Abstract. We obtain some weighted Sobolev interpolation inequalities on \( \mathbb{R}^n \)
and domains satisfying the Boman chain condition for doubling weights satisfying a weighted Poincaré inequality.

1. Introduction

Recently, there has been a significant number of papers on weighted Sobolev interpolation inequalities, for example, Brown and Hilton [3-5], Gutierrez and Wheeden [18], and Chua [9]. In this paper, we will study weighted Sobolev interpolation inequalities with weights satisfying the following inequality:

\[
\|f - f_Q\|_{L^p(Q)} \leq A(Q)\|\nabla f\|_{L^p(Q)}
\]

as in [18] where \( f_Q = \frac{\int_Q f \, dx}{|Q|} \). Let us note that some sufficient conditions have been obtained for (1.1); see [27], [26], or [7].

Definition 1.1 [19]. An open set \( \Omega \) in \( \mathbb{R}^n \) is said to be a member of \( \mathcal{F}(\sigma, N) \), \( \sigma \geq 1 \), \( N \geq 1 \), if there exists a covering \( W \) of \( \Omega \) consisting of cubes such that:

(i) \( \sum_{Q \in W} \chi_{\sigma Q}(x) \leq N \chi_{\Omega}(x) \quad \forall x \in \mathbb{R}^n \).

(ii) There is a ‘central cube’ \( Q_0 \in W \) that can be connected with every cube \( Q \in W \) by a finite chain of cubes \( Q_0, Q_1, \ldots, Q_{k(Q)} = Q \) from \( W \) such that \( Q \subset N Q_j \) for \( j = 0, 1, \ldots, k(Q) \). Moreover, \( Q_j \cap Q_{j+1} \) contains a cube \( R_j \) such that \( Q_j \cup Q_{j+1} \subset NR_j \).

We say that \( \Omega \) satisfies the Boman chain condition if \( \Omega \in \mathcal{F}(\sigma, N) \) for some \( N, \sigma \geq 1 \). There are many types of domains that satisfy the Boman chain condition, for example, balls, cubes, and John domains (see [19]). Moreover, it is easy to check that bounded \( (c, \infty) \) domains (see [20] or [9] for the definition) satisfy the Boman chain condition. Hence, so do bounded Lipschitz domains. In what follows, \( Q \) is always a cube and \( l(Q) \) will be its edgelength. If \( 1 < p < \)}
$p'$ will denote $p/(p - 1)$. By a weight $w$, we mean a nonnegative locally integrable function on $\mathbb{R}^n$. By abusing notation, we will also write $w$ for the measure induced by $w$. Sometimes we write $dw$ to denote $w \, dx$. We say that $w$ is doubling if $w(2Q) \leq Cw(Q)$ for every cube $Q$, where $2Q$ denotes the cube with the same center as $Q$ and twice its edgelength. By $w \in A_p$, we mean $w$ satisfies the Muckenhoupt $A_p$ condition, i.e.,

$$ \frac{1}{|Q|} \left( \int_Q w \, dx \right)^{1/p} \left( \int_Q w^{-1/(p-1)} \, dx \right)^{1/p'} \leq C \quad \text{when } 1 < p < \infty, $$

and

$$ \frac{1}{|Q|} \int_Q w(x) \, dx \leq C \inf_{x \in Q} w(x) \quad \text{when } p = 1, $$

for all cubes $Q$ in $\mathbb{R}^n$. Note that $w$ is doubling when it is in $A_p$.

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$. If $\alpha$ is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we will denote $\sum_{j=1}^n \alpha_j$ by $|\alpha|$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We denote by $\nabla$ the vector $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ and by $\nabla^m$ the vector of all possible $m$th-order derivatives for $m \in \mathbb{N}$. A locally integrable function $f$ on $\mathcal{D}$ (we will write $f \in L^1_{loc}(\mathcal{D})$) has a weak derivative of order $\alpha$ if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$ \int_{\mathcal{D}} f(D^\alpha \varphi) \, dx = (-1)^{|\alpha|} \int_{\mathcal{D}} (D^\alpha f) \varphi \, dx $$

for all $C^\infty$ functions $\varphi$ with compact support in $\mathcal{D}$ (we will write $\varphi \in C^\infty_{0}(\mathcal{D})$).

