

QUOTIENT RINGS OF NOETHERIAN MODULE FINITE RINGS

A. W. CHATTERS AND C. R. HAJARNAVIS

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ABSTRACT. Let R be a Noetherian ring which is a finite module over its centre C . We obtain a necessary and sufficient condition for R to have a full quotient ring. We also show that when such a quotient ring does exist, it is obtained by inverting those elements of C which are regular in R .

1. INTRODUCTION

The problem of deciding when a Noetherian ring possesses a classical ring of quotients was posed by Goldie [4]. Apart from the general characterisation given by Ore's theorem, this question is still open. The most significant result in this area so far has been that of Stafford [13], who obtained a criterion to determine when a Noetherian ring is its own quotient ring. Stafford also comments [13, p. 385] on the difficulty in tackling the more general problem of finding necessary and sufficient conditions for a Noetherian ring to have a full ring of quotients, pointing out that this question is intimately connected with that of localising at a prime ideal where an easy characterisation is unlikely. In this article we restrict our attention to those Noetherian rings which are finitely generated as modules over their centres. This class is a rich source of examples both with and without quotient rings [2, 11]. Here the obstruction mentioned above does not apply since in this case an easy criterion given by Müller [10] to determine the localisability of a prime ideal does exist.

Let R be a Noetherian ring finitely generated as a module over its centre C . Let $\{P_1, \dots, P_n\}$ be the set of regular primes in R and \mathcal{S} the set of all elements in C which are regular in R . We show that R has a full quotient ring if and only if $\{P_1, \dots, P_n\}$ is precisely the set of primes lying over $\{P_1 \cap C, \dots, P_n \cap C\}$ (Proposition 3.3). This allows us to prove that the full quotient ring of R , when it exists, coincides with the partial quotient ring $R_{\mathcal{S}}$ (Theorem 3.4). In particular this gives us an easy test to apply to examples. It is normally straightforward to locate and invert \mathcal{S} and obtain the ring $R_{\mathcal{S}}$. Theorem 3.4 shows that R has a quotient ring if and only if $R_{\mathcal{S}}$ is its own quotient ring.

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We also obtain a sufficiency criterion for R to have a quotient ring in terms of the projective dimension of R as a C -module. We prove that R has a quotient ring if R is torsionfree and has finite projective dimension as a C -module (Proposition 3.6).

The final section of the paper contains illustrative examples.

2. NOTATION AND PRELIMINARIES

Let R be a not necessarily commutative ring and I an ideal of R . We denote $\mathcal{E}(I) = \{c \in R \mid c + I \text{ is regular in the ring } R/I\}$. In particular, $\mathcal{E}(0)$ is the set of regular elements of R . When X is a nonempty subset of a commutative ring R , $\text{ann}(X)$ will denote the *annihilator* of X , i.e., $\text{ann}(X) = \{r \in R \mid Xr = 0\}$.

The ring Q is called a *quotient ring* if every regular element of Q is a unit of Q . Q is said to be the *right quotient ring* of R if

- (i) R is a subring of Q ;
- (ii) every regular element of R is a unit of Q ; and
- (iii) every element of Q is expressible as ac^{-1} for some $a, c \in R$ with c regular.

For emphasis we sometimes refer to Q as the *full* right quotient ring to distinguish it from partial quotient rings.

A multiplicatively closed set \mathcal{S} of R is said to be a *right Ore set* if given $a \in R$ and $c \in \mathcal{S}$ there exist $a_1 \in R$ and $c_1 \in \mathcal{S}$ such that $ac_1 = ca_1$. It is well known that by Ore's theorem R has a right quotient ring if and only if $\mathcal{E}(0)$ is a right Ore set.

The right quotient ring of R coincides with the left quotient ring of R when they both exist.

Throughout R will denote a Noetherian ring *module finite* over its centre C , i.e., R will be finitely generated as a C -module. In this situation C is also a Noetherian ring by Eisenbud's theorem [8, Corollary 10.1.11]. In particular R is an integral extension of C , and therefore by [1, Propositions 1.2, 1.3 and Corollary 1.7] the standard properties 'going up', 'lying over', and 'incomparability' hold between R and C . Also for a prime ideal ρ of C there are only finitely many prime ideals of R lying over ρ .

