ON THE SIZE OF FINITE SIDON SEQUENCES

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Abstract. Let $h \geq 2$ be an integer. A set of positive integers $B$ is called a $B_h$-sequence, or a Sidon sequence of order $h$, if all sums $a_1 + a_2 + \cdots + a_h$, where $a_i \in B \ (i = 1, 2, \ldots, h)$, are distinct up to rearrangements of the summands. Let $F_h(n)$ be the size of the maximum $B_h$-sequence contained in \{1, 2, \ldots, n\}. We prove that

$$F_{2(r-1)}(n) \leq ((r!)^2 n)^{1/(2r-1)} + O(n^{1/(4r-2)}).$$

Let $h \geq 2$ be an integer. A set of positive integers $B$ is called a $B_h$-sequence if all sums $a_1 + a_2 + \cdots + a_h$, where $a_i \in B \ (i = 1, 2, \ldots, h)$, are distinct up to rearrangements of the summands.

A $B_h$-sequence is also called a Sidon sequence of order $h$ [6]. We say that $B$ is a $B_h$-sequence for $\mathbb{Z}/(n)$ if $B$ is a finite $B_h$-sequence and all sums are distinct modulo $n$.

Let $F_h(n)$ denote the size of maximum $B_h$-sequences contained in the set of integers \{1, 2, \ldots, n\} and $f_h(n)$ the size of maximum $B_h$-sequence for $\mathbb{Z}/(n)$. Then it follows from a simple combinatorial argument that

$$F_h(n) \leq (h \cdot h!)^{1/h} n^{1/h} \quad \text{and} \quad f_h(n) \leq (h!)^{1/h} n^{1/h}.$$

Erdős and Turan [2] proved that $F_2(n) < \sqrt{n} + O(n^{1/4})$. On the other hand, Bose and Chowla [1] showed that, for every $h \geq 2$, there exists a $B_h$-sequence $B$ for $\mathbb{Z}/(m^h - 1)$ with $|B| = m$, where $m$ is a prime power. This implies that

$$F_h(n) \geq (1 + o(1)) n^{1/h}.$$

Therefore, $F_2(n) = (1 + o(1)) \sqrt{n}$.

Erdős conjectured that $F_2(n) = \sqrt{n} + O(1)$. For $h = 3$, Lee [4] obtained that

$$F_3(n) \leq \left( 1 - \frac{1}{6 \log_2 n} \right) 4n^{1/3}.$$

For $h = 4$, Lindström [5] proved that

$$F_4(n) \leq (8n)^{1/4} + O(n^{1/8}).$$
When \( h = 2r \) \((r \geq 1)\), Jia [3] showed that
\[
F_{2r}(n) \leq (\ell!)^2 r^n \frac{1}{2r} + O(n^{1/4r})
\]
and
\[
f_{2r}(n) \leq (\ell!)^2 n^{1/2r} + O(n^{1/4r}).
\]

In this paper, we shall obtain a similar upper bound for \( F_{2r-1}(n) \) and \( f_h(n) \) as well.

**Theorem 1.** For all \( r \geq 1 \),
\[
F_{2r-1}(n) \leq (\ell!)^2 n^{1/(2r-1)} + O(n^{1/(4r-2)}).
\]

**Theorem 2.** For all \( r \geq 1 \),
\[
f_{2r}(n) \leq (\ell!)^2 n^{1/2r} + O(1)
\]
and
\[
f_{2r-1}(n) \leq (\ell!(r-1)n^{1/(2r-1)} + O(1).
\]

First, some notation. Let \( B \) be a \( P_{2r-1} \)-sequence. Let \( A = rB \) where \( rB \) denotes the set of all sums of \( r \) not necessarily distinct elements in \( B \). We have
\[
t = |A| = \binom{k + r - 1}{r} \geq \frac{k^r}{r!}.
\]

Let
\[
V = \{ (a, b) | a, b \in rB \} = V_0 \cup V_1,
\]
where \( V_1 = V \setminus V_0 \) and \( V_0 \) consists of all elements \( (a, b) \) such that \( a = \sum_{i=1}^r a_i \) and \( b = \sum_{j=1}^r b_j \) with \( a_i, b_j \in B \) and \( a_i \neq b_j \) for all \( 1 \leq i, j \leq r \).

**Lemma.** For any integer \( d \), there are at most \( k/r \) elements \( (a, b) \) in \( V_0 \) such that \( a - b = d \).

