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ON WEIGHTED SOBOLEV INTERPOLATION INEQUALITIES

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ABSTRACT. We obtain some weighted Sobolev interpolation inequalities on $\mathbb{R}^n$ and domains satisfying the Boman chain condition for doubling weights satisfying a weighted Poincaré inequality.

1. INTRODUCTION

Recently, there has been a significant number of papers on weighted Sobolev interpolation inequalities, for example, Brown and Hilton [3–5], Gutierrez and Wheeden [18], and Chua [9]. In this paper, we will study weighted Sobolev interpolation inequalities with weights satisfying the following inequality:

$$\|f - f_Q\|_{L^q_w(Q)} \leq A(Q)\|\nabla f\|_{L^p_v(Q)} \tag{1.1}$$

as in [18] where $f_Q = \int_Q f\,dx/|Q|$. Let us note that some sufficient conditions have been obtained for (1.1); see [27], [26], or [7].

Definition 1.1 [19]. An open set $\mathscr{D}$ in $\mathbb{R}^n$ is said to be a member of $\mathscr{F}(\sigma, N)$, $\sigma \geq 1$, $N \geq 1$, if there exists a covering W of $\mathscr{D}$ consisting of cubes such that:

(i)

$$\sum_{Q \in W} \chi_{\sigma Q}(x) \leq N\chi_{\mathscr{D}}(x) \quad \forall x \in \mathbb{R}^n.$$

(ii) There is a 'central cube' $Q_0 \in W$ that can be connected with every cube $Q \in W$ by a finite chain of cubes $Q_0, Q_1, \ldots, Q_{k(Q)} = Q$ from W such that $Q \subset NQ_j$ for $j = 0, 1, \ldots, k(Q)$. Moreover, $Q_j \cap Q_{j+1}$ contains a cube R_j such that $Q_j \cup Q_{j+1} \subset NR_j$.

We say that $\mathscr{D}$ satisfies the Boman chain condition if $\mathscr{D} \in \mathscr{F}(\sigma, N)$ for some N, $\sigma \geq 1$. There are many types of domains that satisfy the Boman chain condition, for example, balls, cubes, and John domains (see [19]). Moreover, it is easy to check that bounded (ε, ∞) domains (see [20] or [9] for the definition) satisfy the Boman chain condition. Hence, so do bounded Lipschitz domains. In what follows, Q is always a cube and $l(Q)$ will be its edgelength. If $1 < p <$

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∞, p' will denote $p/(p-1)$. By a weight w, we mean a nonnegative locally integrable function on $\mathbb{R}^n$. By abusing notation, we will also write w for the measure induced by w. Sometimes we write dw to denote $w\,dx$. We say that w is doubling if $w(2Q) \le Cw(Q)$ for every cube Q, where $2Q$ denotes the cube with the same center as Q and twice its edgelength. By $w \in A_p$, we mean w satisfies the Muckenhoupt A_p condition, i.e.,

$$\frac{1}{|Q|}\left(\int_Q w\,dx\right)^{1/p}\left(\int_Q w^{-1/(p-1)}\,dx\right)^{1/p'} \le C \quad \text{when } 1 < p < \infty,$$

and

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C \operatorname*{ess\,inf}_{x\in Q} w(x) \quad \text{when } p = 1,$$

for all cubes Q in $\mathbb{R}^n$. Note that w is doubling when it is in A_p.

Let $\mathscr{D}$ be an open set in $\mathbb{R}^n$. If α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$, we will denote $\sum_{j=1}^n \alpha_j$ by $|\alpha|$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. We denote by ∇ the vector $(\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$ and by ∇^m the vector of all possible mth-order derivatives for $m \in \mathbb{N}$. A locally integrable function f on $\mathscr{D}$ (we will write $f \in L^1_{\mathrm{loc}}(\mathscr{D})$) has a weak derivative of order α if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$\int_{\mathscr{D}} f(D^\alpha \varphi)\,dx = (-1)^{|\alpha|}\int_{\mathscr{D}} (D^\alpha f)\varphi\,dx$$

for all C^∞ functions φ with compact support in $\mathscr{D}$ (we will write $\varphi \in C_0^\infty(\mathscr{D})$).

