ON WEIGHTED SOBOLEV INTERPOLATION INEQUALITIES

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Abstract. We obtain some weighted Sobolev interpolation inequalities on \( \mathbb{R}^n \) and domains satisfying the Boman chain condition for doubling weights satisfying a weighted Poincaré inequality.

1. Introduction

Recently, there has been a significant number of papers on weighted Sobolev interpolation inequalities, for example, Brown and Hilton [3-5], Gutierrez and Wheeden [18], and Chua [9]. In this paper, we will study weighted Sobolev interpolation inequalities with weights satisfying the following inequality:

\[
\|f - f_Q\|_{L^p(Q)} \leq A(Q)\|
\nabla f\|_{L^q(Q)}
\]

as in [18] where \( f_Q = \int_Q f \, dx / |Q| \). Let us note that some sufficient conditions have been obtained for (1.1); see [27], [26], or [7].

Definition 1.1 [19]. An open set \( \mathcal{D} \) in \( \mathbb{R}^n \) is said to be a member of \( \mathcal{F}(\sigma, N) \), \( \sigma \geq 1 \), \( N \geq 1 \), if there exists a covering \( W \) of \( \mathcal{D} \) consisting of cubes such that:

(i) \( \sum_{Q \in W} \chi_{\sigma Q}(x) \leq N \chi_{\mathcal{D}}(x) \quad \forall x \in \mathbb{R}^n \).

(ii) There is a 'central cube' \( Q_0 \in W \) that can be connected with every cube \( Q \in W \) by a finite chain of cubes \( Q_0, Q_1, \ldots, Q_k(Q) = Q \) from \( W \) such that \( Q \subset NQ_j \) for \( j = 0, 1, \ldots, k(Q) \). Moreover, \( Q_j \cap Q_{j+1} \) contains a cube \( R_j \) such that \( Q_j \cup Q_{j+1} \subset NR_j \).

We say that \( \mathcal{D} \) satisfies the Boman chain condition if \( \mathcal{D} \in \mathcal{F}(\sigma, N) \) for some \( N \), \( \sigma \geq 1 \). There are many types of domains that satisfy the Boman chain condition, for example, balls, cubes, and John domains (see [19]). Moreover, it is easy to check that bounded \( (\varepsilon, \infty) \) domains (see [20] or [9] for the definition) satisfy the Boman chain condition. Hence, do bounded Lipschitz domains. In what follows, \( Q \) is always a cube and \( l(Q) \) will be its edgelength. If \( 1 < p < \)
$p'$ will denote $p/(p-1)$. By a weight $w$, we mean a nonnegative locally integrable function on $\mathbb{R}^n$. By abusing notation, we will also write $w$ for the measure induced by $w$. Sometimes we write $dw$ to denote $w \, dx$. We say that $w$ is doubling if $w(2Q) \leq Cw(Q)$ for every cube $Q$, where $2Q$ denotes the cube with the same center as $Q$ and twice its edgelength. By $w \in A_p$, we mean $w$ satisfies the Muckenhoupt $A_p$ condition, i.e.,

$$\frac{1}{|Q|} \left( \int_Q w \, dx \right)^{1/p} \left( \int_Q w^{-1/(p-1)} \, dx \right)^{1/p'} \leq C \quad \text{when } 1 < p < \infty,$$

and

$$\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \text{ ess inf}_{x \in Q} w(x) \quad \text{when } p = 1,$$

for all cubes $Q$ in $\mathbb{R}^n$. Note that $w$ is doubling when it is in $A_p$.

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$. If $\alpha$ is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we will denote $\sum_{j=1}^n \alpha_j$ by $|\alpha|$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We denote by $\nabla$ the vector $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ and by $\nabla^m$ the vector of all possible $m$th-order derivatives for $m \in \mathbb{N}$. A locally integrable function $f$ on $\mathcal{D}$ (we will write $f \in L^1_{\text{loc}}(\mathcal{D})$) has a weak derivative of order $\alpha$ if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$\int_{\mathcal{D}} f(D^\alpha \varphi) \, dx = (-1)^{|\alpha|} \int_{\mathcal{D}} (D^\alpha f) \varphi \, dx$$

for all $C^\infty$ functions $\varphi$ with compact support in $\mathcal{D}$ (we will write $\varphi \in C^\infty(\mathcal{D})$).