**Proof.** Let \( (a_i, b_i) \) be elements in \( V_0 \), \( i = 1, 2, \ldots, s \), such that \( a_i - b_i = d \) for all \( 1 \leq i \leq s \). Suffice to show that, if \( s > k/r \), at least two of the \( (a_i, b_i) \)'s are the same.

Now assume \( s > k/r \). Write \( a_i = \sum_{j=1}^r a_{ij} \) and \( b_i = \sum_{j=1}^r b_{ij} \) where \( a_{ij}, b_{ij} \in B \). Since \( |B| = k \) and \( sr > k \), there are at least two distinct pairs \((i, j)\) and \((i', j')\) \((1 \leq i, i' \leq s \text{ and } 1 \leq j, j' \leq r)\) such that \( a_{ij} = a_{i'j'} \).

By the definition of \( V_0 \), \( i \neq i' \). But then we have \( a_i - a_{ij} - b_i = a_{i'} - a_{i'j'} - b_{i'} \). Hence \( a_i - a_{ij} + b_{i'} = a_{i'} - a_{i'j'} + b_i \). As \( B \) is a \( P_{2r-1} \)-sequence and \( a_{ij} \)'s and \( b_{ij} \)'s are all mutually distinct, we have \{\( a_{i_1}, a_{i_2}, \ldots, a_{i_r} \}\) and \{\( b_{i_1}, b_{i_2}, \ldots, b_{i_r} \)\} \(=\) \{\( a_{i'1}, a_{i'2}, \ldots, a_{i'r} \)\} and \{\( b_{i'1}, b_{i'2}, \ldots, b_{i'r} \)\}. Therefore, \( (a_i, b_i) = (a_{i'}, b_{i'}) \). This completes the proof of the Lemma.

**Proof of Theorem 1.** Let \( u = \lfloor n^{(4r-3)/(4r-2)} \rfloor \) and \( I_m = [-u + m, -1 + m] \), \( m = 1, 2, \ldots, rn + u \), \( C_m = I_m \cap B \), and \( c_m = |C_m| \). Then
\[
(1) \quad (tu)^2 = \left( \sum_{m=1}^{rn+u} c_m \right)^2 \leq (rn + u) \sum_{m=1}^{rn+u} c_m^2.
\]
Note \( c_m^2 \) is the number of elements \((a, b) \in V\) such that \( a, b \in C_m \). Hence \(-u < a - b < u\).

For any integer \( d \), \(-u < d < u\), by the Lemma, there are at most \( k/r \) elements \((a, b) \in V_0\) such that \( a - b = d \). And each such pair is counted \( u - |d| \) times toward the sum \( \sum_{m=1}^{r^r+u} c_m^2 \). Hence, as \( |V| \leq O(k^{2r-1}) \) and \( k \leq O(n^{1/(2r-1)}) \),

\[
(tu)^2 \leq (rn + u) \left( \sum_{-u < d < u} \frac{k}{r} (u - |d|) + O(k^{2r-1}) \right)
= (rn + u) \left( \frac{k}{r} u^2 + O(k^{2r-1}) \right).
\]

So,

\[
t^2 \leq nk(1 + O(n^{-1/(4r-2)})).
\]

Hence,

\[
k \leq (\frac{r!}{2})^2 n(1 + O(n^{-1/(4r-2)}))^{1/(2r-1)} \leq (\frac{r!}{2})^2 n^{1/(2r-1)} + O(n^{1/(4r-2)}).
\]

This proves Theorem 1. \( \Box \)

**Proof of Theorem 2.** In the case \( h = 2r - 1\), we use the same settings \( A \) and \( V \). Since, for any \( d \in Z/(n)\), there exist at most \( k/r \) elements \((a, b) \in V_0\) such that \( a - b = d \), we have

\[
t^2 = |V| = |V_0| + |V_1| \leq \frac{k}{r} n + O(k^{2r-1}).
\]

Hence,

\[
\frac{k^{2r}}{(r!)^2} \leq \frac{k}{r} n + O(k^{2r-1}) = \frac{k}{r} n(1 + O(n^{-1/(2r-1)})),
\]

and

\[
k \leq (r!(r-1)!n)^{1/(2r-1)}(1 + O(n^{-1/(2r-1)}))^{1/(2r-1)}
= (r!(r-1)!n)^{1/(2r-1)} + O(1).
\]

This shows that \( f_{2r-1}(n) \leq (r!(r-1)!n)^{1/(2r-1)} + O(1) \). Similarly, we have \( f_{2r}(n) \leq ((r!)^2 n)^{1/(2r)} + O(1) \), which completes the proof of Theorem 2. \( \Box \)

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**References**


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