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A NOTE CONCERNING A THEOREM OF CRAMER

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ABSTRACT. In this paper we consider H. Cramer's Theorem 5.2 from *Structural and statistical problems for a class of stochastic processes*, Princeton Univ. Press, Princeton, NJ, 1971, and show that the proof of this theorem is not valid.

Let $x(t)$, $t \in (a, b) \subset R$, be a second-order real-valued process with $E[x(t)] = 0$ for each t. Let $H(x, t)$ be the linear closure generated by $x(s)$, $s \in (a, t]$, in the Hilbert space H of all random variables with finite variance. We will suppose that $x(t)$, $t \in (a, b)$, is continuous to the left and purely nondeterministic (i.e., $\prod_{t>a} H(x, t) = 0$).

According to the theory of selfadjoint linear operators in a separable Hilbert space (see [1]), there is a representation

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t, u)\, dz_n(u), \qquad u \le t, t \in (a, b), \tag{1}$$

where:

(1) The processes $z_n(u)$, $n = 1, \dots, N$, are mutually orthogonal with orthogonal increments such that $E[z_n(u)] = 0$ and $E[z_n^2(u)] = F_n(u)$, where $F_n(u)$, $n = 1, \dots, N$, are never decreasing and everywhere continuous to the left, $u \in (a, b)$.

(2) The nonrandom functions $g_n(t, u)$, $u \le t$, are from the spaces $L^2(dF_n(u))$, $n = 1, \dots, N$, i.e.,

$$\sum_{n=1}^{N} \int_a^t g_n^2(t, u)\, dF_n(u) < \infty \quad \text{for each } t \in (a, b).$$

(3) $dF_1 > dF_2 > \cdots > dF_n$; the relation $>$ means absolute continuity between measures.

(4) $H(x, t) = \sum_{n=1}^{N} \oplus H(z_n, t)$, $\quad t \in (a, b)$.

The expansion (1) satisfying conditions (1)–(4) is called the canonical or Cramer one for the process $x(t)$. The number N (finite or infinite) is called the multiplicity of $x(t)$, and N is uniquely determined by the process $x(t)$. But the processes $z_n(u)$ and the functions $g_n(t, u)$ are not uniquely determined.

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A representation (1) satisfying conditions (1)–(3) is canonical if and only if for every $t \in T$ it is impossible to find $y(t)$ from the space $H(z_1, t) \oplus H(z_2, t) \oplus \cdots \oplus H(z_n, t)$ such that $0 < E[y^2] < \infty$ and y is orthogonal to $x(s)$ for all $a < s \leq t$ (Lemma 3.1 from [1]).

One of the main problems here is to determine the class of processes with multiplicity $N = 1$. In this connection, Cramer proved the following theorem.

Theorem 5.1 (from [1]). *Let X be the class of all $x(t)$ processes admitting a canonical representation* (1) *in which each term in the second member satisfies the following regularity conditions*:

- (R_1) *The functions $g_n(t, u)$ and $\partial g_n(t, u)/\partial t$ are bounded and continuous for $u \leq t$, $u, t \in T$.*
- (R_2) $g_n(t, t) = 1$ *for all* $t \in T$, $n = 1, \dots, N$.
- (R_3) *The function $F_n(u) = E[z_n^2(u)]$ is absolutely continuous and not identically constant, and $f_n(u) = \partial F_n(u)/\partial u$ has at most a finite number of discontinuity points in any finite subinterval of (a, b), $n = 1, \dots, N$.*

Then every $x(t) \in X$ has multiplicity $N = 1$.

If we assume $g_n(t, t) > 0$ for all n and t, condition (R_2) implies no further restriction of generality (see [1]).

Considering the case of an $x(t)$ process given by the expression

$$x(t) = \int_a^t g(t, u)\, dz(u), \qquad u \leq t, \ t \in (a, b), \tag{2}$$

Cramer gave

Theorem 5.2. *Let $x(t)$ be given for $t \in (a, b)$ by* (2) *with a second member satisfying conditions* (1) *and* (2) *as well as the regularity conditions. If a is finite, $x(t)$ has multiplicity $N = 1$ and expression* (2) *is a canonical representation of $x(t)$.*

The proof of this theorem contains a mistake. It should be shown that expression (2) is the canonical representation of $x(t)$. Then it immediately follows by Theorem 5.1 that the multiplicity is $N = 1$. In this connection, we take

$$y(t) = \int_a^t h(u)\, dz(u) \in H(z, t), \qquad t \in (a, b),$$

such that

$$E[yx(s)] = \int_a^s h(u) g(s, u)\, dF(u) = 0 \quad \text{for all } s \in (a, t],$$

and where $F(u) = E[z^2(u)]$. Now it is enough to show that $h(u) = 0$ for all $s \in (a, t]$. According to (R_3) (i.e., $dF(u) = f(u)du$), the previous equation becomes

$$\int_a^s h(u) g(s, u) f(u)\, du = 0, \qquad s \in (a, t].$$

According to (R_1) and (R_2) we obtain

$$\int_a^s h(u) f(u) \partial g(s, u)/\partial u\, du + h(s) f(s) = 0, \qquad s \in (a, t].$$

From the classical theory this integral homogeneous equation with unknown function $h(s)f(s)$ has the unique solution $h(s)f(s) = 0$ (i.e., $h(s) = 0$) for all $s \in (a, t]$ in $L^2(du)$ if $g(t, u)$ is a complete kernel in $L^2(du)$, and in Theorem 5.2 this is not assumed. So (2) is not the canonical representation, and the multiplicity of $x(t)$ remains unknown.

The next simple example shows that there exists a process $x(t)$ represented by (2) satisfying the regularity conditions as well as conditions (1) and (2), and a process $y(t)$, $y(t) \in H(z, t)$, such that $E[yx(s)] = 0$ for all $s \in (a, t]$ and $0 < E[y^2] < \infty$.

Example. All processes given by

$$x(t) = \int_0^t [-kt + (k+1)u]\, dz(u),$$

$$u \le t,\ u,\ t \in [0, \beta],\ k \in N,\ E[z^2(u)] = u,$$

have no complete kernels $g_k(t, u) = -kt + (k+1)u$ in $L^2(du)$. For each process $x(t)$ there exists a process $y(t)$:

$$y(t) = \int_0^t u^{k-1}\, dz(u) \in H(z, t)$$

such that

$$E[yx(s)] = \int_0^s g_k(s, u) u^{k-1}\, du = 0 \quad \text{for all } s \in (a, t].$$

References

1. H. Cramer, *Structural and statistical problems for a class of stochastic processes*, Princeton Univ. Press, Princeton, NJ, 1971, p. 30.

2. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York, 1958, p. 423.

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