For $1 \le p < \infty$, $k \in \mathbb{N}$, and any weight w, $L^p_{w,k}(\mathscr{D})$ and $E^p_{w,k}(\mathscr{D})$ are the spaces of functions having weak derivatives of all orders α, $|\alpha| \le k$, and satisfying

$$\|f\|_{L^p_{w,k}(\mathscr{D})} = \sum_{0\le|\alpha|\le k} \|D^\alpha f\|_{L^p_w(\mathscr{D})} = \sum_{0\le|\alpha|\le k}\left(\int_{\mathscr{D}} |D^\alpha f|^p\,dw\right)^{1/p} < \infty$$

and

$$\|f\|_{E^p_{w,k}(\mathscr{D})} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p_w(\mathscr{D})} < \infty,$$

respectively. Moreover, in the case when $w \equiv 1$, we will denote $L^p_{w,k}(\mathscr{D})$ and $E^p_{w,k}(\mathscr{D})$ by $L^p_k(\mathscr{D})$ and $E^p_k(\mathscr{D})$, respectively. Finally, let $\Lambda^k(\mathscr{D})$ be the collection of all functions f on $\mathscr{D}$ such that all its weak derivatives of order $\le k$ exist.

We will prove that

Theorem 1.2. *Let $\mathscr{D} \in \mathscr{F}(\sigma, N)$, and let W be a covering of $\mathscr{D}$ satisfying the Boman chain condition. Let $1 \le p \le q < \infty$. If v is a weight and w is a doubling weight such that* (1.1) *holds for all $Q \in W$ and $f \in \Lambda^1(\mathscr{D})$, then*

$$\begin{aligned}\|\nabla f\|_{L^q_w(\mathscr{D})} \le{}& Cw(Q_0)^{1/q} l(Q_0)^{-n}(l(Q_0)^{-1}\|f\|_{L^1(Q_0)} + l(Q_0)\|\nabla^2 f\|_{L^1(Q_0)})\\ &+ CA_0\|\nabla^2 f\|_{L^p_v(\mathscr{D})}\end{aligned}$$

for all $f \in \Lambda^2(\mathscr{D})$ where $A_0 = \sup_{Q\in W} A(Q)$, Q_0 is the 'central' cube in W, and C is independent of f and v.

Theorem 1.3. *Let $1 \le p \le q < \infty$. Suppose that v is a weight and w is a doubling weight such that*

$$\|f - f_Q\|_{L^q_w(Q)} \le C_0 w(Q)^{1/q} v'(Q)^{1/p'} l(Q)^{-n+1} \|\nabla f\|_{L^p_v(Q)} \tag{1.2}$$

for all cubes Q and $f \in \Lambda^1(\mathbb{R}^n)$ where $v' = v^{-1/(p-1)}$ ($v'(Q)^{1/p'} = \operatorname{ess\,sup}_{x\in Q} v^{-1}(x)$ when $p = 1$). Then

$$\begin{aligned}\|\nabla^k f\|_{L^q_w(Q)} \le{}& C w(Q)^{1/q} l(Q)^{-n-k} \|f\|_{L^1(Q)} \\ &+ C w(Q)^{1/q} v'(Q)^{1/p'} l(Q)^{-n+1} \|\nabla^{k+1} f\|_{L^p_v(Q)}\end{aligned}$$

for all cubes Q and $f \in \Lambda^{k+1}(\mathbb{R}^n)$ where C is independent of f.

Moreover, if $\|\nabla^{k+1} f\|_{L^p_v(\mathbb{R}^n)} \ne 0$ and there exist $a < 1$, $b > (1-k)/2$, $1 \le p_0 \le q$, and weight v_0 such that

$$l(Q)^{(2b-1-n)} w(Q)^{1/q} v_0'(Q)^{1/p_0'} + l(Q)^{(2a-1-n)} w(Q)^{1/q} v'(Q)^{1/p'} \le C \tag{1.3}$$

for all cubes Q, then

$$\|\nabla^k f\|_{L^q_w(\mathbb{R}^n)} \le C \|f\|^{1-(2b+k-1)/(1+k+2(b-a))}_{L^{p_0}_{v_0}(\mathbb{R}^n)} \|\nabla^{k+1} f\|^{(2b+k-1)/(1+k+2(b-a))}_{L^p_v(\mathbb{R}^n)}.$$

In particular, under the assumptions stated above, we have

$$\|\nabla^k f\|_{L^q_w(\mathbb{R}^n)} \le C \|f\|^{1-(2a+k-1)/(1+k)}_{L^p_v(\mathbb{R}^n)} \|\nabla^{k+1} f\|^{(2a+k-1)/(1+k)}_{L^p_v(\mathbb{R}^n)}.$$

These theorems have some interesting corollaries.