For $1 \leq p < \infty$, $k \in \mathbb{N}$, and any weight $w$, $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ are the spaces of functions having weak derivatives of all orders $\alpha$, $|\alpha| \leq k$, and satisfying

$$ \|f\|_{L^p_{w,k}(\mathcal{D})} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^p_{w}(\mathcal{D})} = \sum_{0 \leq |\alpha| \leq k} \left( \int_{\mathcal{D}} |D^\alpha f|^p \, dw \right)^{1/p} < \infty $$

and

$$ \|f\|_{E^p_{w,k}(\mathcal{D})} = \sum_{|\alpha| = k} \|D^\alpha f\|_{L^p_{w,k}(\mathcal{D})} < \infty, $$

respectively. Moreover, in the case when $w \equiv 1$, we will denote $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ by $L^p_k(\mathcal{D})$ and $E^p_k(\mathcal{D})$, respectively. Finally, let $\Lambda^k(\mathcal{D})$ be the collection of all functions $f$ on $\mathcal{D}$ such that all its weak derivatives of order $\leq k$ exist.

We will prove that

**Theorem 1.2.** Let $\mathcal{D} \in \mathcal{F}(\sigma, N)$, and let $W$ be a covering of $\mathcal{D}$ satisfying the Boman chain condition. Let $1 \leq p \leq q < \infty$. If $w$ is a weight and $w$ is a doubling weight such that (1.1) holds for all $Q \in W$ and $f \in \Lambda^1(\mathcal{D})$, then

$$ \|\nabla f\|_{L^q_{w}(\mathcal{D})} \leq C w(Q_0)^{1/q} l(Q_0)^{-n} (l(Q_0))^{-1} \|f\|_{L^1(Q_0)} + l(Q_0) \|\nabla^2 f\|_{L^1(Q_0)} + C A_0 \|\nabla^2 f\|_{L^1(Q_0)} $$

for all $f \in \Lambda^2(\mathcal{D})$ where $A_0 = \sup_{Q \in W} A(Q)$, $Q_0$ is the 'central' cube in $W$, and $C$ is independent of $f$ and $\sigma$. 

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Theorem 1.3. Let $1 \leq p \leq q < \infty$. Suppose that $v$ is a weight and $w$ is a
doubling weight such that
\begin{equation}
\|f - f_Q\|_{L^q_d(Q)} \leq C_0 w(Q)^{1/q} v'(Q)^{1/p'} l(Q)^{-n+1} \|\nabla f\|_{L^p_d(Q)}
\end{equation}
for all cubes $Q$ and $f \in \Lambda^1(\mathbb{R}^n)$ where $v' = v^{-1/(p-1)}$ ($v'(Q)^{1/p'} = \text{ess sup}_{x \in Q} v^{-1}(x)$ when $p = 1$). Then
\begin{equation}
\|\nabla^k f\|_{L^q_d(Q)} \leq C w(Q)^{1/q} l(Q)^{-n-k} \|f\|_{L^1(Q)} + C w(Q)^{1/q} v'(Q)^{1/p'} l(Q)^{-n+1} \|\nabla^{k+1} f\|_{L^q_d(Q)}
\end{equation}
for all cubes $Q$ and $f \in \Lambda^{k+1}(\mathbb{R}^n)$ where $C$ is independent of $f$.

Moreover, if $\|\nabla^{k+1} f\|_{L^q_d(\mathbb{R}^n)} \neq 0$ and there exist $a < 1$, $b > (1-k)/2$,
$1 < p_0 < q$, and weight $v_0$ such that
\begin{equation}
l(Q)^{(2b-1-n)} w(Q)^{1/q} v_0(Q)^{1/p_0} + l(Q)^{(2a-1-n)} w(Q)^{1/q} v'(Q)^{1/p'} \leq C
\end{equation}
for all cubes $Q$, then
\begin{equation}
\|\nabla^k f\|_{L^q_d(\mathbb{R}^n)} \leq C \|f\|_{L^1(Q)}^{1-(2b+k-1)/(1+k+2(b-a))} \|\nabla^{k+1} f\|_{L^q_d(\mathbb{R}^n)}^{(2b+k-1)/(1+k+2(b-a))}
\end{equation}

In particular, under the assumptions stated above, we have
\begin{equation}
\|\nabla^k f\|_{L^q_d(\mathbb{R}^n)} \leq C \|f\|_{L^1(\mathbb{R}^n)}^{1-(2a+k-1)/(1+k)} \|\nabla^{k+1} f\|_{L^q_d(\mathbb{R}^n)}^{(2a+k-1)/(1+k)}
\end{equation}

These theorems have some interesting corollaries.

