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FINDING A BOUNDARY FOR A MENGER MANIFOLD

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ABSTRACT. We give a characterization of k-dimensional $(k \geq 1)$ Menger manifolds admitting boundaries in the sense of Chapman and Siebenmann.

In [5] Chapman and Siebenmann considered the problem of putting a boundary on a Hilbert cube manifold (similar problems in the cases of smooth and piecewise linear manifolds were considered in [2, 14]; a parametrical version of the above problem was considered in [13]). It was proved there that if a Q-manifold M satisfies certain minimal necessary homotopy-theoretic conditions (finite type and tameness at ∞), then there are two obstructions $\sigma_\infty(M)$ and $\tau_\infty(M)$ to M having a boundary. The first one is an element of the group $\varprojlim\{\widetilde{\mathscr{K}_0}\pi_1(M - A) : A \subseteq M \text{ compact}\}$, where $\widetilde{\mathscr{K}_0}\pi_1$ is the projective class group functor. If $\sigma_\infty(M) = 0$, then the second obstruction can be defined as an element of the first derived limit of the inverse system $\varprojlim\{\mathscr{W}h\pi_1(M-A) : A \subseteq M \text{ compact}\}$, where $\mathscr{W}h\pi_1$ is the Whitehead group functor. Further, it was proved in [5] that the different boundaries that can be put on M constitute a whole shape class and that a classification of all possible ways of putting boundaries on M can be done in terms of the group $\varprojlim\{\mathscr{W}h\pi_1(M-A) : A \subseteq M \text{ compact}\}$.

In the present paper we carry out a similar program for the problem of putting boundaries on μ^{n+1}-manifolds, where μ^{n+1} denotes the $(n+1)$-dimensional universal Menger compactum (a μ^{n+1}-manifold M admits a boundary if there exists a compact μ^{n+1}-manifold N such that $M = N - Z$, where Z is a Z-set in N; in this case we shall say that N is a compactification of M corresponding to the boundary Z, and conversely, Z is a boundary of M corresponding to the compactification N. We recall also that a closed subset Z of a space X is said to be a Z-set if for each open cover $\mathscr{U} \in \text{cov}(X)$ there is a map $f: X \to X - Z$ $\mathscr{U}$-close to the identity map of X). Having in mind a deep analogy between the theories of μ^{n+1}-manifolds and Q-manifolds [1, 6–9] it is not surprising that the corresponding results are valid in the case of μ^{n+1}-manifolds as well. However, it should be observed that the situation in the last case is much simpler. For example, the analogies of the above-described

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obstructions always vanish (Theorem 2.7). Moreover, we shall see that the different boundaries that can be put on a μ^{n+1}-manifold constitute a whole n-shape class (Proposition 3.1), but, at the same time, every two compactifications of a μ^{n+1}-manifold are equivalent in the sense of Chapman and Siebenmann (Proposition 3.2). Apparently one of the reasons for these differences is that the natural analogue of Wall's finiteness obstruction [15] vanishes in the n-homotopy category. In the arguments below, this last fact plays the key role although its proof is quite elementary and does not use anything except standard definitions and techniques from [15]. I am absolutely sure that this fact is well known to experts, but unfortunately I could not find any mention of it in the literature. For this reason the proof of this result (Proposition 1.2) is presented in §1.

1. n-HOMOTOPY DOMINATION AND n-TAMENESS AT ∞

Throughout the paper only locally compact metrizable spaces and continuous maps are considered. The letter n denotes an arbitrary (but fixed) nonnegative integer. For information concerning k-dimensional Menger manifolds (briefly, μ^k-manifolds) and k-shapes, see [1] and [7], respectively.

