MINIMAL SURFACES WITH THE RICCI CONDITION
IN 4-DIMENSIONAL SPACE FORMS

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Dedicated to Professor Y. Hatakeyama on his 60th birthday

Abstract. Let $X^N(c)$ denote the $N$-dimensional simply connected space form of constant curvature $c$. We consider a problem to classify those minimal surfaces in $X^N(c)$ which are locally isometric to minimal surfaces in $X^3(c)$. In this paper we solve this problem in the case where $N = 4$, and give a result also in higher codimensional cases.

0. Introduction

Let $X^N(c)$ denote the $N$-dimensional simply connected space form of constant curvature $c$, and let $M$ be a minimal surface in $X^N(c)$ with Gaussian curvature $K$ ($\leq c$) with respect to the induced metric $ds^2$. When $N = 3$, $M$ satisfies the Ricci condition with respect to $c$, that is, the metric $ds^2 = \sqrt{c - K} \, ds^2$ is flat at points where $K < c$. Conversely, every 2-dimensional Riemannian manifold with Gaussian curvature less than $c$ which satisfies the Ricci condition with respect to $c$, can be realized locally as a minimal surface in $X^3(c)$ (see [10]). Then it is an interesting problem to classify those minimal surfaces in $X^N(c)$ which satisfy the Ricci condition with respect to $c$, that is, to classify those minimal surfaces in $X^N(c)$ which are locally isometric to minimal surfaces in $X^3(c)$, or to compare locally the Riemannian structures of minimal surfaces in $X^N(c)$ with those of minimal surfaces in $X^3(c)$. In the case where $c = 0$, Lawson [11] solved this problem completely (cf. Chapter IV of [12]). In [13], with some global assumptions, Naka (= Miyaoka) obtained some results in the case where $c > 0$. In [14] we discussed exceptional minimal surfaces in $X^N(c)$ which satisfy the Ricci condition with respect to $c$.

The main purpose of this paper is to solve the above problem in the case where $N = 4$.

Theorem 1. Let $M$ be a minimal surface in $X^4(c)$ with Gaussian curvature $K$ with respect to the induced metric $ds^2$. Suppose that the metric $ds^2 = \sqrt{c - K} \, ds^2$ is flat at points where $K < c$. Then $M$ lies in a totally geodesic $X^3(c)$.
Remark 1. (i) When $c = 0$, Theorem 1 is included in [11].

(ii) For $c > 0$, there are flat minimal surfaces in $X^5(c)$ not lying in any totally geodesic $X^4(c)$, which automatically satisfy the Ricci condition with respect to $c$ (see [2, 8]). So, in the case where $c > 0$, Theorem 1 is not true if we replace $X^4(c)$ by $X^5(c)$.

We cannot apply the method used to prove Theorem 1 to higher codimensional cases directly. However in §3, with an additional assumption, we will give a result in higher codimensional cases.

1. Preliminaries

Let $M$ be a 2-dimensional Riemannian manifold isometrically immersed in $X^N(c)$ with Gaussian curvature $K$ with respect to the induced metric. Let $A$ be the second fundamental form of $M$. We denote by $T_pM$ and $T^\perp_pM$ the tangent space and the normal space of $M$ at $p$, respectively. A point $p$ on $M$ is called isotropic if the ellipse of curvature $\{A(X, X) \in T_pM; X \in T_pM, |X| = 1\}$ at $p$ is a circle. We say that $M$ is isotropic if each point on $M$ is isotropic.

At each point $p$ on $M$, we choose orthonormal bases $\{e_1, e_2\}$ and $\{e_3, \ldots, e_N\}$ for $T_pM$ and $T^\perp_pM$, respectively. We shall make use of the following convention on the ranges of indices: $1 < i, j, k < 2$, $3 < \alpha, \beta < N$. Let $h_{ij}^\alpha$ be the components of $A$. We denote by $R_{\beta ij}^\alpha$ the components of the normal curvature tensor of $M$. Then

$$R_{\beta ij}^\alpha = \sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta).$$

Following [5], we define the normal scalar curvature $K_n$ of $M$ by

$$K_n = \sum_{i < j, \alpha < \beta} (R_{\beta ij}^\alpha)^2 = \frac{1}{4} \sum_{i, j, \alpha, \beta} (R_{\beta ij}^\alpha)^2.$$

When $N = 4$, we define the normal curvature $K_\nu$ of $M$ by $K_\nu = R^3_{412}$, which changes sign according to the orientation of the bases.