Corollary 1.4. Let $1 \leq p \leq q < \infty$, and let $\varrho$, $W$, $v$, and $w$ be as in Theorem
1.2 such that
\begin{equation}
\|f - f_Q\|_{L^q_d(Q)} \leq A \|\nabla f\|_{L^p_d(Q)}
\end{equation}
for all $Q \in W$ and $f \in \Lambda^1(\varrho)$. Then $E^{p}_{v,k+1}(\varrho) \subset E^{q}_{w,k}(\varrho)$ for all $k \in \mathbb{N}$.

Corollary 1.5. Let $\varrho$ and $W$ be as in Theorem 1.2. Suppose $1 \leq p \leq q < \infty$ and $w$ is a doubling weight such that
\begin{equation}
\|f - f_Q\|_{L^q_d(Q)} \leq A \|\nabla f\|_{L^p_d(Q)}
\end{equation}
for all $Q \in W$ and $f \in \Lambda^1(\varrho)$. Then $f \in E^{p}_{w,k}(\varrho)$ if and only if $f \in L^p_{w,k}(\varrho)$.

Note that when $w \in A_p$ and $\varrho$ is a bounded $(\varepsilon, \infty)$ domain, it has been
obtained in [9] that $E^{p}_{w,k}(\varrho) = L^p_{w,k}(\varrho)$.

Corollary 1.6. Let $1 \leq p \leq q < \infty$, $v \in A_p$, and $w$ be a doubling weight such
that (1.2) holds. Let $i$, $k \in \mathbb{N}$, $1 \leq i < k$, and $-i < a' < k - i$. Then
\begin{equation}
\|\nabla^i f\|_{L^q_d(\mathbb{R}^n)} \leq C \|f\|_{L^1(\mathbb{R}^n)}^{1-(a'+i)/k} \|\nabla^{k+1} f\|_{L^q_d(\mathbb{R}^n)}^{(a'+i)/k}
\end{equation}
for all $f \in \Lambda^k(\mathbb{R}^n)$ and $\|\nabla^k f\|_{L^q_d(\mathbb{R}^n)} \neq 0$ if and only if
\begin{equation}
l(Q)^{a'} w(Q)^{1/q} \leq C v(Q)^{1/p}
\end{equation}
for all cubes $Q$.

Note that when $p > 1$, $i = 1$, and $k = 2$, Corollary 1.6 is first obtained by
Gutierrez and Wheeden [18].

Finally, similar to Theorems 1.8 and 1.9 in [9], we could apply the extension
theorems in [9] to get the following two corollaries.
Corollary 1.7. Let $1 \leq p \leq q < \infty$, $v \in A_p$, and let $\mathcal{D}$ be a bounded $(\varepsilon, \infty)$ domain. Let $i, k \in \mathbb{N}$ such that $1 \leq i < k$. Let $w$ be a doubling weight such that (1.2) and (1.4) hold for $-i < a' < k - i$. If $f \in L^p_{v,k}(\mathcal{D})$ and there exists a cube $Q$ in $\mathcal{D}$ such that $f = 0$ on $Q$ and $\nabla^k f \neq 0$ a.e. on $\mathcal{D}$, then

$$\|\nabla^i f\|_{L^p_v(\mathcal{D})} \leq C \|f\|_{L^p_v(\mathcal{D})}^{(k-a'-i)/k} \|\nabla^k f\|_{L^p_v(\mathcal{D})}^{(a'+i)/k}$$

where $C$ depends only on $\mathcal{D}$, $w$, $v$, $k$, $n$, $p$, $q$, and $Q$.

Corollary 1.8. Let $\mathcal{D}$ be an unbounded $(\varepsilon, \infty)$ domain, and let $v$, $w$, $p$, $q$, $i$, and $k$ be as in the preceding corollary. If $f \in L^p_{v,k}(\mathcal{D})$ and $\nabla^k f \neq 0$ a.e. on $\mathcal{D}$, then

$$\|\nabla^i f\|_{L^p_v(\mathcal{D})} \leq C \|f\|_{L^p_v(\mathcal{D})}^{(k-a'-i)/k} \|\nabla^k f\|_{L^p_v(\mathcal{D})}^{(a'+i)/k}$$

where $C$ depends only on $\varepsilon$, $p$, $q$, $v$, $w$, $k$, and $n$.