The term R is *C -torsion free* will mean that for $0 \neq r \in R$ and c regular in C we have $rc \neq 0$.

Müller's theorem [10, Theorem 7 or 5, Theorems 11.20 and 12.21] provides us with the following convenient localisability criterion: Let P be a prime ideal of R , and let $P = P_1, \dots, P_n$ be the complete list of prime ideals lying over $P \cap C$. Then $\{P_1, \dots, P_n\}$ is precisely the *clique* (clan) of P , and this clique is *localisable*; i.e., $\bigcap_{i=1}^n \mathcal{E}(P_i)$ is an Ore set.

By [12, Corollary 2.3 and Proposition 2.6] R contains a unique, irredundant, finite set of primes P_1, \dots, P_n such that $\mathcal{E}(0) = \mathcal{E}(P_1) \cap \dots \cap \mathcal{E}(P_n)$. These are called the *regular primes* of R .

R is called a *semilocal ring* if R/J is a semisimple Artinian ring where J is the Jacobson radical of R . When R is semilocal, J is a finite intersection of the maximal ideals of R .

Where relevant the absence of the adjectives right or left will imply two-sided assumptions.

3. QUOTIENT RINGS

Lemma 3.1. *Let R be a Noetherian ring which has a full quotient ring Q . Let P_1, \dots, P_n be the regular primes of R . Then $P_i \not\subseteq P_j$ for $i \neq j$.*

Proof. By [13, Corollary 2.7] Q is a semilocal ring. Let M_1, \dots, M_k be the maximal ideals of Q . Noting that units can be lifted over the Jacobson radical, we have $\mathcal{E}(0) = \mathcal{E}(M_1) \cap \dots \cap \mathcal{E}(M_k)$ in Q . This implies that $\mathcal{E}(0) = \mathcal{E}(M_1 \cap R) \cap \dots \cap \mathcal{E}(M_k \cap R)$ in R . It is easily seen that this intersection is not redundant. Clearly $M_i \cap R \not\subseteq M_j \cap R$ for $i \neq j$. By the uniqueness of regular primes it follows that $k = n$ and after suitable renumbering $P_j = M_j \cap R$ for $j = 1, \dots, n$. \square

Lemma 3.2. *Let R be a Noetherian ring which is module finite over its centre C . Suppose that R has a full quotient ring. Let P_1, \dots, P_n be the regular primes of R . Then the set $\{P_1, \dots, P_n\}$ contains all the prime ideals of R which lie over the prime ideals $P_1 \cap C, \dots, P_n \cap C$ of C .*

Proof. By assumption $\mathcal{E}(0)$ is an Ore set. An elementary argument shows that for each prime ideal P of R we have either $\mathcal{E}(0) \subseteq \mathcal{E}(P)$ or $\mathcal{E}(0) \cap P \neq \emptyset$. If P and Q are two primes in the same clique then we have $\mathcal{E}(0) \subseteq \mathcal{E}(P)$ if and only if $\mathcal{E}(0) \subseteq \mathcal{E}(Q)$ by [5, Lemma 12.17]. Take Q to be a prime in the clique of P_1 . We shall show that $Q = P_i$ for some i . This will imply that $\{P_1, \dots, P_n\}$ is a union of cliques. We then have the result, noting that by Müller's theorem a clique in R consists precisely of all primes lying over a fixed prime of C .