Two maps $f, g\colon X \to Y$ are said [5, 9] to be (properly) n-homotopic (notation: $f \overset{n}{\simeq} g$ and $f \underset{p}{\overset{n}{\simeq}} g$ respectively) if for any (proper) map $h\colon Z \to X$ of any at most n-dimensional space Z into X the compostions fh and gh are (properly) homotopic in the usual sense. For the maps between at most $(n+1)$-dimensional LC^n-spaces the concept of (proper) n-homotopy coincides [6, Proposition 2.3] with the concept of (proper) μ-homotopy introduced in [1]. It should be emphasized especially that as of 1941 Fox had a protoype of the notion of n-homotopy [11]. A (proper) map $f\colon X \to Y$ is called (proper) n-homotopy equivalence [16, 9] if it has a (proper) n-homotopy inverse, i.e., a (proper) map $g\colon Y \to X$ such that $gf \overset{n}{\simeq} \mathrm{id}_X$ and $fg \overset{n}{\simeq} \mathrm{id}_Y$ (respectively, $gf \underset{p}{\overset{n}{\simeq}} \mathrm{id}_X$ and $fg \underset{p}{\overset{n}{\simeq}} \mathrm{id}_Y$). In this case we shall say that X and Y are (properly) n-homotopy equivalent. If only the first relation is satisfied then we shall say that X is n-homotopy dominated by Y. Any proper UV^n-map between locally compact LC^n-spaces with at most $(n+1)$-dimensional range can serve as an example of proper n-homotopy equivalence. We recall that X is a UV^n-compactum iff X has a trivial n-shape. For information concerning UV^n-maps see [1] and [12].

Clearly every connected polyhedron is 0-homotopy equivalent to the one-point space. Consequently any map between connected polyhedra is a 0-homotopy equivalence. This simple observation together with the corresponding result of Whitehead [16, Theorem 2] give the following algebraic characterization of n-homotopy equivalences.

Proposition 1.1. *A map $f\colon X \to Y$ between at most $(n+1)$-dimensional locally finite polyhedra is an n-homotopy equivalence iff it induces isomorphisms of homotopy groups of dimension $\leq n$, i.e., f induces a bijection between the components of X and Y and a homomorphism $\pi_k(f')\colon \pi_k(C_X) \to \pi_k(C_Y)$ is an isomorphism for each $k \leq n$ and each pair of components $C_X \subseteq X$ and $C_Y \subseteq Y$ with $f(C_X) \subseteq C_Y$, where $f'\colon C_X \to C_Y$ denotes the restriction of f.*

Proposition 1.2. *Let M be at most $(n+1)$-dimensional locally finite polyhedron. Suppose that there exists at most $(n+1)$-dimensional finite polyhedron K and two maps $f: M \to K$ and $g: K \to M$ such that $gf \overset{n}{\simeq} \mathrm{id}_M$. Then there exists at most $(n+1)$-dimensional finite polyhedron T, containing K as a subpolyhedron, and an n-homotopy equivalence $h: T \to M$, extending g such that f is an n-homotopy inverse of h.*

Proof. It suffices to consider only connected polyhedra. Consequently, the case $n = 0$ is trivial. If $n = 1$, then, by the assumptions, $\pi_1(g): \pi_1(K) \to \pi_1(M)$ is an epimorphism and $\mathrm{Ker}(\pi_1(g))$ is a finitely generated group. Select finitely many generators of $\mathrm{Ker}(\pi_1(g))$, and use them to attach 2-cells to K and to extend g over these cells. In this way we obtain a 2-dimensional finite polyhedron T, containing K as a subpolyhedron, and a map $h: T \to M$, extending g, which induces an isomorphism of fundamental groups. By Proposition 1.1, h is a 1-homotopy equivalence.

Assume, by induction, that the proposition is already proved in the cases $n \leq m$, $m \geq 1$, and consider the case $n = m + 1$. Without loss of generality we can suppose that $f(M^i) \subseteq K^i$ and $g(K^i) \subseteq M^i$ for each $i \leq m+1$. Since $gf \overset{m+1}{\simeq} \mathrm{id}_M$ it follows easily that $gf/M^{m+1} \overset{m}{\simeq} \mathrm{id}_{M^{m+1}}$. By the inductive hypothesis, there are an $(m+1)$-dimensional finite polyhedron R, containing K^{m+1} as a subpolyhedron, and an m-homotopy equivalence $r: R \to M^{m+1}$, extending g/K^{m+1}. Sewing together the polyhedra K and R along naturally embedded copies of K^{m+1} we obtain the $(m+2)$-dimensional finite polyhedron L, containing K and R as subpolyhedra, and the map $s: L \to M$ which coincides with g on K and with r on R, whence $sf = gf \overset{m+1}{\simeq} \mathrm{id}_M$ and $fs/L^{m+1} = fr \overset{m}{\simeq} \mathrm{id}_{L^{m+1}}$. By these conditions, we conclude that $\pi_i(s): \pi_i(L) \to \pi_i(M)$ is an isomorphism for each $i \leq m$ and an epimorphism for $i = m+1$. One can easily verify that in this situation $\mathrm{Ker}(\pi_{m+1}(s))$ is a finitely generated $\mathbb{Z}(\pi_1(L))$-module. Select $\mathbb{Z}(\pi_1(L))$-generators for $\mathrm{Ker}(\pi_{m+1}(s))$ and use them to attach $(m+2)$-cells to L and to extend s over these cells. Let T denote the resulting $(m+2)$-dimensional finite polyhedron, containing L as a subpolyhedron, and $h: T \to M$ the corresponding extension of s. Then $\pi_i(h)$ is an isomorphism for each $i \leq m+1$. Again, by Proposition 1.1, h is an $(m+1)$-homotopy equivalence. This performs the inductive step and finishes the proof.