Remark 1.9. (a) Let $\mathcal{D} \subset \mathcal{F}(\sigma, N)$ for some $\sigma, N \geq 1$ and $M \subset \partial \mathcal{D}$ (the boundary of $\mathcal{D}$). Suppose $w(x) = \text{dist}(x, M) = \inf_{y \in M} |x - y|$. Let $W$ be a covering of $\mathcal{D}$ that satisfies the chain condition. Let $\alpha \in \mathbb{R}$. Then it is clear that if $1 \leq p \leq q < \infty$, then

$$\|f - f_Q\|_{L^p_{\omega^\alpha}(Q)} \leq C_l(Q) \|\nabla f\|_{L^p_{\omega^\alpha}(Q)},$$

and indeed, when $1 - (\frac{p}{q} - \frac{n}{q}) > 0$,

$$\|f - f_Q\|_{L^p_{\omega^\alpha}(Q)} \leq C_l(Q)^{1-(n/p-n/q)} \text{dist}(Q, M)^{\alpha/q - \beta/p} \|\nabla f\|_{L^p_{\omega^\beta}(Q)}$$

for $f \in \Lambda^1(\mathbb{R}^n)$ and $Q \in W$ with $C$ depending only on $\sigma, N, n, p, \alpha, \beta,$ and $q$. These estimates can easily be obtained by the fact that $w$ is comparable to $\text{dist}(Q, M)$ on $Q$ and the unweighted Poincaré type estimate.

We can now apply Theorem 1.2 to conclude that when $w^\alpha$ is doubling,

$$\|\nabla^k f\|_{L^p_{\omega^\alpha}(\mathcal{D})} \leq C \|f\|_{L^p(\mathcal{D})} + C \|\nabla^k + 1 f\|_{L^p(\mathcal{D})} + C \|\nabla^k + 1 f\|_{L^p_{\omega^\beta}(\mathcal{D})}$$

provided $1 - (\frac{1}{p} - \frac{1}{q})n + \frac{\alpha}{q} - \frac{\beta}{p} > 0$ and $1 - (\frac{n}{p} - \frac{n}{q}) \geq 0$ with $C$ depending only on $\sigma, N, n, p, \alpha, \beta,$ and $q$. Hence, for all $k \in \mathbb{N}$, $E^p_{w^\alpha,k+1}(\mathcal{D}) \subset E^p_{w^\alpha,k}(\mathcal{D})$ for such $p, q, \alpha,$ and $\beta$. Moreover, if $p = q$ and $\alpha = \beta$, we have $E^p_{w^\alpha,k}(\mathcal{D}) = L^p_{w^\alpha,k}(\mathcal{D})$.

(b) Furthermore, if $w(x) = s(\text{dist}(x, M))$ where $s$ is a positive and continuous function on the positive real numbers that satisfies certain properties described in Kufner [21], a similar conclusion can be obtained by Theorem 1.2 if we know that $w$ is doubling.

(c) We do not know exactly when the weights $w$ defined as above are doubling. However, in the case that $M$ is just a finite subset of $\partial \mathcal{D}$, it is easy to see that $\text{dist}(x, M)^\alpha$ is doubling if and only if $\alpha > -n$.

2. Preliminaries

In what follows, $C$ denotes various positive constants. They may differ even in the same string of estimates. Moreover, sometimes, we will use $C(\alpha, \beta, \ldots)$ instead of $C$ to emphasize that the constant is depending on $\alpha, \beta, \ldots$.

Since one of our main tools will be a projection of functions into polynomials, first let us state an inequality on polynomials.
Theorem 2.1. Let \( F, Q \) be cubes such that \( F \subset Q \) and \( |F| > \gamma |Q| \). If \( w \) is a doubling weight, \( 1 \leq q < \infty \), and \( p \) is a polynomial of degree \( m \), then

\[
\|p\|_{L^q_w(E)} \leq C(\gamma, m, n, w) \left( \frac{w(E)}{w(F)} \right)^{1/q} \|p\|_{L^q_w(F)}
\]

for all measurable sets \( E \subset Q \).

This theorem is just a consequence of the following two lemmas.