Suppose first that $Q \not\subseteq P_i$ for $1 \leq i \leq n$. Then $Q \cap \mathcal{E}(P_i) \neq \emptyset$. So, by [3, Lemma 13.4], Q contains an element of $\bigcap_{i=1}^n \mathcal{E}(P_i) = \mathcal{E}(0)$. Thus $\mathcal{E}(0) \not\subseteq \mathcal{E}(Q)$, so by the argument above $\mathcal{E}(0) \not\subseteq \mathcal{E}(P_1)$. This is a contradiction since $\bigcap_{i=1}^n \mathcal{E}(P_i) = \mathcal{E}(0)$. Therefore, $Q \subseteq P_i$ for some i . Now since Q belongs to the clique of P_1 , by Müller's theorem we have $Q \cap C = P_1 \cap C$. However, $Q \subseteq P_i$, so $P_1 \cap C = Q \cap C \subseteq P_i \cap C$. Hence by 'going up' there exists a prime ideal Q' of R such that $P_1 \subseteq Q'$ and $Q' \cap C = P_i \cap C$. Now by Müller's theorem Q' belongs to the clique of P_i . So as with Q and P_1 above, it follows that $Q' \subseteq P_j$ for some j . Thus $P_1 \subseteq Q' \subseteq P_j$. We have $P_1 = P_j$ by Lemma 3.1, so $P_1 = Q'$ and $P_1 \cap C = P_i \cap C$. Therefore, $Q \cap C = P_i \cap C$ with $Q \subseteq P_i$. By 'incomparability' it follows that $Q = P_i$. \square

Proposition 3.3. *Let R be a Noetherian ring which is module finite over its centre C . Let P_1, \dots, P_n be the regular primes of R . Then R has a full quotient ring if and only if the set $\{P_1, \dots, P_n\}$ contains all the prime ideals of R which lie over the prime ideals $P_1 \cap C, \dots, P_n \cap C$ of C .*

Proof. If R has a full quotient ring then Lemma 3.2 gives the result one way.

Conversely suppose that the set $\{P_1, \dots, P_n\}$ satisfies the given condition. Then by Müller's theorem $\{P_1, \dots, P_n\}$ is a union of cliques. Renumbering suitably, let P_1, \dots, P_k be a clique. Then these are all the primes lying over $P_1 \cap C$. Let $a, c \in R$ with $c \in \mathcal{E}(0)$. We have $c \in \mathcal{E}(P_i)$ for all i , $1 \leq i \leq n$, so $c \in \bigcap_{i=1}^k \mathcal{E}(P_i)$. Set $K = \{x \in R \mid ax \in cR\}$. Since the clique is localisable, $\bigcap_{i=1}^k \mathcal{E}(P_i)$ is an Ore set, so $K \cap \bigcap_{i=1}^k \mathcal{E}(P_i) \neq \emptyset$. This works for all the cliques in $\{P_1, \dots, P_n\}$, so $K \cap \bigcap_{i=1}^n \mathcal{E}(P_i) \neq \emptyset$ by [3, Lemma 13.4]. Thus

$K \cap \mathcal{E}(0) \neq \emptyset$, and R has the right Ore condition with respect to $\mathcal{E}(0)$. The left Ore condition is analogous. \square

Theorem 3.4. *Let R be a Noetherian ring which is module finite over its centre C . Suppose that R has a full quotient ring Q . Then Q can be formed by inverting those elements of C which are regular as elements of R .*

Proof. Let \mathcal{S} denote the elements of C which are regular as elements of R . Let $R_{\mathcal{S}}$ be the partial quotient ring of R formed by inverting the elements of \mathcal{S} . Then $R_{\mathcal{S}}$ is a subring of Q , and we shall prove the theorem by showing that $Q = R_{\mathcal{S}}$. It is clearly enough to show that $R_{\mathcal{S}}$ is a quotient ring. Note that $R_{\mathcal{S}}$ is a finitely generated module over its centre $C_{\mathcal{S}}$.

Let P_1, \dots, P_n be the regular primes of R . By Lemma 3.1 we know that $P_i \not\subseteq P_j$ for $i \neq j$. Also by Lemma 3.2 the set $\{P_1, \dots, P_n\}$ contains all the prime ideals of R which lie over any prime ideal of C of the form $P_i \cap C$. So 'going up' gives $P_i \cap C \not\subseteq P_j \cap C$ for $i \neq j$. Clearly $\mathcal{S} = \{c \in C \mid c \notin P_i \text{ for } i = 1, \dots, n\}$. Hence $C_{\mathcal{S}}$ is a commutative Noetherian semilocal ring with maximal ideals the distinct members from the list $(P_1 \cap C)C_{\mathcal{S}}, \dots, (P_n \cap C)C_{\mathcal{S}}$. Now each $P_i R_{\mathcal{S}}$ is a prime ideal of $R_{\mathcal{S}}$ and lies over the maximal ideal $P_i R_{\mathcal{S}} \cap C_{\mathcal{S}} = (P_i \cap C)C_{\mathcal{S}}$ of $C_{\mathcal{S}}$. Since $R_{\mathcal{S}}$ is finite over $C_{\mathcal{S}}$, by 'incomparability', $P_i R_{\mathcal{S}}$ is a maximal ideal of $R_{\mathcal{S}}$.