For $1 \leq p < \infty$, $k \in \mathbb{N}$, and any weight $w$, $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ are the spaces of functions having weak derivatives of all orders $\alpha$, $|\alpha| \leq k$, and satisfying

$$\|f\|_{L^p_{w,k}(\mathcal{D})} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^p_w(\mathcal{D})} = \sum_{0 \leq |\alpha| \leq k} \left( \int_{\mathcal{D}} |D^\alpha f|^p \, dw \right)^{1/p} < \infty$$

and

$$\|f\|_{E^p_{w,k}(\mathcal{D})} = \sum_{|\alpha| = k} \|D^\alpha f\|_{L^p_w(\mathcal{D})} < \infty,$$

respectively. Moreover, in the case when $w \equiv 1$, we will denote $L^p_{w,k}(\mathcal{D})$ and $E^p_{w,k}(\mathcal{D})$ by $L^p_k(\mathcal{D})$ and $E^p_k(\mathcal{D})$, respectively. Finally, let $\Lambda^k(\mathcal{D})$ be the collection of all functions $f$ on $\mathcal{D}$ such that all its weak derivatives of order $\leq k$ exist.

We will prove that

**Theorem 1.2.** Let $\mathcal{D} \in \mathcal{F}(\sigma, N)$, and let $W$ be a covering of $\mathcal{D}$ satisfying the Boman chain condition. Let $1 \leq p \leq q < \infty$. If $v$ is a weight and $w$ is a doubling weight such that (1.1) holds for all $Q \in W$ and $f \in \Lambda^1(\mathcal{D})$, then

$$\|\nabla f\|_{L^q_w(\mathcal{D})} \leq C w(Q_0)^{1/q} l(Q_0)^{-n} (l(Q_0))^{-1} \|f\|_{L^1(Q_0)} + l(Q_0) \|\nabla^2 f\|_{L^1(\mathcal{D})} + C A_0 \|\nabla f\|_{L^1(\mathcal{D})}$$

for all $f \in \Lambda^2(\mathcal{D})$ where $A_0 = \sup_{Q \in W} A(Q)$, $Q_0$ is the 'central' cube in $W$, and $C$ is independent of $f$ and $v$.
Theorem 1.3. Let \( 1 \leq p \leq q < \infty \). Suppose that \( v \) is a weight and \( w \) is a doubling weight such that

\[
\| f - f_Q \|_{L^p_w(Q)} \leq C_0 w(Q)^{1/q} v'(Q)^{1/p'} l(Q)^{-n+1} \| \nabla f \|_{L^q_v(Q)}
\]

for all cubes \( Q \) and \( f \in \Lambda^1(\mathbb{R}^n) \) where \( v' = v^{-1/(p-1)} \) (\( v'(Q)^{1/p'} = \text{ess sup}_{x \in Q} v^{-1}(x) \) when \( p = 1 \)). Then

\[
\| \nabla^k f \|_{L^p_v(Q)} \leq C w(Q)^{1/q} l(Q)^{-n-k} \| f \|_{L^1_v(Q)} + C w(Q)^{1/q} v'(Q)^{1/p'} l(Q)^{-n+1} \| \nabla^{k+1} f \|_{L^q_v(Q)}
\]

for all cubes \( Q \) and \( f \in \Lambda^{k+1}(\mathbb{R}^n) \) where \( C \) is independent of \( f \).

Moreover, if \( \| \nabla^{k+1} f \|_{L^q_v(Q)} \neq 0 \) and there exist \( a < 1 \), \( b > (1 - k)/2 \), \( 1 \leq p_0 \leq q \), and weight \( v_0 \) such that

\[
l(Q)^{(2b-1-n)} w(Q)^{1/q} v_0'(Q)^{1/p_0} + l(Q)^{(2a-1-n)} w(Q)^{1/q} v'(Q)^{1/p'} \leq C
\]

for all cubes \( Q \), then

\[
\| \nabla^k f \|_{L^p_v(Q)} \leq C \| f \|_{L^p_v(Q)}^{1-(2b+k-1)/(1+k+2(b-a))} \| \nabla^{k+1} f \|_{L^q_v(Q)}^{(2b+k-1)/(1+k+2(b-a))}
\]

In particular, under the assumptions stated above, we have

\[
\| \nabla^k f \|_{L^p_v(Q)} \leq C \| f \|_{L^p_v(Q)}^{1-(2a+k-1)/(1+k)} \| \nabla^{k+1} f \|_{L^q_v(Q)}^{(2a+k-1)/(1+k)}
\]

These theorems have some interesting corollaries.

