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MINIMAL SURFACES WITH THE RICCI CONDITION IN 4-DIMENSIONAL SPACE FORMS

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Dedicated to Professor Y. Hatakeyama on his 60th birthday

ABSTRACT. Let $X^N(c)$ denote the N-dimensional simply connected space form of constant curvature c. We consider a problem to classify those minimal surfaces in $X^N(c)$ which are locally isometric to minimal surfaces in $X^3(c)$. In this paper we solve this problem in the case where $N = 4$, and give a result also in higher codimensional cases.

0. INTRODUCTION

Let $X^N(c)$ denote the N-dimensional simply connected space form of constant curvature c, and let M be a minimal surface in $X^N(c)$ with Gaussian curvature K $(\leq c)$ with respect to the induced metric ds^2. When $N = 3$, M satisfies the Ricci condition with respect to c, that is, the metric $d\hat{s}^2 = \sqrt{c-K}\, ds^2$ is flat at points where $K < c$. Conversely, every 2-dimensional Riemannian manifold with Gaussian curvature less than c which satisfies the Ricci condition with respect to c, can be realized locally as a minimal surface in $X^3(c)$ (see [10]). Then it is an interesting problem to classify those minimal surfaces in $X^N(c)$ which satisfy the Ricci condition with respect to c, that is, to classify those minimal surfaces in $X^N(c)$ which are locally isometric to minimal surfaces in $X^3(c)$, or to compare locally the Riemannian structures of minimal surfaces in $X^N(c)$ with those of minimal surfaces in $X^3(c)$. In the case where $c = 0$, Lawson [11] solved this problem completely (cf. Chapter IV of [12]). In [13], with some global assumptions, Naka (= Miyaoka) obtained some results in the case where $c > 0$. In [14] we discussed exceptional minimal surfaces in $X^N(c)$ which satisfy the Ricci condition with respect to c.

The main purpose of this paper is to solve the above problem in the case where $N = 4$.

Theorem 1. *Let M be a minimal surface in $X^4(c)$ with Gaussian curvature K with respect to the induced metric ds^2. Suppose that the metric $d\hat{s}^2 = \sqrt{c-K}\, ds^2$ is flat at points where $K < c$. Then M lies in a totally geodesic $X^3(c)$.*

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Remark 1. (i) When $c = 0$, Theorem 1 is included in [11].

(ii) For $c > 0$, there are flat minimal surfaces in $X^5(c)$ not lying in any totally geodesic $X^4(c)$, which automatically satisfy the Ricci condition with respect to c (see [2, 8]). So, in the case where $c > 0$, Theorem 1 is not true if we replace $X^4(c)$ by $X^5(c)$.

We cannot apply the method used to prove Theorem 1 to higher codimensional cases directly. However in §3, with an additional assumption, we will give a result in higher codimensional cases.

1. Preliminaries

Let M be a 2-dimensional Riemannian manifold isometrically immersed in $X^N(c)$ with Gaussian curvature K with respect to the induced metric. Let A be the second fundamental form of M. We denote by T_pM and $T_p^\perp M$ the tangent space and the normal space of M at p, respectively. A point p on M is called isotropic if the ellipse of curvature $\{A(X, X) \in T_p^\perp M; X \in T_pM, |X| = 1\}$ at p is a circle. We say that M is isotropic if each point on M is isotropic.

At each point p on M, we choose orthonormal bases $\{e_1, e_2\}$ and $\{e_3, \dots, e_N\}$ for T_pM and $T_p^\perp M$, respectively. We shall make use of the following convention on the ranges of indices: $1 \le i, j, k \le 2$, $3 \le \alpha, \beta \le N$. Let h^α_{ij} be the components of A. We denote by $R^\alpha_{\beta ij}$ the components of the normal curvature tensor of M. Then

$$R^\alpha_{\beta ij} = \sum_k (h^\alpha_{ik} h^\beta_{jk} - h^\alpha_{jk} h^\beta_{ik}).$$

Following [5], we define the normal scalar curvature K_n of M by

$$K_n = \sum_{i<j,\, \alpha<\beta} (R^\alpha_{\beta ij})^2 = \frac{1}{4} \sum_{i, j, \alpha, \beta} (R^\alpha_{\beta ij})^2.$$

When $N = 4$, we define the normal curvature K_ν of M by $K_\nu = R^3_{412}$, which changes sign according to the orientation of the bases.