Lemma 2.2 [29, Chapter 3, Lemma 7]. If \( w \) is a doubling measure and \( m \) is a positive integer, then there exists \( s_0(n, m, w) \) such that if \( s < s_0 \), then for all cubes \( Q, \lambda > 0 \) such that

\[
w(\{ x \in Q : |p(x)| > \lambda \}) \leq s w(Q)
\]

we have \( \sup_{x \in Q} |p(x)| \leq C \lambda \), where \( p \) is any polynomial of degree \( m \) and \( C \) is a constant independent of \( \lambda, Q, \) and \( p \).

It follows from Chebyshev's inequality and this lemma that given \( m \) and a polynomial \( p \) of degree \( m \),

\[
\|p\|_{L^\infty(Q)} \leq \frac{C}{w(Q)} \|p\|_{L^\infty_w(Q)}
\]

with \( C \) independent of \( Q \) and \( p \).

Lemma 2.3 [9, Theorem 2.2]. Let \( Q \) be a cube, and let \( E \) be a measurable set in \( Q \) with \( |E| > \gamma |Q| \). If \( p \) is a polynomial of degree \( m \), then

\[
\|p\|_{L^\infty(E)} \geq C(\gamma, m) \|p\|_{L^\infty(Q)}.
\]

Next, let us state Markov's inequality; see, for example, [1].

Theorem 2.4. Let \( p \) be any polynomial of order less than \( k \). Then there exists a constant \( C \) depending only on \( k \) and the dimension \( n \) such that

\[
\|\nabla p\|_{L^\infty(Q)} \leq C l(Q)^{-1} \|p\|_{L^\infty(Q)}
\]

for all cubes \( Q \) in \( \mathbb{R}^n \) with \( \sigma Q \subset \mathcal{D} \).

Finally, the following is now a consequence of Markov's inequality and Lemma 2.2.

Theorem 2.5. Let \( p \) be a polynomial of order less than \( k \) and \( 1 \leq q \leq \infty \). If \( w \) is doubling, then

\[
\|\nabla p\|_{L^q_w(Q)} \leq C l(Q)^{-1} \|p\|_{L^q_w(Q)}
\]

for all cubes \( Q \) in \( \mathbb{R}^n \), where \( C \) depends only on \( k, w, q, \) and \( n \).

Now let us state a theorem from [12].

Theorem 2.6. Let \( \sigma, N \geq 1, 1 \leq p \leq q < \infty, k \in \mathbb{N}, \) and \( \mathcal{D} \in \mathcal{F}(\sigma, N) \), and let \( f, g \) be measurable functions defined on \( \mathcal{D} \). Also, let \( v \) be a weight, and let \( w \) be a doubling weight. Suppose that for each cube \( Q \) with \( \sigma Q \subset \mathcal{D} \), there exists a polynomial \( P(f, Q) \) of degree \( k \) such that

\[
(2.1) \quad \|f - P(f, Q)\|_{L^q_w(\sigma Q)} \leq A \|g\|_{L^q_w(\sigma Q)}
\]

with \( A \) independent of \( Q \). Then there exists a polynomial \( P(f, \mathcal{D}) \) of degree \( k \) such that

\[
(2.2) \quad \|f - P(f, \mathcal{D})\|_{L^q_w(\mathcal{D})} \leq CA \|g\|_{L^q_w(\mathcal{D})}
\]
where $C$ depends only on $n, q, w, \sigma, k,$ and $N$. Moreover, we can take $P(f, Q) = P(f, Q_0)$ where $Q_0$ is the 'central' cube in $Q$.

Let $\mathcal{R}_k$ be the collection of all polynomials with degree $< k$ on $\mathbb{R}^n$. Now, let us state a theorem concerning the projection of function into polynomials.

**Theorem 2.7.** Let $\mathcal{D}$ be an open set. For each $k \in \mathbb{N}$ and cubes $Q \subset \mathcal{D}$, there exists a projection $\pi_k(Q): \Lambda^k(\mathcal{D}) \to \mathcal{R}_k$ such that

$$\text{ess sup}_{x \in Q} |\pi_k(Q)f(x)| \leq C l(Q)^{-n} \|f\|_{L^1(Q)}$$

with $C$ independent of $f$ and $Q$. Moreover, $\pi_k(Q)$ is linear and $\pi_k(Q)p = p$ for all $p \in \mathcal{R}_k$.