Corollary 1.3. *If a μ^{n+1}-manifold M is n-homotopy dominated by at most $(n+1)$-dimensional LC^n-compactum, then M is n-homotopy equivalent to a compact μ^{n+1}-manifold.*

Proof. By the triangulation theorem for μ^{n+1}-manifolds [7], there exists a proper UV^n-retraction, $r: M \to P$ onto some locally finite polyhedron. Note that, by [8, Proposition 1.4] r is an n-homotopy equivalence. Let X be at most $(n+1)$-dimensional LC^n-compactum which n-homotopy dominates M. By [8, Proposition 1.5], X is n-homotopy equivalent to an at most $(n+1)$-dimensional finite polyhedron L. Consequently, L n-homotopy dominates P. By Proposition 1.2, there is an $(n+1)$-dimensional finite polyhedron T (containing L) n-homotopy equivalent to P. Consider now a UV^n-surjection $f: N \to T$ of some compact μ^{n+1}-manifold N onto T [10]. It only remains to note that M and N are n-homotopy equivalent. The proof is finished.

The following concept is an analogue of the well-known notion of tameness at ∞.

Definition 1.4. A space X is said to be n-tame at ∞ if for each compactum $A \subseteq X$ there exists a larger compactum $B \subseteq X$ such that the inclusion $X - B \to X - A$ factors up to n-homotopy through an at most $(n+1)$-dimensional finite polyhedron.

Proposition 1.5. *If a μ^{n+1}-manifold is n-tame at ∞, then it is n-homotopy equivalent to a compact μ^{n+1}-manifold.*

Proof. Fix a proper UV^n-retraction $r\colon M \to P$ of a given μ^{n+1}-manifold M onto some $(n+1)$-dimensional polyhedron P [7]. It follows from the well-known properties of proper UV^n-maps [12] that P is n-tame at ∞ as well. Using [6, Proposition 2.2] instead of the usual homotopy extension theorem and repeating the proof of Lemma 5.1 from [5] we can conclude that P is n-homotopy dominated by an at most $(n+1)$-dimensional finite polyhedron. Corollay 1.3 finishes the proof.

2. The main result

Let us recall that a Q-manifold M lying in a larger Q-manifold N is said to be clean in N [5] if M is closed in N and the topological frontier of M in N is collared both in M and $N - \operatorname{Int} M$. By obvious dimensional reasons we cannot define directly the corresponding notion in the case of μ^{n+1}-manifolds. Nevertheless, the following notion will be useful for us.

Definition 2.1. A μ^{n+1}-manifold M lying in a μ^{n+1}-manifold N is said to be n-clean (in N) provided that M is closed in N and there exists a closed subspace $\delta(M)$ of M such that the following conditions are satisfied:

(i) $\delta(M)$ is a μ^{n+1}-manifold;
(ii) $(N - M) \cup \delta(M)$ is a μ^{n+1}-manifold;
(iii) $\delta(M)$ is a Z-set in M;
(iv) $\delta(M)$ is a Z-set in $(N - M) \cup \delta(M)$; and
(v) $M - \delta(M)$ is open in N.

Let us indicate the standard situation when n-clean submanifolds arise naturally. Suppose that L is a submanifold of a combinatorial PL-manifold P. Fix a UV^n-map $f\colon N \to P$ of some μ^{n+1}-manifold N onto P constructed in [10, Theorem 1.3] (see also [9, Theorem 1.6]). Using the properties of f it is easy to see that $M = f^{-1}(L)$ is an n-clean submanifold of N with $\delta(M) = f^{-1}(\partial L)$. Generally speaking f is not an open map and consequently $\delta(M)$ does not necessarily coincide with the topological frontier of M in N.