Assume further that M is a minimal surface. Then we may choose the bases $\{e_i\}$ and $\{e_\alpha\}$ so that the components h^α_{ij} satisfy

$$(h^3_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h^\alpha_{ij}) = (0) \qquad \text{for } \alpha \ge 5,$$

for some λ and μ. It is easy to see that $(c - K)^2 - K_n \ge 0$, and the point p is isotropic if and only if $|\lambda| = |\mu|$, which is equivalent to that $(c - K)^2 - K_n = 0$. Similarly, when $N = 4$, $c - K \ge |K_\nu|$, and p is isotropic if and only if $c - K = |K_\nu|$.

2. Proof of Theorem 1

Proof of Theorem 1. Assume that M lies fully in $X^4(c)$, namely, does not lie in a totally geodesic $X^3(c)$. Then $K < c$ and $K_\nu \ne 0$ open densely, where K_ν denotes the normal curvature of M (see Lemma 2 of [7]). As the metric $d\hat{s}^2 = \sqrt{c - K}\, ds^2$ is flat at points where $K < c$, we have

$$\frac{K}{\sqrt{c - K}} - \frac{1}{2\sqrt{c - K}} \Delta \log \sqrt{c - K} = 0,$$

or equivalently,

$$\Delta \log(c - K) = 4K \tag{1}$$

at points where $K < c$, where Δ denotes the Laplacian of M with respect to ds^2.

If M is isotropic, then noting that an isotropic minimal surface lying fully in $X^4(c)$ is exceptional in the sense of [6], we have by Theorem A of [6],

$$\Delta \log(c - K) = 6K - 2c \tag{2}$$

at points where $K < c$. From (1) and (2) we have a contradiction. So M is not isotropic.

Set

$$M_1 = \{p \in M\,;\, K < c,\ K_\nu \neq 0, p \text{ is not isotropic}\},$$

which is an open dense subset of M. By Theorem 1 of [4],

$$\Delta \log(c - K + K_\nu) = 2(2K - K_\nu) \tag{3}$$

and

$$\Delta \log(c - K - K_\nu) = 2(2K + K_\nu) \tag{4}$$

on M_1. Set $F = K_\nu/(c - K)$. Then by (1), (3), and (4),

$$\Delta F = -2(c - K)F(1 + F^2) \tag{5}$$

and

$$|\nabla F|^2 = 2(c - K)F^2(1 - F^2) \tag{6}$$

on M_1, where ∇ is the Riemannian connection of M with respect to ds^2. We denote by $\tilde{K}$, $\tilde{\nabla}$, and $\tilde{\Delta}$ the Gaussian curvature, the Riemannian connection, and the Laplacian of M_1 with respect to the metric $d\tilde{s}^2 = (c - K)\,ds^2$, respectively. We note that the metric $d\tilde{s}^2$ is nondegenerate on M_1. Then

$$\tilde{K} = \frac{K}{c - K} - \frac{1}{2(c - K)}\Delta \log(c - K) = \frac{K}{K - c} \tag{7}$$

on M_1, where we use (1) for the second equality. Equations (5) and (6) are rewritten as follows:

$$\tilde{\Delta} F = -2F(1 + F^2) =: P(F) \tag{8}$$

and

$$|\tilde{\nabla} F|^2 = 2F^2(1 - F^2) =: Q(F) \tag{9}$$

on M_1. As $0 < |F| < 1$ on M_1, F is not constant on M_1 by (9). Hence we have

$$Q\tilde{K} + (P - Q')(P - \tfrac{1}{2}Q') + Q(P' - \tfrac{1}{2}Q'') = 0 \tag{10}$$

on M_1, where the prime denotes the differentiation with respect to F (see [3, p. 164; 9, p. 136]). Noting that $0 < |F| < 1$ on M_1, we have by (7)–(10), $K = 8c/9$ on M_1 and, by continuity, on M. As $K < c$ on M_1, we find that $c > 0$. Now we have a contradiction because there are no minimal surfaces with constant curvature $8c/9$ in $X^4(c)$, where $c > 0$ (see [2, 9]).

