+H)/(X_\alpha+H) \cong X_{\alpha+1}/(X_{\alpha+1} \cap (X_\alpha+H)) = X_{\alpha+1}/(X_\alpha+H_{\alpha+1}).$$

We claim that $X_\alpha + H_{\alpha+1}$ is a B_2-group which is TEP in $X_{\alpha+1}$. The TEP property follows if one observes that $H_{\alpha+1}$ is TEP in $X_{\alpha+1}$ and that $(X_\alpha + H_{\alpha+1})/H_{\alpha+1} \cong G_\alpha$ is TEP in $X_{\alpha+1}/H_{\alpha+1} \cong G_{\alpha+1}$. Moreover, since both $H_{\alpha+1}$ and $(X_\alpha + H_{\alpha+1})/H_{\alpha+1} \cong G_\alpha$ are B_2-groups and $X_\alpha + H_{\alpha+1}$ is finitely Butler, Lemma 3 implies that $X_\alpha + H_{\alpha+1}$ is a B_2-group. Thus we get a TEP exact sequence

$$0 \to X_\alpha + H_{\alpha+1} \to X_{\alpha+1} \to G_{\alpha+1}/G_\alpha \to 0$$

where the first two groups are B_2-groups and all the groups have cardinality $< \kappa$. By induction hypothesis, $G_{\alpha+1}/G_\alpha$ is therefore a B_2-group. Lemma 3 then implies that G_α fits into a B_2-filtration of $G_{\alpha+1}$. Consequently, $G = \bigcup_{\alpha<\kappa} G_\alpha$ is a B_2-group.

Suppose κ is a singular cardinal. Now the family $\mathbb{C}^* = \{Y \in \mathbb{C} : |Y| < \kappa\}$ is easily seen to be a κ-family in the sense of Definition 6.1 of [DHR]. Since each member of $\mathbb{C}^*$ is, by induction, a B_2-group, Observation 2 implies that G is a B_2-group.

Corollary 5. *Let $G = C/H$ be a B_1-group, where C is completely decomposable and H is balanced in C. Then G is a B_2-group if H is.*

Proof. Just observe that the balanced subgroup H is TEP in C since G is a B_1-group. Then Theorem 4 applies.

By [AH], the group H above will always be a B_2-group if $|H| \leq \aleph_1$. Then Corollary 5 yields the following

Corollary 6 [DHR]. *A B_1-group of cardinality $\leq \aleph_1$ is a B_2-group.*

To consider the next case in our question, we begin with the following generalization of Theorem 7.5 of [DHR]. We first state a useful lemma.

Lemma 7 [DHR]. *Let A, B, and H be pure subgroups of a torsion-free group G.*

(a) *If A is balanced in G and $A \| H$, then $A \cap H$ is balanced in H.*
(b) *If $A \| H$ and $H + A \| B$ then $H \| A + B$.*

Proof. See the proof of Lemma 7.2 of [DHR].

Proposition 8. *Let $C = \bigcup_{\alpha<\tau} C_\alpha$ be a fixed B_2-filtration of a B_2-group C and let H be a pure subgroup of C. Then there exists a $G(2^{\aleph_0})$-family of pure decent subgroups of the form $C(S)$, where S is a closed set of ordinals $< \tau$ such that $C(S)\|H$.*