Corollary 1.4. *Let $1 \le p \le q < \infty$, and let $\mathscr{D}$, W, v, and w be as in Theorem 1.2 such that*

$$\|f - f_Q\|_{L^q_w(Q)} \le A \|\nabla f\|_{L^p_v(Q)}$$

for all $Q \in W$ and $f \in \Lambda^1(\mathscr{D})$. Then $E^p_{v,k+1}(\mathscr{D}) \subset E^q_{w,k}(\mathscr{D})$ for all $k \in \mathbb{N}$.

Corollary 1.5. *Let $\mathscr{D}$ and W be as in Theorem 1.2. Suppose $1 \le p \le q < \infty$ and w is a doubling weight such that*

$$\|f - f_Q\|_{L^p_w(Q)} \le A \|\nabla f\|_{L^p_w(Q)}$$

for all $Q \in W$ and $f \in \Lambda^1(\mathscr{D})$. Then $f \in E^p_{w,k}(\mathscr{D})$ if and only if $f \in L^p_{w,k}(\mathscr{D})$.

Note that when $w \in A_p$ and $\mathscr{D}$ is a bounded (ε, ∞) domain, it has been obtained in [9] that $E^p_{w,k}(\mathscr{D}) = L^p_{w,k}(\mathscr{D})$.

Corollary 1.6. *Let $1 \le p \le q < \infty$, $v \in A_p$, and w be a doubling weight such that (1.2) holds. Let $i, k \in \mathbb{N}$, $1 \le i < k$, and $-i < a' < k - i$. Then*

$$\|\nabla^i f\|_{L^q_w(\mathbb{R}^n)} \le C \|f\|^{1-(a'+i)/k}_{L^p_v(\mathbb{R}^n)} \|\nabla^k f\|^{(a'+i)/k}_{L^p_v(\mathbb{R}^n)}$$

for all $f \in \Lambda^k(\mathbb{R}^n)$ and $\|\nabla^k f\|_{L^q_v(\mathbb{R}^n)} \ne 0$ if and only if

$$l(Q)^{a'} w(Q)^{1/q} \le C v(Q)^{1/p} \tag{1.4}$$

for all cubes Q.

Note that when $p > 1$, $i = 1$, and $k = 2$, Corollary 1.6 is first obtained by Gutierrez and Wheeden [18].

Finally, similar to Theorems 1.8 and 1.9 in [9], we could apply the extension theorems in [9] to get the following two corollaries.

Corollary 1.7. *Let $1 \le p \le q < \infty$, $v \in A_p$, and let $\mathscr{D}$ be a bounded (ε, ∞) domain. Let $i, k \in \mathbb{N}$ such that $1 \le i < k$. Let w be a doubling weight such that* (1.2) *and* (1.4) *hold for $-i < a' < k - i$. If $f \in L^p_{v,k}(\mathscr{D})$ and there exists a cube Q in $\mathscr{D}$ such that $f = 0$ on Q and $\nabla^k f \neq 0$ a.e. on $\mathscr{D}$, then*

$$\|\nabla^i f\|_{L^q_w(\mathscr{D})} \le C\|f\|^{(k-a'-i)/k}_{L^p_v(\mathscr{D})}\|\nabla^k f\|^{(a'+i)/k}_{L^p_v(\mathscr{D})} \tag{1.5}$$

where C depends only on $\mathscr{D}$, w, v, k, n, p, q, and Q.

Corollary 1.8. *Let $\mathscr{D}$ be an unbounded (ε, ∞) domain, and let v, w, p, q, i, and k be as in the preceding corollary. If $f \in L^p_{v,k}(\mathscr{D})$ and $\nabla^k f \neq 0$ a.e. on $\mathscr{D}$, then*

$$\|\nabla^i f\|_{L^q_w(\mathscr{D})} \le C\|f\|^{(k-a'-i)/k}_{L^p_v(\mathscr{D})}\|\nabla^k f\|^{(a'+i)/k}_{L^p_v(\mathscr{D})}$$

where C depends only on ε, p, q, v, w, k, and n.