For the proof, please refer to [9] or [10].

Finally, let us state the weighted Poincaré inequality for $A_p$ weights. For the proof, see [9].

**Theorem 2.8.** If $1 \leq p < \infty,$ and $v \in A_p$, then

$$\|f - f_Q\|_{L^p(Q)} \leq C l(Q) \|\nabla f\|_{L^p(Q)}$$

for all cubes $Q$ and $f \in \Lambda^1(\mathbb{R}^n)$ where $C$ depends only on $p, v,$ and $n$.

### 3. Proof of main results

**Proof of Theorem 1.2.** First let us fix $f \in \Lambda^2(\mathcal{D})$ such that $|\nabla^2 f| \in L^p(\mathcal{D})$. Next, we let $P_{Q_0}f$ be the polynomial of degree 1 such that $\int_{Q_0} D^\alpha(f - P_{Q_0}f) \, dx = 0$ for all $|\alpha| \leq 1$. Now, by Theorem 2.6,

$$\|\nabla (f - P_{Q_0}f)\|_{L^q_0(\mathcal{D})} \leq C A_2 \|\nabla^2 f\|_{L^q_0(\mathcal{D})},$$

where $A_2 = \sup_{Q \in W} A(Q)$. Next, let us note that

$$\|\nabla P_{Q_0}f\|_{L^q_0(\mathcal{D})} \leq C \|\nabla P_{Q_0}f\|_{L^q_0(Q_0)} \quad \text{(by Theorem 2.1)}$$

$$\leq C \|\nabla (P_{Q_0}f - \pi_2(Q_0)f)\|_{L^q_0(Q_0)} + \|\nabla \pi_2(Q_0)f\|_{L^q_0(Q_0)}$$

$$\leq Cl(Q_0)^{-1} \|P_{Q_0}f - \pi_2(Q_0)f\|_{L^q_0(Q_0)}$$

$$+ Cl(Q_0)^{-1} \|\nabla \pi_2(Q_0)f\|_{L^q_0(Q_0)} \quad \text{(by Theorem 2.5)}$$

$$\leq Cl(Q_0)^{-1-n} w(Q_0)^{1/q} \|f - P_{Q_0}f\|_{L^1(Q_0)}$$

$$+ Cl(Q_0)^{-1-n} w(Q_0)^{1/q} \|f\|_{L^1(Q_0)} \quad \text{(by Theorem 2.7)}$$

$$\leq Cl(Q_0)^{-1-n} w(Q_0)^{1/q} \|\nabla^2 f\|_{L^1(Q_0)} + Cl(Q_0)^{-1-n} w(Q_0)^{1/q} \|f\|_{L^1(Q_0)}$$

by the nonweighted Poincaré inequality. Hence

$$\|\nabla f\|_{L^q_0(\mathcal{D})} \leq C \|\nabla P_{Q_0}f\|_{L^q_0(\mathcal{D})} + C A_2 \|\nabla^2 f\|_{L^q_0(\mathcal{D})}$$