Proof. We claim $\mathbb{C} = \{C(S) : S$ a closed set and $C(S)\|H\}$ is a $G(2^{\aleph_0})$-family. The same proof of Theorem 7.5 of [DHR] with easy modifications works here. We shall indicate the proof for the sake of completeness. Since, by [AH], arbitrary union of closed sets are closed, conditions (a) and (b) of a $G(2^{\aleph_0})$-family hold (see the preliminaries above). To verify the condition (c), let $C(J) \in \mathbb{C}$ and let A be a subset of C with cardinality $\leq 2^{\aleph_0}$. Observe that every infinite subset of C can be embedded in a subgroup of the form $C(S)$ having the same cardinality. By induction on n, we define closed subsets Z_n of cardinality $\leq 2^{\aleph_0}$. Let Z_1 be minimal with $A \subset C(Z_1)$. Clearly $|Z_1| \leq 2^{\aleph_0}$. Suppose Z_n has already been defined. Let $g \in C(Z_n)$. Consider the coset $g + (H + C(J))$. Since there are only $2^{\aleph_0}$ height sequences, we can find a subset of at most $2^{\aleph_0}$ elements $\{h_{\alpha,g}, \alpha < 2^{\aleph_0}\} \subset H+C(J)$ so that for each $y \in H+C(J)$ there is an $h_{\alpha,g}$ such that $\chi(g+y) \leq \chi(g+h_{\alpha,g})$. Let Z' be a closed set minimal with respect to $\{h_{\alpha,g} : \alpha < 2^{\aleph_0},\ g \in C(Z_n)\} \subset C(Z')$. Then define $Z_{n+1} = Z_n \cup Z'$. If we let $Z = \bigcup_{n<\omega} Z_n$, then $|C(Z)| \leq 2^{\aleph_0}$ and $H + C(J)\|C(Z)$. Also $H\|C(J)$, since $C(J) \in \mathbb{C}$. Then, by Lemma 7(b), $H\|C(J)+C(Z)$. If $J^* = J \cup Z$, then $C(J^*) = C(J \cup Z) = C(J)+C(Z)$ contains $C(J)$ and A, $|C(J^*)/C(J)| \leq 2^{\aleph_0}$ and $C(J^*)\|H$. Thus $\mathbb{C}$ is a $G(2^{\aleph_0})$-family.

We are now ready to investigate whether, in the exact sequence (1), the group H will be a B_2-group if C and G are both B_2-groups. The answer is in the affirmative if we assume Continuum Hypothesis (CH) and that the sequence (1) is balanced.

Theorem 9 (CH). *Suppose $0 \to H \to C \to G \to 0$ is a balanced exact sequence. If C and G are B_2-groups, so is H.*

Proof. We shall consider H a subgroup of C with $C/H = G$. By [AH] and [R], G has an axiom-3 family $\mathbb{R}$ or pure decent TEP subgroups each of which is a B_2-group. Proposition 8 above yields a $G(\aleph_1)$-family $\mathbb{C}'$ of pure decent TEP subgroups of C of the form $C(S)$, where S a closed set such that $C(S)\|H$. Then $\mathbb{C}^* = \{X \in \mathbb{C}' : (X+H)/H \in \mathbb{R}\}$ is a $G(\aleph_1)$-family. Let $\mathbb{C} = \{H \cap X : X \in \mathbb{C}^*\}$. By Lemma 7(a), $H \cap X$ is balanced in X for all $X \in \mathbb{C}^*$.

We apply induction on the cardinality κ of H. If $\kappa \leq \aleph_1$, then replacing C, if necessary, by a suitable $C(S) \in \mathbb{C}'$, we may assume that $|C| = \kappa$ and then H is a B_2-group by [DHR, Proposition 3.11]. Assume $\kappa > \aleph_1$ and that the theorem holds for cardinalities $< \kappa$. Suppose κ is regular. By using the $G(\aleph_1)$-family $\mathbb{C}^*$, build a smooth κ-filtration $H = \bigcup_{\alpha<\kappa} H_\alpha$, where, for each $\alpha < \kappa$, $H_\alpha \in \mathbb{C}$, say $H_\alpha = H \cap X_\alpha$ for some $X_\alpha \in \mathbb{C}^*$, $|H_\alpha| < \kappa$ and $|H_{\alpha+1}/H_\alpha| \leq \aleph_1$. Since H_α is balanced in X_α and X_α/H_α is a B_2-group, H_α is a TEP subgroup of X_α. Since X_α is TEP in C, H_α is TEP in C and hence in $H_{\alpha+1}$. Then $H_{\alpha+1}/H_\alpha$ is a Butler group of cardinality $\leq \aleph_1$ and so, by [DHR], a B_2-group. Then, by Lemma 3, $\bigcup H_\alpha$ refines to a B_2-filtration of H.

Suppose κ is a singular cardinal. Since $\mathbb{C}^*$ is a $G(\aleph_1)$-family, $\mathbb{C}^{**} = \{Y :$

$Y \in \mathbb{C}$ and $|Y| < \kappa\}$ is readily seen to a κ-family in the sense of Definition 6.1 of [DHR] and, by induction hypothesis, members of $\mathbb{C}^{**}$ are all B_2-groups. By observation 2 above, H is then a B_2-group. Hence the theorem.