Corollary 1.4. Let \( 1 \leq p \leq q < \infty \), and let \( \mathcal{D} \), \( W \), \( v \), and \( w \) be as in Theorem 1.2 such that

\[
\| f - f_Q \|_{L^p_w(Q)} \leq A \| \nabla f \|_{L^q_v(Q)}
\]

for all \( Q \in W \) and \( f \in \Lambda^1(\mathcal{D}) \). Then \( E^p_{v,k+1}(\mathcal{D}) \subseteq E^q_{w,k}(\mathcal{D}) \) for all \( k \in \mathbb{N} \).

Corollary 1.5. Let \( \mathcal{D} \) and \( W \) be as in Theorem 1.2. Suppose \( 1 \leq p \leq q < \infty \) and \( w \) is a doubling weight such that

\[
\| f - f_Q \|_{L^p_w(Q)} \leq A \| \nabla f \|_{L^q_v(Q)}
\]

for all \( Q \in W \) and \( f \in \Lambda^1(\mathcal{D}) \). Then \( f \in E^p_{w,k}(\mathcal{D}) \) if and only if \( f \in L^p_{w,k}(\mathcal{D}) \).

Note that when \( w \in A_p \) and \( \mathcal{D} \) is a bounded \((\varepsilon, \infty)\) domain, it has been obtained in [9] that \( E^p_{w,k}(\mathcal{D}) = L^p_{w,k}(\mathcal{D}) \).

Corollary 1.6. Let \( 1 \leq p \leq q < \infty \), \( v \in A_p \), and \( w \) be a doubling weight such that (1.2) holds. Let \( i, k \in \mathbb{N}, \ 1 \leq i < k \), and \( -i < a' < k - i \). Then

\[
\| \nabla^i f \|_{L^p_v(Q)} \leq C \| f \|_{L^p_v(Q)}^{1-(a'+i)/k} \| \nabla^k f \|_{L^q_v(Q)}^{(a'+i)/k}
\]

for all \( f \in \Lambda^k(\mathbb{R}^n) \) and \( \| \nabla^k f \|_{L^q_v(Q)} \neq 0 \) if and only if

\[
l(Q)^{a'} w(Q)^{1/q} \leq C v(Q)^{1/p}
\]

for all cubes \( Q \).

Note that when \( p > 1 \), \( i = 1 \), and \( k = 2 \), Corollary 1.6 is first obtained by Gutierrez and Wheeden [18].

Finally, similar to Theorems 1.8 and 1.9 in [9], we could apply the extension theorems in [9] to get the following two corollaries.
Corollary 1.7. Let \( 1 \leq p \leq q < \infty, \ v \in A_p, \) and let \( \mathcal{D} \) be a bounded \((\varepsilon, \infty)\) domain. Let \( i, k \in \mathbb{N} \) such that \( 1 \leq i < k \). Let \( w \) be a doubling weight such that (1.2) and (1.4) hold for \( -i < a' < k - i \). If \( f \in L^p_{v,k}(\mathcal{D}) \) and there exists a cube \( Q \) in \( \mathcal{D} \) such that \( f = 0 \) on \( Q \) and \( \nabla^k f \neq 0 \) a.e. on \( \mathcal{D} \), then

\[
\|\nabla^i f\|_{L^q_w(\mathcal{D})} \leq C \|f\|_{L^p_w(\mathcal{D})}^{(k-a')/k} \|\nabla^k f\|_{L^p_w(\mathcal{D})}^{(a'+i)/k}
\]

where \( C \) depends only on \( \mathcal{D}, w, v, k, n, p, q, \) and \( Q \).

Corollary 1.8. Let \( \mathcal{D} \) be an unbounded \((\varepsilon, \infty)\) domain, and let \( v, w, p, q, i, \) and \( k \) be as in the preceding corollary. If \( f \in L^p_{v,k}(\mathcal{D}) \) and \( \nabla^k f \neq 0 \) a.e. on \( \mathcal{D} \), then

\[
\|\nabla^i f\|_{L^q_w(\mathcal{D})} \leq C \|f\|_{L^p_w(\mathcal{D})}^{(k-a')/k} \|\nabla^k f\|_{L^p_w(\mathcal{D})}^{(a'+i)/k}
\]

where \( C \) depends only on \( \varepsilon, p, q, v, w, k, \) and \( n \).