$$\leq Cl(Q_0)^{-1-n} w(Q_0)^{1/q} \|\nabla^2 f\|_{L^1(Q_0)}$$

$$+ Cl(Q_0)^{-1-n} w(Q_0)^{1/q} \|f\|_{L^1(Q_0)} + C A_2 \|\nabla^2 f\|_{L^q_0(\mathcal{D})}.$$
Proof of Theorem 1.3. Similar to the proof as above, for any cube $Q$, we have
\[
\|\nabla^k f\|_{L^q(Q)} \leq Cw(Q)^{1/q} l(Q)^{-n-k} \|f\|_{L^1(Q)} + Cw(Q)^{1/q} l(Q)^{1-n} \|\nabla^{k+1} f\|_{L^q(Q)} + Cw(Q)^{1/q} v(Q)^{1/p} l(Q)^{-n+1} \|\nabla^{k+1} f\|_{L^q_0(Q)}
\]
by Hölder's inequality. Next, if (1.3) holds, then for all cubes $Q$ we have
\[
\|\nabla^k f\|_{L^q(Q)} \leq C l(Q)^{-2b-k+1} \|f\|_{L^q_0(Q)} + C l(Q)^{2-a} \|\nabla^{k+1} f\|_{L^q(Q)}.
\]
Now, for all $e > 0$, we will cover $\mathbb{R}^n$ by cubes of length $e$. Let $W'$ be any nonoverlapping cover of cubes with length $e$. Summing up the cubes, we have
\[
\|\nabla f\|_{L^q_0(\mathbb{R}^n)} \leq C e^{-2b-k+1} \|f\|_{L^q_0(\mathbb{R}^n)} + C e^{2-a} \sum_{Q \in W'} \|\nabla^{k+1} f\|_{L^q_0(Q)}^q.
\]
Thus
\[
\|\nabla f\|_{L^q_0(\mathbb{R}^n)} \leq C e^{-2b-k+1} \left( \sum_{Q \in W'} \|f\|_{L^q_0(Q)}^p \right)^{1/p} + C e^{2-a} \left( \sum_{Q \in W'} \|\nabla^{k+1} f\|_{L^q_0(Q)}^p \right)^{1/p}
\]
since $q \geq p_0, p$. Hence,
\[
\|\nabla f\|_{L^q_0(\mathbb{R}^n)} \leq C e^{-2b-k+1} \|f\|_{L^q_0(\mathbb{R}^n)} + C e^{2-a} \|\nabla^{k+1} f\|_{L^q_0(\mathbb{R}^n)}.
\]
To complete the proof just choose $e = \left( \|f\|_{L^q_0(\mathbb{R}^n)}/\|\nabla^{k+1} f\|_{L^q_0(\mathbb{R}^n)} \right)^{1/(1+k+2(b-a))}$.

Next, let us note that Corollaries 1.4 and 1.5 are just immediate consequences of Theorem 1.2.

Proof of Corollary 1.6. First note that by similar arguments as above, for any cube $Q$ and $f \in L^q_0(\mathbb{R}^n)$, let $P_Q f$ be the unique polynomial of degree $< k$ such that $\int_Q D^\alpha (f - P_Q f) \, dx = 0$ for all $|\alpha| < k$, we have
\[
\|\nabla^i P_Q f\|_{L^q_0(Q)} \leq C w(Q)^{1/q} l(Q)^{-i-n} \|f\|_{L^1(Q)} + C w(Q)^{1/q} l(Q)^{1-n-k+i} \|\nabla^k f\|_{L^1(Q)}
\]
\[
\leq C w(Q)^{1/q} l(Q)^{-i} v(Q)^{-1/p} \|P_Q f\|_{L^q_0(Q)} + C w(Q)^{1/q} v(Q)^{-1/p} l(Q)^{k-i} \|\nabla^k f\|_{L^q_0(Q)}
\]
since $v \in A_p$ . Also,
\[
\|\nabla^i (f - P_Q f)\|_{L^q_0(Q)} \leq C l(Q) w(Q)^{1/q} v(Q)^{-1/p} \|\nabla^{i+1} (f - P_Q f)\|_{L^q_0(Q)} \quad \text{(by (1.2))}
\]
\[
\leq C l(Q)^{-i} w(Q)^{1/q} v(Q)^{-1/p} \|\nabla^k f\|_{L^q_0(Q)}
\]
by Theorem 2.8. Hence by the triangle inequality and (1.4),
\[
\|\nabla^i f\|_{L^q_0(Q)} \leq \|\nabla^i (f - P_Q f)\|_{L^q_0(Q)} + \|\nabla^i P_Q f\|_{L^q_0(Q)}
\]
\[
\leq C l(Q)^{-a' + k - i} \|f\|_{L^q_0(Q)} + C l(Q)^{-a' + k - i} \|\nabla^k f\|_{L^q_0(Q)}.
\]
We can now just follow as before to show
\[ \|\nabla^i f\|_{L^p_{u,k}(\mathbb{R}^n)} \leq C \|f\|_{L^2_{u,k}(\mathbb{R}^n)}^{1-(d'+i)/k} \|\nabla^k f\|_{L^p_{u,k}(\mathbb{R}^n)}^{(d'+i)/k} \]
for all \( f \in L^p_{u,k}(\mathbb{R}^n) \) with \( \|\nabla^k f\|_{L^p_{u,k}(\mathbb{R}^n)} \neq 0 \). Finally, the reader can refer to the proof of Theorem 5 in Gutierrez and Wheeden [18] for the converse.

Finally, let us note that the proof of Corollaries 1.7 and 1.8 are almost identical to the proof of Theorems 1.8 and 1.9 in [9] with the help of Corollary 1.6.

References

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