Remark 1.9. (a) Let $\mathscr{D} \subset \mathscr{F}(\sigma, N)$ for some $\sigma, N \ge 1$ and $M \subset \partial\mathscr{D}$ (the boundary of $\mathscr{D}$). Suppose $w(x) = \operatorname{dist}(x, M) = \inf_{y \in M} |x - y|$. Let W be a covering of $\mathscr{D}$ that satisfies the chain condition. Let $\alpha \in \mathbb{R}$. Then it is clear that if $1 \le p \le q < \infty$, then

$$\|f - f_Q\|_{L^q_{w^\alpha}(Q)} \le Cl(Q)\|\nabla f\|_{L^q_{w^\alpha}(Q)}, \tag{1.6}$$

and indeed, when $1 - (\frac{n}{p} - \frac{n}{q}) \ge 0$,

$$\|f - f_Q\|_{L^q_{w^\alpha}(Q)} \le Cl(Q)^{1-(n/p-n/q)}\operatorname{dist}(Q, M)^{\alpha/q-\beta/p}\|\nabla f\|_{L^p_{w^\beta}(Q)} \tag{1.7}$$

for $f \in \Lambda^1(\mathbb{R}^n)$ and $Q \in W$ with C depending only on σ, N, n, p, α, β, and q. These estimates can easily be obtained by the fact that w is comparable to $\operatorname{dist}(Q, M)$ on Q and the unweighted Poincaré type estimate.

We can now apply Theorem 1.2 to conclude that when w^α is doubling,

$$\|\nabla^k f\|_{L^q_{w^\alpha}(\mathscr{D})} \le C\|f\|_{L^1(Q_0)} + C\|\nabla^{k+1} f\|_{L^1(Q_0)} + C\|\nabla^{k+1} f\|_{L^p_{w^\beta}(\mathscr{D})} \tag{1.8}$$

provided $1-(\frac{1}{p}-\frac{1}{q})n+\frac{\alpha}{q}-\frac{\beta}{p} \ge 0$ and $1-(\frac{n}{p}-\frac{n}{q}) \ge 0$ with C depending only on σ, N, n, p, α, β, and q. Hence, for all $k \in \mathbb{N}$, $E^p_{w^\beta,k+1}(\mathscr{D}) \subset E^p_{w^\alpha,k}(\mathscr{D})$ for such p, q, α, and β. Moreover, if $p = q$ and $\alpha = \beta$, we have $E^p_{w^\alpha,k}(\mathscr{D}) = L^p_{w^\alpha,k}(\mathscr{D})$.

(b) Furthermore, if $w(x) = s(\operatorname{dist}(x, M))$ where s is a positive and continuous function on the positive real numbers that satisfies certain properties described in Kufner [21], a similar conclusion can be obtained by Theorem 1.2 if we know that w is doubling.

(c) We do not know exactly when the weights w defined as above are doubling. However, in the case that M is just a finite subset of $\partial\mathscr{D}$, it is easy to see that $\operatorname{dist}(x, M)^\alpha$ is doubling if and only if $\alpha > -n$.

2. Preliminaries

In what follows, C denotes various positive constants. They may differ even in the same string of estimates. Moreover, sometimes, we will use $C(\alpha, \beta, \dots)$ instead of C to emphasize that the constant is depending on α, β, ...

Since one of our main tools will be a projection of functions into polynomials, first let us state an inequality on polynomials.

Theorem 2.1. *Let F, Q be cubes such that $F \subset Q$ and $|F| > \gamma|Q|$. If w is a doubling weight, $1 \le q < \infty$, and p is a polynomial of degree m, then*

$$\|p\|_{L^q_w(E)} \le C(\gamma, m, n, w)\left(\frac{w(E)}{w(F)}\right)^{1/q}\|p\|_{L^q_w(F)}$$

for all measurable sets $E \subset Q$.

This theorem is just a consequence of the following two lemmas.

Lemma 2.2 [29, Chapter 3, Lemma 7]. *If w is a doubling measure and m is a positive integer, then there exists $s_0(n, m, w)$ such that if $s < s_0$, then for all cubes Q, $\lambda > 0$ such that*

$$w(\{x \in Q\colon |p(x)| > \lambda\}) \le sw(Q)$$

we have $\sup_{x\in Q}|p(x)| \le C\lambda$, where p is any polynomial of degree m and C is a constant independent of λ, Q, and p.