Remark 1.9. (a) Let \( \mathcal{D} \subset \overline{\mathcal{D}}(\sigma, N) \) for some \( \sigma, N \geq 1 \) and \( M \subset \partial \mathcal{D} \) (the boundary of \( \mathcal{D} \)). Suppose \( w(x) = \text{dist}(x, M) = \inf_{y \in M} |x - y| \). Let \( W \) be a covering of \( \mathcal{D} \) that satisfies the chain condition. Let \( \alpha \in \mathbb{R} \). Then it is clear that if \( 1 \leq p \leq q < \infty \), then

\[
\|f - f_Q\|_{L^q_{w^\alpha}(Q)} \leq C l(Q) \|\nabla f\|_{L^q_w(Q)},
\]

and indeed, when \( 1 - \left(\frac{n}{p} - \frac{n}{q}\right) \geq 0 \),

\[
\|f - f_Q\|_{L^q_{w^\alpha}(Q)} \leq C l(Q)^{1-(n/p-n/q)} \text{dist}(Q, M)^{\alpha/q-\beta/p} \|\nabla f\|_{L^q_w(\mathcal{D})}
\]

for \( f \in \Lambda^1(\mathbb{R}^n) \) and \( Q \in W \) with \( C \) depending only on \( \sigma, N, n, p, \alpha, \beta, \) and \( q \). These estimates can easily be obtained by the fact that \( w \) is comparable to \( \text{dist}(Q, M) \) on \( Q \) and the unweighted Poincaré type estimate.

We can now apply Theorem 1.2 to conclude that when \( w^\alpha \) is doubling,

\[
\|\nabla^k f\|_{L^q_{w^\alpha}(\mathcal{D})} \leq C \|f\|_{L^1(Q_0)} + C \|\nabla^{k+1} f\|_{L^1(Q_0)} + C \|\nabla^{k+1} f\|_{L^q_w(\mathcal{D})}
\]

provided \( 1 - \left(\frac{1}{p} - \frac{1}{q}\right)n + \frac{\alpha}{q} - \frac{\beta}{p} \geq 0 \) and \( 1 - \left(\frac{n}{p} - \frac{n}{q}\right) \geq 0 \) with \( C \) depending only on \( \sigma, N, n, p, \alpha, \beta, \) and \( q \). Hence, for all \( k \in \mathbb{N} \), \( E^p_{w^\alpha,k+1}(\mathcal{D}) \subset E^p_{w^\alpha,k}(\mathcal{D}) \) for such \( p, q, \alpha, \) and \( \beta \). Moreover, if \( p = q \) and \( \alpha = \beta \), we have \( E^p_{w^\alpha,k}(\mathcal{D}) = L^p_{w^\alpha,k}(\mathcal{D}) \).

(b) Furthermore, if \( w(x) = s(\text{dist}(x, M)) \) where \( s \) is a positive and continuous function on the positive real numbers that satisfies certain properties described in Kufner [21], a similar conclusion can be obtained by Theorem 1.2 if we know that \( w \) is doubling.

(c) We do not know exactly when the weights \( w \) defined as above are doubling. However, in the case that \( M \) is just a finite subset of \( \partial \mathcal{D} \), it is easy to see that \( \text{dist}(x, M)^\alpha \) is doubling if and only if \( \alpha > -n \).

2. Preliminaries

In what follows, \( C \) denotes various positive constants. They may differ even in the same string of estimates. Moreover, sometimes, we will use \( C(\alpha, \beta, \ldots) \) instead of \( C \) to emphasize that the constant is depending on \( \alpha, \beta, \ldots \).

Since one of our main tools will be a projection of functions into polynomials, first let us state an inequality on polynomials.
Theorem 2.1. Let $F, Q$ be cubes such that $F \subset Q$ and $|F| > \gamma|Q|$. If $w$ is a doubling weight, $1 \leq q < \infty$, and $p$ is a polynomial of degree $m$, then
\[
\|p\|_{L^q_w(F)} \leq C(\gamma, m, n, w) \left( \frac{w(E)}{w(F)} \right)^{1/q} \|p\|_{L^q_w(F)}
\]
for all measurable sets $E \subset Q$.

This theorem is just a consequence of the following two lemmas.