Therefore, M lies in a totally geodesic $X^3(c)$.

3. Higher codimensional cases

In this section we prove the following.

Theorem 2. *Let M be a minimal surface in $X^N(c)$ with Gaussian curvature K with respect to the induced metric ds^2. Suppose that the metric $d\hat{s}^2 = \sqrt{c-K}\,ds^2$ is flat at points where $K < c$ and the normal scalar curvature of M is constant. Then either* (i) *M lies in a totally geodesic $X^3(c)$, or* (ii) *$c > 0$ and M is flat.*

Remark 2. (i) A minimal surface M in $X^N(c)$ lies in a totally geodesic $X^3(c)$ if and only if the normal scalar curvature of M is identically zero (see Lemma 2 of [7]).

(ii) Flat minimal surfaces in $X^N(c)$ where $c > 0$ are classified (see [2]). By Theorem 3.1(2) of [2] and Proposition 1(iii) of [8], we find that all of them have constant normal scalar curvature.

(iii) Minimal surfaces with constant normal scalar curvature in space forms are studied in [1] and [5].

Proof of Theorem 2. Assume that M does not lie in a totally geodesic $X^3(c)$. Then $K < c$ open densely and the normal scalar curvature K_n of M is a positive constant (see Remark 2(i)).

When M is isotropic, we have $(c-K)^2 - K_n = 0$ on M, and K is a constant less than c. From the hypothesis that the metric $d\hat{s}^2 = \sqrt{c-K}\,ds^2$ is flat at points where $K < c$, we have $c > 0$ and $K = 0$ on M.

When M is not isotropic, set $M_2 = \{p \in M\,;\, K < c,\ p \text{ is not isotropic}\}$, which is an open dense subset of M. As the metric $d\hat{s}^2 = \sqrt{c-K}\,ds^2$ is flat at points where $K < c$, we have

$$\Delta \log(c-K) = 4K \tag{11}$$

on M_2, where Δ denotes the Laplacian of M with respect to ds^2. The argument to get (2.12) of [5] is valid on minimal surfaces in $X^N(c)$ except at isotropic points. Hence we have by (2.12) of [5] under our notation,

$$\Delta \log\{(c-K)^2 - K_n\} = 8K \tag{12}$$

on M_2. By (11) and (12),

$$\Delta K = 6K^2 - 6cK - 2K_n - 2cK_n/(K-c) =: R(K)$$

and

$$|\nabla K|^2 = 2K^3 - 4cK^2 + 2(c^2 - K_n)K =: S(K)$$

on M_2, where ∇ is the Riemannian connection of M with respect to ds^2. If K is not constant on M_2, then

$$SK + (R - S')(R - \tfrac{1}{2}S') + S(R' - \tfrac{1}{2}S'') = 0 \tag{13}$$

on M_2, where the prime denotes the differentiation with respect to K (see [3, p. 164; 9, p. 136]). By the computation we find that (13) is a nontrivial equation for K. So K must be constant on M_2, which is a contradiction. Hence K is a constant less than c on M_2, and by continuity, on M. From the hypothesis that the metric $d\hat{s}^2 = \sqrt{c-K}\,ds^2$ is flat at points where $K < c$, we have $c > 0$ and $K = 0$ on M.

Therefore, either (i) M lies in a totally geodesic $X^3(c)$ or (ii) $c > 0$ and M is flat.

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