Let M be a maximal ideal of $R_{\mathcal{S}}$. We shall show that $M = P_i R_{\mathcal{S}}$ for some i . Now $M \cap R$ is a prime ideal of R , and by 'going up' $M \cap C_{\mathcal{S}}$ is a maximal ideal of $C_{\mathcal{S}}$. Hence $M \cap C_{\mathcal{S}} = (P_i \cap C)C_{\mathcal{S}}$ for some i . Therefore, $P_i \cap C \subseteq M \cap C$. We have $(M \cap R)R_{\mathcal{S}} = M \neq R_{\mathcal{S}}$; so $M \cap R \cap \mathcal{S} = \emptyset$. Thus $M \cap C \cap \mathcal{S} = \emptyset$. Hence $M \cap C \subseteq \bigcup_{j=1}^n (P_j \cap C)$. It follows by [6, Theorem 81] that $M \cap C \subseteq P_j \cap C$ for some j . Therefore, we have $P_i \cap C \subseteq M \cap C \subseteq P_j \cap C$. Since, as shown above, the inclusion cannot be strict, $P_i \cap C = M \cap C = P_j \cap C$. Thus $M \cap R$ lies over $P_i \cap C$. Hence, by Lemma 3.2, $M \cap R = P_k$ for some k . Therefore, $M = (M \cap R)R_{\mathcal{S}} = P_k R_{\mathcal{S}}$, and we have now shown that $P_1 R_{\mathcal{S}}, \dots, P_n R_{\mathcal{S}}$ are precisely the maximal ideals of $R_{\mathcal{S}}$.

Now let $a \in \mathcal{E}(0)$ in $R_{\mathcal{S}}$. Then $as \in R$ for some $s \in \mathcal{S}$. Clearly as is a regular element of R . Hence $as \in \mathcal{E}(P_i)$ in R for all i so that $as \in \mathcal{E}(P_i R_{\mathcal{S}})$ in $R_{\mathcal{S}}$. Since $R_{\mathcal{S}}$ is a semilocal ring with maximal ideals $P_1 R_{\mathcal{S}}, \dots, P_n R_{\mathcal{S}}$, it follows that a is a unit of $R_{\mathcal{S}}$. Therefore, $R_{\mathcal{S}}$ is a quotient ring, and $Q = R_{\mathcal{S}}$ as required. \square

Lemma 3.5. *Let R be a commutative Noetherian ring which is a quotient ring. Let M be a finitely generated R -module of finite projective dimension. Then M is projective.*

Proof. The regular elements of R are the elements which are regular modulo the maximal associated primes of the ideal $\{0\}$ in a primary decomposition of $\{0\}$. Thus R is semilocal, and its maximal ideals are annihilator ideals. Let P be a maximal ideal of R . It is easily seen that in the local ring R_P the Jacobson radical is an annihilator. Hence, by [6, Theorem 191], $M_P = M \otimes_R R_P$ is a projective R_P -module. It follows by [9, Theorem 7.12] that M is a projective R -module. \square

Proposition 3.6. *Let R be a Noetherian ring which is finitely generated, torsion-free, and of finite projective dimension over its centre C . Then R has a full quotient ring which is a projective module over its centre.*