Lemma 2.2 [29, Chapter 3, Lemma 7]. If $w$ is a doubling measure and $m$ is a positive integer, then there exists $s_0(n, m, w)$ such that if $s < s_0$, then for all cubes $Q$, $\lambda > 0$ such that
\[
\lambda w(\{x \in Q : |p(x)| > \lambda\}) \leq s w(Q)
\]
we have $\sup_{x \in Q} |p(x)| \leq C \lambda$, where $p$ is any polynomial of degree $m$ and $C$ is a constant independent of $\lambda$, $Q$, and $p$.

It follows from Chebyshev's inequality and this lemma that given $m$ and a polynomial $p$ of degree $m$,
\[
\|p\|_{L^\infty(Q)} \leq \frac{C}{w(Q)} \|p\|_{L^1(Q)}
\]
with $C$ independent of $Q$ and $p$.

Lemma 2.3 [9, Theorem 2.2]. Let $Q$ be a cube, and let $E$ be a measurable set in $Q$ with $|E| > \gamma|Q|$. If $p$ is a polynomial of degree $m$, then
\[
\|p\|_{L^\infty(E)} \geq C(\gamma, m)\|p\|_{L^1(Q)}
\]

Next, let us state Markov's inequality; see, for example, [1].

Theorem 2.4. Let $p$ be any polynomial of order less than $k$. Then there exists a constant $C$ depending only on $k$ and the dimension $n$ such that
\[
\|\nabla p\|_{L^\infty(Q)} \leq C(\gamma, Q)^{-1}\|p\|_{L^\infty(Q)}
\]
for all cubes $Q$ in $\mathbb{R}^n$.

Finally, the following is now a consequence of Markov's inequality and Lemma 2.2.

Theorem 2.5. Let $p$ be a polynomial of order less than $k$ and $1 \leq q < \infty$. If $w$ is doubling, then
\[
\|\nabla p\|_{L^q_w(Q)} \leq C(\gamma, Q)^{-1}\|p\|_{L^q_w(Q)}
\]
for all cubes $Q$ in $\mathbb{R}^n$, where $C$ depends only on $k$, $w$, $q$, and $n$.

Now let us state a theorem from [12].

Theorem 2.6. Let $\sigma$, $N \geq 1$, $1 \leq p, q < \infty$, $k \in \mathbb{N}$, and $\mathcal{D} \in \mathcal{F}(\sigma, N)$, and let $f, g$ be measurable functions defined on $\mathcal{D}$. Also, let $\nu$ be a weight, and let $w$ be a doubling weight. Suppose that for each cube $Q$ with $\sigma Q \subset \mathcal{D}$, there exists a polynomial $P(f, Q)$ of degree $k$ such that
\[
\|f - P(f, Q)\|_{L^q_w(Q)} \leq A\|g\|_{L^1_w(\sigma Q)}
\]
with $A$ independent of $Q$. Then there exists a polynomial $P(f, \mathcal{D})$ of degree $k$ such that
\[
\|f - P(f, \mathcal{D})\|_{L^q_w(\mathcal{D})} \leq CA\|g\|_{L^1_w(\mathcal{D})}
\]
where $C$ depends only on $n, q, w, \sigma, k, \mbox{ and } N$. Moreover, we can take
$P(f, \mathcal{D}) = P(f, Q_0)$ where $Q_0$ is the 'central' cube in $\mathcal{D}$.

Let $\mathcal{P}_k$ be the collection of all polynomials with degree $< k$ on $\mathbb{R}^n$. Now,
let us state a theorem concerning the projection of function into polynomials.

**Theorem 2.7.** Let $\mathcal{D}$ be an open set. For each $k \in \mathbb{N}$ and cubes $Q \subset \mathcal{D}$, there
exists a projection $\pi_k(Q) : \Lambda^k(\mathcal{D}) \to \mathcal{P}_k$ such that

$$\text{ess sup } |\pi_k(Q)f(x)| \leq C l(Q)^{-n} \|f\|_{L^1(Q)}$$

with $C$ independent of $f$ and $Q$. Moreover, $\pi_k(Q)$ is linear and $\pi_k(Q)p = p$ for all $p \in \mathcal{P}_k$.

For the proof, please refer to [9] or [10].

Finally, let us state the weighted Poincaré inequality for $A_p$ weights. For the
proof, see [9].

**Theorem 2.8.** If $1 < p < \infty$, and $v \in A_p$, then

$$\|f - f_Q\|_{L^p(Q)} \leq C l(Q) \|\nabla f\|_{L^p(Q)}$$

for all cubes $Q$ and $f \in \Lambda^1(\mathbb{R}^n)$ where $C$ depends only on $p, v, \mbox{ and } n$.

