ON THE SIZE OF FINITE SIDON SEQUENCES

SHENG CHEN

(Communicated by William W. Adams)

Abstract. Let \( h \geq 2 \) be an integer. A set of positive integers \( B \) is called a \( B_h \)-sequence, or a Sidon sequence of order \( h \), if all sums \( a_1 + a_2 + \ldots + a_h \), where \( a_i \in B \ (i = 1, 2, \ldots, h) \), are distinct up to rearrangements of the summands. Let \( F_h(n) \) be the size of the maximum \( B_h \)-sequence contained in \( \{1, 2, \ldots, n\} \). We prove that

\[
F_{2r-1}(n) \leq \left( \frac{r!}{2} n \right)^{1/(2r-1)} + \frac{O(n^{1/(4r-2)})}{n^{1/(4r-2)}}.
\]

Let \( h \geq 2 \) be an integer. A set of positive integers \( B \) is called a \( B_h \)-sequence if all sums \( a_1 + a_2 + \ldots + a_h \), where \( a_i \in B \ (i = 1, 2, \ldots, h) \), are distinct up to rearrangements of the summands.

A \( B_h \)-sequence is also called a Sidon sequence of order \( h \) [6]. We say that \( B \) is a \( B_h \)-sequence for \( \mathbb{Z}/n \mathbb{Z} \) if \( B \) is a finite \( B_h \)-sequence and all sums are distinct modulo \( n \).

Let \( F_h(n) \) denote the size of maximum \( B_h \)-sequences contained in the set of integers \( \{1, 2, \ldots, n\} \) and \( f_h(n) \) the size of maximum \( B_h \)-sequence for \( \mathbb{Z}/n \mathbb{Z} \). Then it follows from a simple combinatorial argument that

\[
F_h(n) \leq (h \cdot h!)^{1/h} n^{1/h} \quad \text{and} \quad f_h(n) \leq (h!)^{1/h} n^{1/h}.
\]

Erdős and Turan [2] proved that \( F_2(n) < \sqrt{n} + O(n^{1/4}) \). On the other hand, Bose and Chowla [1] showed that, for every \( h \geq 2 \), there exists a \( B_h \)-sequence \( B \) for \( \mathbb{Z}/(m^h - 1) \) with \( |B| = m \), where \( m \) is a prime power. This implies that

\[
F_h(n) \geq (1 + o(1)) n^{1/h}.
\]

Therefore, \( F_2(n) = (1 + o(1)) \sqrt{n} \).

Erdős conjectured that \( F_2(n) = \sqrt{n} + O(1) \). For \( h = 3 \), Lee [4] obtained that

\[
F_3(n) \leq \left( \frac{1}{6 \log_2 n} \right)^{1/3} 4n.
\]

For \( h = 4 \), Lindström [5] proved that

\[
F_4(n) \leq (8n)^{1/4} + O(n^{1/8}).
\]
When \( h = 2r \) \((r \geq 1)\), Jia [3] showed that
\[
F_{2r}(n) \leq ((r!)^2 n^{1/2r}) + O(n^{1/4r})
\]
and
\[
f_{2r}(n) \leq ((r!)^2 n^{1/2r}) + O(n^{1/4r}).
\]

In this paper, we shall obtain a similar upper bound for \( F_{2r-1}(n) \) and \( f_h(n) \) as well.

**Theorem 1.** For all \( r \geq 1 \),
\[
F_{2r-1}(n) \leq ((r!)^2 n^{1/(2r-1)}) + O(n^{1/(4r-2)}).
\]

**Theorem 2.** For all \( r \geq 1 \),
\[
f_{2r}(n) \leq ((r!)^2 n^{1/2r}) + O(1)
\]
and
\[
f_{2r-1}(n) \leq (r!(r-1)n^{1/(2r-1)}) + O(1).
\]

First, some notation. Let \( B \) be a \( B_{2r-1} \)-sequence. Let \( A = rB \) where \( rB \) denotes the set of all sums of \( r \) not necessarily distinct elements in \( B \). We have
\[
t = |A| = \binom{k + r - 1}{r} \geq \frac{k^r}{r!}.
\]

Let
\[
V = \{(a, b) | a, b \in rB\} = V_0 \cup V_1,
\]
where \( V_1 = V \setminus V_0 \) and \( V_0 \) consists of all elements \((a, b)\) such that \( a = \sum_{i=1}^{r} a_i \) and \( b = \sum_{j=1}^{r} b_j \) with \( a_i, b_j \in B \) and \( a_i \neq b_j \) for all \( 1 \leq i, j \leq r \).

**Lemma.** For any integer \( d \), there are at most \( k/r \) elements \((a, b)\) in \( V_0 \) such that \( a - b = d \).