Proof. Let \mathcal{S} be the set of all regular elements of C . Since R is C -torsionfree, the elements of \mathcal{S} are regular in R . We shall show that $R_{\mathcal{S}}$ is its own quotient ring and hence is the full quotient ring of R . Now $C_{\mathcal{S}}$ is the full quotient ring of C ; it is also the centre of $R_{\mathcal{S}}$, and $R_{\mathcal{S}}$ has finite projective dimension as a $C_{\mathcal{S}}$ -module. By Lemma 3.5 it follows that $R_{\mathcal{S}}$ is $C_{\mathcal{S}}$ -projective and $C_{\mathcal{S}}$ is a semilocal ring. Let $1 = e_1 + \dots + e_n$ be a sum of primitive orthogonal idempotents in $C_{\mathcal{S}}$. Then each $e_i R_{\mathcal{S}}$ is a finitely generated projective module over its centre $e_i C_{\mathcal{S}}$. Now $e_i C_{\mathcal{S}}$ is an indecomposable commutative Noetherian semilocal ring. So, by [7, Corollary 3.6 and Theorem 3.7], $e_i R_{\mathcal{S}}$ is a free $e_i C_{\mathcal{S}}$ -module. Since $e_i C_{\mathcal{S}}$ is its own quotient ring, so also is $e_i R_{\mathcal{S}}$ by [13, Proposition 5.2]. Therefore, $R_{\mathcal{S}}$ is its own quotient ring as required. \square

Of course, R has finite projective dimension over C when C is a Noetherian ring of finite global dimension. However, in this case C is a semiprime ring, and then, if R is a torsion free C -module, it is easy to see that R has an Artinian quotient ring. A nontrivial situation when the above proposition is applicable is given in Example 4.2.

Note that as the examples in [2, 11] and Example 4.1 show, neither of the two conditions, ‘finite projective dimension’ and ‘torsionfree’ can be dropped to get the conclusion of Proposition 3.6.

4. EXAMPLES

Let R be a Noetherian ring module finite over its centre C . Let \mathcal{S} be the set of elements of C which are regular as elements of R . We can always form the partial quotient ring $R_{\mathcal{S}}$ by inverting the elements of \mathcal{S} . Theorem 3.4 shows that $R_{\mathcal{S}}$ must be the quotient ring of R if this exists. In particular when C is its own quotient ring then $R_{\mathcal{S}} = R$ so that in this case either R is a quotient ring or R has no quotient ring at all. Our next example shows that the latter possibility can occur.

Example 4.1. Let F be a field and $S = F[X, Y]$ the ring of polynomials in commuting indeterminates X and Y . Let T be the localisation of S at the ideal generated by X and Y . Consider the factor ring $C = T/Y(XT + YT)$ of T . Set $y = Y + Y(XT + YT)$, $M = (XT + YT)/Y(XT + YT)$, and $P = yC$. Then C is a commutative Noetherian ring with unique maximal ideal M , and P is the only other prime ideal of C . We have $M = \text{ann}(P)$. From this it follows that regular elements of C are regular mod (M) and hence are units of C . Thus C is its own quotient ring. Let bars denote images in $\bar{C} = C/P$. Set $R = \begin{bmatrix} C & \bar{C} \\ 0 & \bar{C} \end{bmatrix}$. The centre $Z(R)$ of R consists of all matrices of the form $\begin{bmatrix} c & \bar{0} \\ 0 & \bar{c} \end{bmatrix}$ for $c \in C$. Hence the centre of R is isomorphic to C . Identifying the centre with C , the ring R is a finite module over C . Also R is C -torsionfree since the regular elements of C are units. By Theorem 3.4 we know that if R had a quotient ring it would be R itself. Now select w to be a nonzero nonunit of \bar{C} . Then the matrix $\begin{bmatrix} 1 & \bar{0} \\ 0 & w \end{bmatrix}$ is a regular element of R but not a unit. Therefore, R is not a quotient ring, so R does not have a two-sided quotient ring. Note that R does have a right quotient ring. Let K be the quotient field of \bar{C} . Set $Q = \begin{bmatrix} C & K \\ 0 & K \end{bmatrix}$. Then $Q \neq R$, and Q is the right quotient ring of R but not the left quotient ring.