It follows from Chebyshev's inequality and this lemma that given m and a polynomial p of degree m,

$$\|p\|_{L^\infty(Q)} \le \frac{C}{w(Q)}\|p\|_{L^1_w(Q)}$$

with C independent of Q and p.

Lemma 2.3 [9, Theorem 2.2]. *Let Q be a cube, and let E be a measurable set in Q with $|E| > \gamma|Q|$. If p is a polynomial of degree m, then*

$$\|p\|_{L^\infty(E)} \ge C(\gamma, m)\|p\|_{L^\infty(Q)}.$$

Next, let us state Markov's inequality; see, for example, [1].

Theorem 2.4. *Let p be any polynomial of order less than k. Then there exists a constant C depending only on k and the dimension n such that*

$$\|\nabla p\|_{L^\infty(Q)} \le Cl(Q)^{-1}\|p\|_{L^\infty(Q)} \quad \textit{for all cubes } Q \textit{ in } \mathbb{R}^n.$$

Finally, the following is now a consequence of Markov's inequality and Lemma 2.2.

Theorem 2.5. *Let p be a polynomial of order less than k and $1 \le q \le \infty$. If w is doubling, then*

$$\|\nabla p\|_{L^q_w(Q)} \le Cl(Q)^{-1}\|p\|_{L^q_w(Q)}$$

for all cubes Q in $\mathbb{R}^n$, where C depends only on k, w, q, and n.

Now let us state a theorem from [12].

Theorem 2.6. *Let $\sigma, N \ge 1$, $1 \le p \le q < \infty$, $k \in \mathbb{N}$, and $\mathscr{D} \in \mathscr{F}(\sigma, N)$, and let f, g be measurable functions defined on $\mathscr{D}$. Also, let v be a weight, and let w be a doubling weight. Suppose that for each cube Q with $\sigma Q \subset \mathscr{D}$, there exists a polynomial $P(f, Q)$ of degree k such that*

$$\|f - P(f, Q)\|_{L^q_w(Q)} \le A\|g\|_{L^p_v(\sigma Q)} \tag{2.1}$$

with A independent of Q. Then there exists a polynomial $P(f, \mathscr{D})$ of degree k such that

$$\|f - P(f, \mathscr{D})\|_{L^q_w(\mathscr{D})} \le CA\|g\|_{L^p_v(\mathscr{D})} \tag{2.2}$$

where C depends only on n, q, w, σ, k, and N. Moreover, we can take $P(f, \mathscr{D}) = P(f, Q_0)$ where Q_0 is the 'central' cube in $\mathscr{D}$.

Let $\mathscr{P}_k$ be the collection of all polynomials with degree $< k$ on $\mathbb{R}^n$. Now, let us state a theorem concerning the projection of function into polynomials.

Theorem 2.7. *Let $\mathscr{D}$ be an open set. For each $k \in \mathbb{N}$ and cubes $Q \subset \mathscr{D}$, there exists a projection $\pi_k(Q)\colon \Lambda^k(\mathscr{D}) \to \mathscr{P}_k$ such that*

$$\operatorname*{ess\,sup}_{x \in Q} |\pi_k(Q) f(x)| \le C l(Q)^{-n} \|f\|_{L^1(Q)}$$

with C independent of f and Q. Moreover, $\pi_k(Q)$ is linear and $\pi_k(Q)p = p$ for all $p \in \mathscr{P}_k$.

For the proof, please refer to [9] or [10].

Finally, let us state the weighted Poincaré inequality for A_p weights. For the proof, see [9].

Theorem 2.8. *If $1 \le p < \infty$, and $v \in A_p$, then*

$$\|f - f_Q\|_{L^p_v(Q)} \le C l(Q) \|\nabla f\|_{L^p_v(Q)}$$

for all cubes Q and $f \in \Lambda^1(\mathbb{R}^n)$ where C depends only on p, v, and n.