**Proof.** Let \((a_i, b_i)\) be elements in \( V_0 \), \( i = 1, 2, \ldots, s \), such that \( a_i - b_i = d \) for all \( 1 \leq i \leq s \). Suffice to show that, if \( s > k/r \), at least two of the \((a_i, b_i)\)'s are the same.

Now assume \( s > k/r \). Write \( a_i = \sum_{j=1}^{r} a_{ij} \) and \( b_i = \sum_{j=1}^{r} b_{ij} \) where \( a_{ij}, b_{ij} \in B \). Since \(|B| = k \) and \( sr > k \), there are at least two distinct pairs \((i, j)\) and \((i', j')\) \((1 \leq i, i' \leq s \) and \( 1 \leq j, j' \leq r \) such that \( a_{ij} = a_{i'j'} \). By the definition of \( V_0 \), \( i \neq i' \). But then we have \( a_i - a_{ij} - b_i = a_i - a_{i'j'} - b_i \). Hence \( a_i - a_{ij} + b_{i'} = a_i - a_{i'j'} + b_{i'} \). As \( B \) is a \( B_{2r-1} \)-sequence and \( a_{ij} \)'s and \( b_{ij} \)'s are all mutually distinct, we have \( \{a_{i1}, a_{i2}, \ldots, a_{ir}\} = \{a_{i'1}, a_{i'2}, \ldots, a_{i'r}\} \) and \( \{b_{i1}, b_{i2}, \ldots, b_{ir}\} = \{b_{i'1}, b_{i'2}, \ldots, b_{i'r}\} \). Therefore, \((a_i, b_i) = (a_{i'}, b_{i'})\). This completes the proof of the Lemma.

**Proof of Theorem 1.** Let \( u = \lfloor n^{(4r-3)/(4r-2)} \rfloor \) and \( I_m = [-u + m, -1 + m] \), \( m = 1, 2, \ldots, rn + u \), \( C_m = I_m \cap B \), and \( c_m = |C_m| \). Then
\[
(tu)^2 = \left( \sum_{m=1}^{rn+u} c_m \right)^2 \leq (rn + u) \sum_{m=1}^{rn+u} c_m^2.
\]
Note \( c^2_m \) is the number of elements \((a, b) \in V\) such that \( a, b \in C_m \). Hence \(-u < a - b < u\).

For any integer \( d, \ -u < d < u, \) by the Lemma, there are at most \( k/r \) elements \((a, b)\) in \( V_0\) such that \( a - b = d \). And each such pair is counted \( u - |d| \) times toward the sum \( \sum_{m=1}^{rn+u} c^2_m \). Hence, as \( |V_1| \leq O(k^{2r-1}) \) and \( k \leq O(n^{1/(2r-1)}) \),

\[
(tu)^2 \leq (rn + u) \left( \sum_{-u < d < u} \frac{k}{r} (u - |d|) + O(k^{2r-1}) \right)
\]

\[
= (rn + u) \left( \frac{k}{r} u^2 + O(k^{2r-1}) \right).
\]

So,

\[
t^2 \leq nk(1 + O(n^{-1/(4r-2)})).
\]

Hence,

\[
k \leq \left( \frac{(r!)^2 n (1 + O(n^{-1/(4r-2)})))}{(2r-1)} \right) k \leq \left( \frac{(r!)^2 n}{n^{1/(2r-1)}} \right) + O(n^{1/(4r-2)}).
\]

This proves Theorem 1. \( \Box \)

**Proof of Theorem 2.** In the case \( h = 2r - 1 \), we use the same settings \( A \) and \( V \). Since, for any \( d \in \mathbb{Z}/n \), there exist at most \( k/r \) elements \((a, b) \in V_0\) such that \( a - b = d \), we have

\[
t^2 = |V| = |V_0| + |V_1| \leq \frac{k}{r} n + O(k^{2r-1}).
\]

Hence,

\[
\frac{k^{2r}}{(r!)^2} \leq \frac{k}{r} n + O(k^{2r-1}) = \frac{k}{r} n (1 + O(n^{-1/(2r-1)})),
\]

and

\[
k \leq (r!(r - 1)! n)^{1/(2r-1)} (1 + O(n^{-1/(2r-1)}))^{1/(2r-1)}
\]

\[
= (r!(r - 1)! n)^{1/(2r-1)} + O(1).
\]

This shows that \( f_{2r-1}(n) \leq (r!(r - 1)! n)^{1/(2r-1)} + O(1) \). Similarly, we have \( f_{2r}(n) \leq ((r!)^2 n)^{1/(2r)} + O(1) \), which completes the proof of Theorem 2. \( \Box \)

**ACKNOWLEDGMENT**

The author would like to thank W. Gu and X.-D. Jia for their valuable and intriguing discussion.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, SOUTHWEST TEXAS STATE UNIVERSITY, SAN MARCOS, TEXAS 78666

E-mail address: sc03@swtexas.bitnet