An element $\begin{bmatrix} a & \bar{b} \\ 0 & \bar{c} \end{bmatrix}$ with $a, b, c \in C$ is regular in R if and only if a is a unit of C and \bar{c} is regular in \bar{C} . Thus the regular primes of R are $P_1 = \begin{bmatrix} M & \bar{C} \\ 0 & \bar{C} \end{bmatrix}$

and $P_2 = \begin{bmatrix} C & \bar{C} \\ 0 & 0 \end{bmatrix}$. Set $\mathcal{P}_2 = Z(R) \cap P_2$. Then $\mathcal{P}_2 = \begin{bmatrix} P & \bar{0} \\ 0 & 0 \end{bmatrix}$. Set $P_3 = \begin{bmatrix} P & \bar{C} \\ 0 & C \end{bmatrix}$. Then P_3 is a prime ideal of R , and P_3 lies over $\mathcal{P}_2 = Z(R) \cap P_2$. However, P_2 is a regular prime of R whereas P_3 is not. Thus we can also see from Proposition 3.3 that R does not possess a two-sided quotient ring.

Our next example gives a nontrivial application of Proposition 3.6.

Example 4.2. Let A be any commutative Noetherian ring. Set $C = A[X, Y]$ the ring of polynomials in commuting indeterminates. Let $M = xC + yC$. It is easily seen that the projective dimension of M cannot exceed 1. We shall now show that M is not projective as a C -module. Let $f: M \rightarrow C$ be a C -homomorphism. Since $f(xy) = f(x)y = f(y)x$ it follows that $f(xy) \in xyC$, and hence $f(x) \in xC$. Similarly $f(y) \in yC$, and we have $f(M) \subseteq M$. Now if M were projective, the dual basis lemma would give $M = M^2$ which is a contradiction. Thus M is a module of projective dimension 1. Now set $R = \begin{bmatrix} C & M \\ 0 & C \end{bmatrix}$. Then R is a Noetherian ring. Since M is a faithful C -module, it is easily seen that the centre of R consists of all matrices of the form $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ where $c \in C$ so that the centre of R is isomorphic to C , and we shall identify it with C .

Thus R is a Noetherian ring which is finitely generated, torsion-free, and of finite homological dimension over its centre C . Therefore, the hypotheses of Proposition 3.6 are satisfied with $C = A[X, Y]$ for any commutative Noetherian ring A .

In view of Müller's theorem, we can interpret Proposition 3.3 as follows: If R is a Noetherian ring which is module-finite over its centre, then R has a full quotient ring if and only if the set of regular primes of R is closed under the formation of links. The following example shows that this is not true for general Noetherian rings.

Example 4.3. As in [5, Example 2ZC, p. 41], let D be a Noetherian integral domain with a unique proper nonzero ideal I such that D/I is a field, say, F . Then D does not satisfy the Ore condition with respect to $\mathcal{E}(I)$ since I is an idempotent ideal and the elements of $\mathcal{E}(I)$ are regular in D . Set $R = \begin{bmatrix} D & F \\ 0 & F \end{bmatrix}$. Then R is a Noetherian ring. The regular elements of R have the form $\begin{bmatrix} d & x \\ 0 & y \end{bmatrix}$ where $d \in \mathcal{E}(I) \subseteq D$ and $x, y \in F$ with $y \neq 0$. Since D does not satisfy the Ore condition with respect to $\mathcal{E}(I)$, it follows that R does not have a full quotient ring. Also it is easily seen that the regular primes of R are $P_1 = \begin{bmatrix} I & F \\ 0 & F \end{bmatrix}$ and $P_2 = \begin{bmatrix} D & F \\ 0 & 0 \end{bmatrix}$. Note that R/P_1 and R/P_2 are both Artinian rings. (In fact, they are isomorphic to F). The only other prime ideal of R is $P = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. However, R/P is not Artinian, so P is not linked to either P_1 or P_2 . Therefore the set $\{P_1, P_2\}$ of regular primes of R is link-closed.

Added in proof. T. Lenagan has pointed out that in contrast to Example 4.3, the methods of this paper can easily be extended to prove the proposed extension of Proposition 3.3 to Noetherian rings satisfying the second layer condition.

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SCHOOL OF MATHEMATICS, UNIVERSITY WALK, BRISTOL BS8 1TW, ENGLAND

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, ENGLAND

E-mail address: crh@maths.warwick.ac.uk