3. Proof of main results

Proof of Theorem 1.2. First let us fix $f \in \Lambda^2(\mathscr{D})$ such that $|\nabla^2 f| \in L^p_v(\mathscr{D})$. Next, we let $P_{Q_0} f$ be the polynomial of degree 1 such that $\int_{Q_0} D^\alpha(f - P_{Q_0} f)\, dx = 0$ for all $|\alpha| \le 1$. Now, by Theorem 2.6,

$$\|\nabla(f - P_{Q_0} f)\|_{L^q_w(\mathscr{D})} \le C A_2 \|\nabla^2 f\|_{L^p_v(\mathscr{D})},$$

where $A_2 = \sup_{Q \in W} A(Q)$. Next, let us note that

$$\begin{aligned}
\|\nabla P_{Q_0} f\|_{L^q_w(\mathscr{D})} &\le C \|\nabla P_{Q_0} f\|_{L^q_w(Q_0)} \quad \text{(by Theorem 2.1)} \\
&\le C \|\nabla(P_{Q_0} f - \pi_2(Q_0) f)\|_{L^q_w(Q_0)} + \|\nabla \pi_2(Q_0) f\|_{L^q_w(Q_0)} \\
&\le C l(Q_0)^{-1} \|P_{Q_0} f - \pi_2(Q_0) f\|_{L^q_w(Q_0)} \\
&\quad + C l(Q_0)^{-1} \|\pi_2(Q_0) f\|_{L^q_w(Q_0)} \quad \text{(by Theorem 2.5)} \\
&\le C l(Q_0)^{-1-n} w(Q_0)^{1/q} \|f - P_{Q_0} f\|_{L^1(Q_0)} \\
&\quad + C l(Q_0)^{-1-n} w(Q_0)^{1/q} \|f\|_{L^1(Q_0)} \quad \text{(by Theorem 2.7)} \\
&\le C l(Q_0)^{1-n} w(Q_0)^{1/q} \|\nabla^2 f\|_{L^1(Q_0)} + C l(Q_0)^{-1-n} w(Q_0)^{1/q} \|f\|_{L^1(Q_0)}
\end{aligned}$$

by the nonweighted Poincaré inequality. Hence

$$\begin{aligned}
\|\nabla f\|_{L^q_w(\mathscr{D})} &\le C \|\nabla P_{Q_0} f\|_{L^q_w(\mathscr{D})} + C A_2 \|\nabla^2 f\|_{L^p_v(\mathscr{D})} \\
&\le C l(Q_0)^{1-n} w(Q_0)^{1/q} \|\nabla^2 f\|_{L^1(Q_0)} \\
&\quad + C l(Q_0)^{-1-n} w(Q_0)^{1/q} \|f\|_{L^1(Q_0)} + C A_2 \|\nabla^2 f\|_{L^p_v(\mathscr{D})}.
\end{aligned}$$

Proof of Theorem 1.3. Similar to the proof as above, for any cube Q, we have

$$\begin{aligned}\|\nabla^k f\|_{L^q_w(Q)} &\le Cw(Q)^{1/q}l(Q)^{-k-n}\|f\|_{L^1(Q)} + Cw(Q)^{1/q}l(Q)^{1-n}\|\nabla^{k+1}f\|_{L^1(Q)}\\ &\quad + Cw(Q)^{1/q}v'(Q)^{1/p'}l(Q)^{-n+1}\|\nabla^{k+1}f\|_{L^p_v(Q)}\\ &\le Cw(Q)^{1/q}l(Q)^{-k-n}v_0'(Q)^{1/p_0'}\|f\|_{L^{p_0}_{v_0}(Q)}\\ &\quad + Cw(Q)^{1/q}v'(Q)^{1/p'}l(Q)^{-n+1}\|\nabla^{k+1}f\|_{L^p_v(Q)}\end{aligned}$$

by Hölder's inequality. Next, if (1.3) holds, then for all cubes Q we have

$$\|\nabla^k f\|_{L^q_w(Q)} \le Cl(Q)^{-2b-k+1}\|f\|_{L^{p_0}_{v_0}(Q)} + Cl(Q)^{2-2a}\|\nabla^{k+1}f\|_{L^p_v(Q)}.$$

Now, for all $\varepsilon > 0$, we will cover $\mathbb{R}^n$ by cubes of length ε. Let W' be any nonoverlapping cover of cubes with length ε. Summing up the cubes, we have

$$\|\nabla f\|^q_{L^q_w(\mathbb{R}^n)} \le C\varepsilon^{(-2b-k+1)q}\sum_{Q\in W'}\|f\|^q_{L^{p_0}_{v_0}(Q)} + C\varepsilon^{(2-2a)q}\sum_{Q\in W'}\|\nabla^{k+1}f\|^q_{L^p_v(Q)}.$$