Lemma 2.2. *Let N be a μ^{n+1}-manifold which is n-tame at ∞. Suppose that M is a compact and n-clean submanifold of N. Then the μ^{n+1}-manifold $(N - M) \cup \delta(M)$ is n-homotopy equivalent to a compact μ^{n+1}-manifold.*

Proof. By Proposition 1.5, it suffices to show that a μ^{n+1}-manifold $(N - M) \cup \delta(M)$ is n-tame at ∞. Let A be a compact subspace of $(N - M) \cup \delta(M)$. Clearly, $K_1 = A \cup M$ is compact. Since N is n-tame at ∞, there exists a compactum K_2 such that $K_1 \subseteq K_2 \subseteq N$ and the inclusion $N - K_2 \to N - K_1$ factors up to n-homotopy through an at most $(n+1)$-dimensional finite polyhedron. Let $B = ((N-M)\cup\delta(M))\cap K_2$. Clearly B is compact and $A \subseteq B$. Note that $((N-M)\cup\delta(M))-B = N-K_2$ and $N-K_1 \subseteq ((N-M)\cup\delta(M))-A$. Consequently, the inclusion $((N-M)\cup\delta(M))-B \to ((N-M)\cup\delta(M))-A$ factors up to n-homotopy through an at most $(n+1)$-dimensional finite polyhedron. Hence, $(N-M)\cup\delta(M)$ is n-tame at ∞. The proof is finished.

Lemma 2.3. *Each μ^{n+1}-manifold M can be written as a union $M = \bigcup(M_i : i \in \omega)$ such that all M_i's are compact and n-clean and $M_i \subseteq M_{i+1} - \delta(M_{i+1})$, $i \in \omega$.*

Proof. It suffices to show that for each compactum $K \subseteq M$ there exists a compact and n-clean $M_1 \subseteq M$ such that $K \subseteq M_1 - \delta(M_1)$. As in [7] fix a proper UV^n-surjection $g: M \to X$, where X is a Q-manifold. By [5] there exists a compact and clean $Y \subseteq X$ such that $g(K) \subseteq \mathrm{Int}_X(Y)$. By the relative triangulation theorem for Q-manifolds [4, Lemma 37.1], there exists a polyhedron P which can be written as a union of two subpolyhedra P_1 and P_2 such that $X = P\times Q$, $Y = P_1\times Q$, $X-\mathrm{Int}_X(Y) = P_2\times Q$, and $\mathrm{Bd}_X(Y) = (P_1\cap P_2)\times Q$. Note also that the subpolyhedron $P_1\cap P_2$ is a Z-set both in P_1 and P_2.

By [9–10], there exists a proper UV^n-surjection $f: N \to P$ of some μ^{n+1}-manifold N onto P satisfying the following two conditions:

(a) If L is a subpolyhedron of P then $f^{-1}(L)$ is a μ^{n+1}-manifold,

(b) If L is a subpolyhedron of P and A is a Z-set in L then $f^{-1}(Z)$ is a Z-set in $f^{-1}(L)$.

Consequently we have two proper UV^n-surjections $f: N \to P$ and $\pi_P g: M \to P$ ($\pi_P: P\times Q \to P$ denotes the natural projection) of two μ^{n+1}-manifolds onto the polyhedron P. Consider an open cover $\mathcal{U} = \{P-\pi_P g(K),\ \mathrm{Int}_P(P_1)\}$ of P. By [1, Remark 5.1.1], there exists a homeomorphism $h: M \to N$ such that the compositions $\pi_P g$ and fh are $\mathcal{U}$-close. Let $M_1 = h^{-1}f^{-1}(P_1)$ and $\delta(M_1) = h^{-1}f^{-1}(P_1\cap P_2)$. By the properties of the map f, M_1 is compact and n-clean. It only remains to note that $K \subseteq M_1 - \delta(M_1)$. This finishes the proof.

The following proposition is a direct consequence of the chacterization theorem for μ^{n+1}-manifolds [1].