Assume further that $M$ is a minimal surface. Then we may choose the bases $\{e_i\}$ and $\{e_\alpha\}$ so that the components $h_{ij}^\alpha$ satisfy

$$(h_{ij}^3) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \quad (h_{ij}^\alpha) = (0) \quad \text{for } \alpha \geq 5,$$

for some $\lambda$ and $\mu$. It is easy to see that $(c - K)^2 - K_n \geq 0$, and the point $p$ is isotropic if and only if $|\lambda| = |\mu|$, which is equivalent to that $(c - K)^2 - K_n = 0$. Similarly, when $N = 4$, $c - K \geq |K_\nu|$, and $p$ is isotropic if and only if $c - K = |K_\nu|$.

2. Proof of Theorem 1

Proof of Theorem 1. Assume that $M$ lies fully in $X^4(c)$, namely, does not lie in a totally geodesic $X^2(c)$. Then $K < c$ and $K_\nu \neq 0$ open densely, where $K_\nu$ denotes the normal curvature of $M$ (see Lemma 2 of [7]). As the metric $d\bar{s}^2 = \sqrt{c - K} \, ds^2$ is flat at points where $K < c$, we have

$$\frac{K}{\sqrt{c - K}} - \frac{1}{2\sqrt{c - K}} \Delta \log \sqrt{c - K} = 0,$$
or equivalently,

(1) \[ \Delta \log(c - K) = 4K \]

at points where \( K < c \), where \( \Delta \) denotes the Laplacian of \( M \) with respect to \( ds^2 \).

If \( M \) is isotropic, then noting that an isotropic minimal surface lying fully in \( \mathcal{X}^4(c) \) is exceptional in the sense of [6], we have by Theorem A of [6],

(2) \[ \Delta \log(c - K) = 6K - 2c \]

at points where \( K < c \). From (1) and (2) we have a contradiction. So \( M \) is not isotropic.

Set \( M_1 = \{ p \in M ; K < c, K_{\nu} \neq 0, p \text{ is not isotropic} \} \), which is an open dense subset of \( M \). By Theorem 1 of [4],

(3) \[ \Delta \log(c - K + K_{\nu}) = 2(2K - K_{\nu}) \]

and

(4) \[ \Delta \log(c - K - K_{\nu}) = 2(2K + K_{\nu}) \]

on \( M_1 \). Set \( F = K_{\nu}/(c - K) \). Then by (1), (3), and (4),

(5) \[ \Delta F = -2(c - K)F(1 + F^2) \]

and

(6) \[ |\nabla F|^2 = 2(c - K)F^2(1 - F^2) \]

on \( M_1 \), where \( \nabla \) is the Riemannian connection of \( M \) with respect to \( ds^2 \). We denote by \( \tilde{K}, \tilde{\nabla}, \) and \( \tilde{\Delta} \) the Gaussian curvature, the Riemannian connection, and the Laplacian of \( M_1 \) with respect to the metric \( ds^2 = (c - K) ds^2 \), respectively. We note that the metric \( ds^2 \) is nondegenerate on \( M_1 \). Then

(7) \[ \tilde{K} = \frac{K}{c - K} - \frac{1}{2(c - K)} \Delta \log(c - K) = \frac{K}{K - c} \]

on \( M_1 \), where we use (1) for the second equality. Equations (5) and (6) are rewritten as follows:

(8) \[ \tilde{\Delta} F = -2F(1 + F^2) =: P(F) \]

and

(9) \[ |\tilde{\nabla} F|^2 = 2F^2(1 - F^2) =: Q(F) \]

on \( M_1 \). As \( 0 < |F| < 1 \) on \( M_1 \), \( F \) is not constant on \( M_1 \) by (9). Hence we have

(10) \[ Q\tilde{K} + (P - Q')(P - \frac{1}{2}Q') + Q(P' - \frac{1}{2}Q'') = 0 \]

on \( M_1 \), where the prime denotes the differentiation with respect to \( F \) (see [3, p. 164; 9, p. 136]). Noting that \( 0 < |F| < 1 \) on \( M_1 \), we have by (7)–(10), \( K = 8c/9 \) on \( M_1 \) and, by continuity, on \( M \). As \( K < c \) on \( M_1 \), we find that \( c > 0 \). Now we have a contradiction because there are no minimal surfaces with constant curvature \( 8c/9 \) in \( \mathcal{X}^4(c) \), where \( c > 0 \) (see [2, 9]).