Thus

$$\begin{aligned}\|\nabla^k f\|_{L^q_w(\mathbb{R}^n)} &\le C\varepsilon^{-2b-k+1}\left(\sum_{Q\in W'}\|f\|^{p_0}_{L^{p_0}_{v_0}(Q)}\right)^{1/p_0}\\ &\quad + C\varepsilon^{(2-2a)}\left(\sum_{Q\in W'}\|\nabla^{k+1}f\|^p_{L^p_v(Q)}\right)^{1/p}\end{aligned}$$

since $q \ge p_0, p$. Hence,

$$\|\nabla f\|_{L^q_w(\mathbb{R}^n)} \le C\varepsilon^{-2b-k+1}\|f\|_{L^{p_0}_{v_0}(\mathbb{R}^n)} + C\varepsilon^{(2-2a)}\|\nabla^{k+1}f\|_{L^p_v(\mathbb{R}^n)}.$$

To complete the proof just choose $\varepsilon = (\|f\|_{L^{p_0}_{v_0}(\mathbb{R}^n)}/\|\nabla^{k+1}f\|_{L^p_v(\mathbb{R}^n)})^{1/(1+k+2(b-a))}$.

Next, let us note that Corollaries 1.4 and 1.5 are just immediate consequences of Theorem 1.2.

Proof of Corollary 1.6. First note that by similar arguments as above, for any cube Q and $f \in L^p_{v,k}(\mathbb{R}^n)$, let $P_Q f$ be the unique polynomial of degree $< k$ such that $\int_Q D^\alpha(f - P_Q f)\,dx = 0$ for all $|\alpha| < k$, we have

$$\begin{aligned}\|\nabla^i P_Q f\|_{L^q_w(Q)} &\le Cw(Q)^{1/q}l(Q)^{-i-n}\|f\|_{L^1(Q)} + Cw(Q)^{1/q}l(Q)^{-n+k-i}\|\nabla^k f\|_{L^1(Q)}\\ &\le Cw(Q)^{1/q}l(Q)^{-i}v(Q)^{-1/p}\|f\|_{L^p_v(Q)}\\ &\quad + Cw(Q)^{1/q}v(Q)^{-1/p}l(Q)^{k-i}\|\nabla^k f\|_{L^p_v(Q)}\end{aligned}$$

since $v \in A_p$. Also,

$$\begin{aligned}\|\nabla^i(f - P_Q f)\|_{L^q_w(Q)} &\le Cl(Q)w(Q)^{1/q}v(Q)^{-1/p}\|\nabla^{i+1}(f - P_Q f)\|_{L^p_v(Q)} \quad \text{(by (1.2))}\\ &\le Cl(Q)^{k-i}w(Q)^{1/q}v(Q)^{-1/p}\|\nabla^k f\|_{L^p_v(Q)}\end{aligned}$$

by Theorem 2.8. Hence by the triangle inequality and (1.4),

$$\begin{aligned}\|\nabla^i f\|_{L^q_w(Q)} &\le \|\nabla^i(f - P_Q f)\|_{L^q_w(Q)} + \|\nabla^i P_Q f\|_{L^q_w(Q)}\\ &\le Cl(Q)^{-a'-i}\|f\|_{L^p_v(Q)} + Cl(Q)^{-a'+k-i}\|\nabla^k f\|_{L^p_v(Q)}.\end{aligned}$$

We can now just follow as before to show

$$\|\nabla^i f\|_{L^q_w(\mathbb{R}^n)} \leq C\|f\|^{1-(a'+i)/k}_{L^p_v(\mathbb{R}^n)}\|\nabla^k f\|^{(a'+i)/k}_{L^p_v(\mathbb{R}^n)}$$

for all $f \in L^p_{v,k}(\mathbb{R}^n)$ with $\|\nabla^k f\|_{L^q_v(\mathbb{R}^n)} \neq 0$. Finally, the reader can refer to the proof of Theorem 5 in Gutierrez and Wheeden [18] for the converse.

Finally, let us note that the proof of Corollaries 1.7 and 1.8 are almost identical to the proof of Theorems 1.8 and 1.9 in [9] with the help of Corollary 1.6.

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