3. Proof of main results

**Proof of Theorem 1.2.** First let us fix $f \in \Lambda^2(\mathcal{D})$ such that $|\nabla^2 f| \in L^p(\mathcal{D})$.
Next, we let $P_{Q_0} f$ be the polynomial of degree 1 such that $\int_{Q_0} D^a (f - P_{Q_0} f) \, dx = 0$ for all $|a| \leq 1$. Now, by Theorem 2.6,

$$\|\nabla (f - P_{Q_0} f)\|_{L^q_q(\mathcal{D})} \leq C A_2 \|\nabla^2 f\|_{L^q_q(\mathcal{D})},$$

where $A_2 = \sup_{Q \in \mathcal{W}} A(Q)$. Next, let us note that

$$\|\nabla P_{Q_0} f\|_{L^q_q(\mathcal{D})} \leq C \|\nabla P_{Q_0} f\|_{L^q_q(Q_0)} \quad \text{(by Theorem 2.1)}$$

$$\leq C \|\nabla (P_{Q_0} f - \pi_2(Q_0) f)\|_{L^q_q(Q_0)} + \|\nabla \pi_2(Q_0) f\|_{L^q_q(Q_0)}$$

$$\leq C l(Q_0)^{-1} \|P_{Q_0} f - \pi_2(Q_0) f\|_{L^q_q(Q_0)}$$

$$+ C l(Q_0)^{-1} \|\pi_2(Q_0) f\|_{L^q_q(Q_0)} \quad \text{(by Theorem 2.5)}$$

$$\leq C l(Q_0)^{-1} \|w(Q_0)^{1/q} \| f - P_{Q_0} f\|_{L^1(Q_0)}$$

$$+ C l(Q_0)^{-1} \|w(Q_0)^{1/q} \| f\|_{L^1(Q_0)} \quad \text{(by Theorem 2.7)}$$

$$\leq C l(Q_0)^{-1} \|w(Q_0)^{1/q} \| \nabla^2 f\|_{L^1(Q_0)} + C l(Q_0)^{-1} \|w(Q_0)^{1/q} \| f\|_{L^1(Q_0)}$$

by the nonweighted Poincaré inequality. Hence

$$\|\nabla f\|_{L^q_q(\mathcal{D})} \leq C \|\nabla P_{Q_0} f\|_{L^q_q(\mathcal{D})} + C A_2 \|\nabla^2 f\|_{L^q_q(\mathcal{D})}$$

$$\leq C l(Q_0)^{-1} \|w(Q_0)^{1/q} \| \nabla^2 f\|_{L^1(Q_0)}$$

$$+ C l(Q_0)^{-1} \|w(Q_0)^{1/q} \| f\|_{L^1(Q_0)} + C A_2 \|\nabla^2 f\|_{L^q_q(\mathcal{D})}.$$
Proof of Theorem 1.3. Similar to the proof as above, for any cube $Q$, we have
\[
\|\nabla^k f\|_{L^q_\ast(Q)} \leq Cw(Q)^{1/q} l(Q)^{-k-n} \|f\|_{L^1(Q)} + Cw(Q)^{1/q} l(Q)^{1-n} \|\nabla^{k+1} f\|_{L^q(Q)} + Cw(Q)^{1/q} v'(Q)^{1/p} l(Q)^{-n+1} \|\nabla^{k+1} f\|_{L^\infty_\ast(Q)}
\]
\[
\leq Cw(Q)^{1/q} l(Q)^{-k-n} v'(Q)^{1/p} l(Q)^{1-n} \|\nabla^{k+1} f\|_{L^q(Q)} + Cw(Q)^{1/q} v'(Q)^{1/p} l(Q)^{-n+1} \|\nabla^{k+1} f\|_{L^\infty_\ast(Q)}
\]
by Hölder's inequality. Next, if (1.3) holds, then for all cubes $Q$ we have
\[
\|\nabla^k f\|_{L^q_\ast(Q)} \leq Cl(Q)^{-b-k+1} \|f\|_{L^p_\ast(Q)} + Cl(Q)^{2-2a} \|\nabla^{k+1} f\|_{L^\infty_\ast(Q)}.
\]
Now, for all $\varepsilon > 0$, we will cover $\mathbb{R}^n$ by cubes of length $\varepsilon$. Let $W'$ be any nonoverlapping cover of cubes with length $\varepsilon$. Summing up the cubes, we have
\[
\|\nabla f\|^q_{L^q_\ast(\mathbb{R}^n)} \leq C\varepsilon^{-(b-k+1)q} \sum_{Q \in W'} \|f\|^q_{L^p_\ast(Q)} + C\varepsilon^{(2-2a)q} \sum_{Q \in W'} \|\nabla^k f\|^q_{L^q_\ast(Q)}.
\]
Thus
\[
\|\nabla^k f\|_{L^q_\ast(\mathbb{R}^n)} \leq C\varepsilon^{-b-k+1} \left( \sum_{Q \in W'} \|f\|^p_{L^p_\ast(Q)} \right)^{1/p} + C\varepsilon^{(2-2a)} \left( \sum_{Q \in W'} \|\nabla^k f\|^p_{L^q_\ast(Q)} \right)^{1/p}
\]
since $q \geq p_0, p$. Hence,
\[
\|\nabla f\|_{L^q_\ast(\mathbb{R}^n)} \leq C\varepsilon^{-b-k+1} \|f\|_{L^p_\ast(\mathbb{R}^n)} + C\varepsilon^{(2-2a)} \|\nabla^k f\|_{L^\infty_\ast(\mathbb{R}^n)}.
\]
To complete the proof just choose $\varepsilon = \left( \|f\|^p_{L^p_\ast(\mathbb{R}^n)}/\|\nabla^k f\|^p_{L^\infty_\ast(\mathbb{R}^n)} \right)^{1/(1+k+2(b-a))}$.