Therefore, \( M \) lies in a totally geodesic \( \mathcal{X}^3(c) \).
3. Higher codimensional cases

In this section we prove the following.

**Theorem 2.** Let $M$ be a minimal surface in $X^N(c)$ with Gaussian curvature $K$ with respect to the induced metric $ds^2$. Suppose that the metric $d\tilde{s}^2 = \sqrt{c - \bar{K}} \, ds^2$ is flat at points where $K < c$ and the normal scalar curvature of $M$ is constant. Then either (i) $M$ lies in a totally geodesic $X^3(c)$, or (ii) $c > 0$ and $M$ is flat.

**Remark 2.** (i) A minimal surface $M$ in $X^N(c)$ lies in a totally geodesic $X^3(c)$ if and only if the normal scalar curvature of $M$ is identically zero (see Lemma 2 of [7]).

(ii) Flat minimal surfaces in $X^N(c)$ where $c > 0$ are classified (see [2]). By Theorem 3.1(2) of [2] and Proposition 1(iii) of [8], we find that all of them have constant normal scalar curvature.

(iii) Minimal surfaces with constant normal scalar curvature in space forms are studied in [1] and [5].

**Proof of Theorem 2.** Assume that $M$ does not lie in a totally geodesic $X^3(c)$. Then $K < c$ open densely and the normal scalar curvature $K_n$ of $M$ is a positive constant (see Remark 2(i)).

When $M$ is isotropic, we have $(c - K)^2 - K_n = 0$ on $M$, and $K$ is a constant less than $c$. From the hypothesis that the metric $d\tilde{s}^2 = \sqrt{c - \bar{K}} \, ds^2$ is flat at points where $K < c$, we have $c > 0$ and $K = 0$ on $M$.

When $M$ is not isotropic, set $M_2 = \{ p \in M ; K < c, \ p \text{ is not isotropic} \}$, which is an open dense subset of $M$. As the metric $d\tilde{s}^2 = \sqrt{c - \bar{K}} \, ds^2$ is flat at points where $K < c$, we have

\[
\Delta \log(c - K) = 4K \tag{11}
\]
on $M_2$, where $\Delta$ denotes the Laplacian of $M$ with respect to $ds^2$. The argument to get (2.12) of [5] is valid on minimal surfaces in $X^N(c)$ except at isotropic points. Hence we have by (2.12) of [5] under our notation,

\[
\Delta \log((c - K)^2 - K_n) = 8K \tag{12}
\]
on $M_2$. By (11) and (12),

\[
\Delta K = 6K^2 - 6cK - 2K_n - 2cK_n/(K - c) =: R(K)
\]

and

\[
|\nabla K|^2 = 2K^3 - 4cK^2 + 2(c - K_n)K =: S(K)
\]
on $M_2$, where $\nabla$ is the Riemannian connection of $M$ with respect to $ds^2$. If $K$ is not constant on $M_2$, then

\[
SK + (R - S')(R - \frac{1}{2}S') + S(R' - \frac{1}{2}S'') = 0 \tag{13}
\]
on $M_2$, where the prime denotes the differentiation with respect to $K$ (see [3, p. 164; 9, p. 136]). By the computation we find that (13) is a nontrivial equation for $K$. So $K$ must be constant on $M_2$, which is a contradiction. Hence $K$ is a constant less than $c$ on $M_2$, and by continuity, on $M$. From the hypothesis that the metric $d\tilde{s}^2 = \sqrt{c - \bar{K}} \, ds^2$ is flat at points where $K < c$, we have $c > 0$ and $K = 0$ on $M$.

Therefore, either (i) $M$ lies in a totally geodesic $X^3(c)$ or (ii) $c > 0$ and $M$ is flat.
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