Corollary 10 (CH). $\mathrm{Bext}^2(G, T) = 0$ *for any* B_2*-group* G *and any torsion group* T.

Theorem 4 and Corollary 10 enable us to answer when a B_1-group is B_2.

Theorem 11 (CH). *Let* G *be a* B_1*-group. Then* G *is* B_2*-group if and only if* $\mathrm{Bext}^2(G, T) = 0$ *for every torsion group* T.

Proof. Suppose $\mathrm{Bext}^2(G, T) = 0$ for all torsion T. Consider a balanced exact sequence

$$0 \to H \xrightarrow{\varphi} C \to G \to 0 \tag{2}$$

where C is completely decomposable. Then for any torsion group T, we obtain induced exact sequences

$$\mathrm{Hom}(C, T) \xrightarrow{\varphi^*} \mathrm{Hom}(H, T) \to \mathrm{Bext}^1(G, T) = 0$$

and

$$\mathrm{Bext}^1(C, T) = 0 \to \mathrm{Bext}^1(H, T) \to \mathrm{Bext}^2(G, T) = 0.$$

Since φ^* is an epimorphism, (2) is a TEP exact sequence and since $\mathrm{Bext}^1(H, T) = 0$, H is a B_1-group. Now there are at most $2^{\aleph_0} = \aleph_1$ types in the typeset of C and so C can be written as the union of a smooth increasing chain direct summands each of which has its typeset at most countable. Intersecting H with these direct summands, H becomes a union of a smooth increasing chain of pure subgroups each with an at most countable typeset. Since H is also a B_1-group, Theorem 3.1 of [FR] implies that H is a B_2-group. Then Theorem 4 shows that G is a B_2-group. The converse follows from Corollary 10.

Remark. Fuchs and Magidor [FM] show, under $(V = L)$, that $\mathrm{Bext}^2(G, T) = 0$ for any torsion-free group G and any torsion group T. This, together with Theorem 11, immediately gives their following main theorem (Theorem 10.1) (thus making most of §§5 through 10 (except §7) in [FM], in some sense, redundant): $(V = L)$. Every B_1-group is a B_2-group.

In [BS] it was shown that a B_2-group is always a B_1-group. From Theorem 11 and Corollary 10, we then obtain the following homological characterization of B_2 groups:

Theorem 12 (CH). *A torsion-free abelian group* G *is a* B_2*-group if and only if* $\mathrm{Bext}^1(G, T) = 0 = \mathrm{Bext}^2(G, T)$ *for every torsion group* T.

References

[AH] U. Albrecht and P. Hill, *Butler groups of infinite rank and axiom*-3, Czechoslavak. Math. J. **3** (1987), 293–309.

[BS] L. Bican and L. Salce, *Butler groups of infinite rank*, Abelian Group Theory, Lecture Notes in Math., vol. 1006, Springer-Verlag, Berlin and New York, 1983, pp. 171–189.

[DR] M. Dugas and K. M. Rangaswamy, *Infinite rank Butler groups*, Trans. Amer. Math. Soc. **305** (1988), 129–142.

[DHR] M. Dugas, P. Hill, and K. M. Rangaswamy, *Infinite rank Butler groups.* II, Trans. Amer. Math. Soc. **32** (1990), 643–664.

[DT] M. Dugas and B. Thome, *The functor Bext under the negation of CH*, Forum Math. **3** (1991), 23–33.

[F] L. Fuchs, *Infinite abelian groups*, Vol. 2, Academic Press, New York, 1973.

[FM] L. Fuchs and M. Magidor, *Butler groups of arbitrary cardinality* (to appear).

[FR] L. Fuchs and K. M. Rangaswamy, *Butler groups that are unions of subgroups with countable typesets*, Arch. Math. **61** (1993), 105–110.

[R] K. M. Rangaswamy, *Butler groups of infinite rank* III, J. Ramanujan Math. Soc. **8** (1993), 17–28.

Department of Mathematics, University of Colorado, Colorado Springs, Colorado 80933

E-mail address: `kmranga@colospgs.bitnet`