Next, let us note that Corollaries 1.4 and 1.5 are just immediate consequences of Theorem 1.2.

Proof of Corollary 1.6. First note that by similar arguments as above, for any cube $Q$ and $f \in L^\infty_{\ast,k}(\mathbb{R}^n)$, let $P_Q f$ be the unique polynomial of degree $< k$ such that $\int_Q D^\alpha (f - P_Q f) \, dx = 0$ for all $|\alpha| < k$, we have
\[
\|\nabla^i P_Q f\|_{L^q_\ast(Q)} \leq Cw(Q)^{1/q} l(Q)^{-i-n} \|f\|_{L^1(Q)} + Cw(Q)^{1/q} l(Q)^{1-n} \|\nabla^{k+i} f\|_{L^1(Q)}
\]
\[
\leq Cw(Q)^{1/q} l(Q)^{-i} v(Q)^{1/p} l(Q)^{-n+i} \|f\|_{L^p_\ast(Q)} + Cw(Q)^{1/q} v'(Q)^{1/p} l(Q)^{1-n} \|\nabla^k f\|_{L^\infty_\ast(Q)}
\]
since $v \in A_p$. Also,
\[
\|\nabla^i (f - P_Q f)\|_{L^q_\ast(Q)} \leq Cl(Q)^{1/q} v(Q)^{-i} \|\nabla^{i+1} (f - P_Q f)\|_{L^\infty_\ast(Q)} \leq Cl(Q)^{1/q} v(Q)^{-i} \|\nabla^k f\|_{L^\infty_\ast(Q)}
\]
by Theorem 2.8. Hence by the triangle inequality and (1.4),
\[
\|\nabla^i f\|_{L^q_\ast(Q)} \leq \|\nabla^i (f - P_Q f)\|_{L^q_\ast(Q)} + \|\nabla^i P_Q f\|_{L^q_\ast(Q)} \leq Cl(Q)^{-a'-i} \|f\|_{L^p_\ast(Q)} + Cl(Q)^{-a'+k-i} \|\nabla^k f\|_{L^\infty_\ast(Q)}.
\]
We can now just follow as before to show
$$\|\nabla^i f\|_{L^p_0(R^n)} \leq C \|f\|_{L^p_1(R^n)}^{1-(a'+i)/k} \|\nabla^k f\|_{L^p_1(R^n)}^{(a'+i)/k}$$
for all $f \in L^p_{\nu, k}(R^n)$ with $\|\nabla^k f\|_{L^p_1(R^n)} \neq 0$. Finally, the reader can refer to the proof of Theorem 5 in Gutierrez and Wheeden [18] for the converse.

Finally, let us note that the proof of Corollaries 1.7 and 1.8 are almost identical to the proof of Theorems 1.8 and 1.9 in [9] with the help of Corollary 1.6.

References


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