Proposition 2.4. *Let a space M be a union of two closed subspaces M_1 and M_2. If M_1, M_2, and $M_0 = M_1\cap M_2$ are μ^{n+1}-manifolds and M_0 is a Z-set both in M_1 and M_2, then M is a μ^{n+1}-manifold.*

Proof. By [1], it suffices to show that for any map $f: X \to M$ of any at most $(n+1)$-dimensional compactum X into M and any open cover $\mathcal{U} \in \mathrm{cov}(M)$ there exists an embedding $g: X \to M$ $\mathcal{U}$-close to f. Let us consider the case when $f(X)\cap M_i \neq \varnothing$ for each $i = 0, 1, 2$. All other possible cases are trivial. By [6, Proposition 2.1], there exists an open cover $\mathcal{V} \in \mathrm{cov}(M)$ refining $\mathcal{U}$ such that the following condition holds:

$(*)_n$ for any at most $(n+1)$-dimensional compactum B, its closed subspace A, and any two $\mathcal{V}$-close maps $\alpha_1, \alpha_2: A \to M$ such that α_1 has an extension $\beta_1: B \to M$, it follows that α_2 also has an extension $\beta_2: B \to M$ which is $\mathcal{U}$-close to β_1.

Let $X_i = f^{-1}(M_i)$, $i = 0, 1, 2$. Since M_0 is a μ^{n+1}-manifold, there is a Z-embedding $g_0: X_0 \to M_0$ such that g_0 and f/X_0 are $\mathcal{V}$-close. By $(*)_n$, there is an extension $h: X \to M$ of g_0 such that h and f are $\mathcal{V}$-close. Since

M_0 is a Z-set both in M_1 and M_2 we conclude that $g_0(X_0)$ is a Z-set both in M_1 and M_2. Consequently, by [1, Chapter 6, the Z-set Approximation Theorem] for each $i = 1, 2$, there is a Z-embedding $g_i\colon X_i \to M_i$ such that $g_i/X_0 = g_0$ and g_i is $\mathscr{U}$-close to h/X_i. At the same time without loss of generality we can assume that one of these maps, say g_1, has the following property: $g_1(X_1 - X_0) \cap M_0 = \varnothing$ (we once again use the fact that M_0 is a Z-set in M_1). Then the map g coinciding with g_i on X_i $(i = 1, 2)$ is an embedding. It only remains to note that g and f are $\mathscr{U}$-close. The proof is finished.

Lemma 2.5. *If a μ^{n+1}-manifold M is n-tame at ∞, then we can write $M = \bigcup(M_i : i \in \omega)$ such that all M_i's are compact and n-clean, $M_i \subseteq M_{i+1} - \delta(M_{i+1})$ and the inclusion $\delta(M_i) \to (M_{i+1} - M_i) \cup \delta(M_i)$ is n-homotopy equivalence for each $i \in \omega$.*

Proof. Choose any compact and n-clean submanifold A of M. By Lemma 2.3, it will suffice to find a compact and n-clean submanifold B of M such that $A \subseteq B - \delta(B)$ and the inclusion $\delta(B) \to (M - B) \cup \delta(B)$ is an n-homotopy equivalence. By Lemma 2.2, the μ^{n+1}-manifold $(M-A)\cup\delta(A)$ is n-homotopy equivalent to some compact μ^{n+1}-manifold X. Fix the corresponding n-homotopy equivalence $\psi_1\colon (M - A) \cup \delta(A) \to X$ and its n-homotopy inverse $\varphi_1\colon X \to (M - A) \cup \delta(A)$. By [1, Theorem 2.3.8] and [6, Lemma 2.1 and Proposition 2.1], there is a map $\psi_2\colon (M - A) \cup \delta(A) \to X$ such that $\psi_2/\delta(A)\colon \delta(A) \to X$ is a Z-embedding and ψ_2 is as close to ψ_1 as we wish. Similarly, there is a Z-embedding $\varphi_2\colon X \to (M-A)\cup\delta(A)$ which is as close to φ_1 as we wish. Particularly, we can assume that ψ_2 and φ_2 are n-homotopy equivalences. If ψ_2 and φ_2 were chosen sufficiently close to ψ_1 and φ_1, respectively, then, by the Z-set unknotting theorem [1], there exists a homeomorphism $h\colon (M - A) \cup \delta(A) \to (M - A) \cup \delta(A)$ which extends the homeomorphism $\varphi_2\psi_2/\delta(A)\colon \delta(A) \to \varphi_2\psi_2(\delta(A))$ and which is sufficiently close to the identity map of $(M - A) \cup \delta(A)$. Particularly, we can assume that h is n-homotopic to $\mathrm{id}_{(M-A)\cup\delta(A)}$. Then an n-homotopy equivalence $\varphi = h^{-1}\varphi_2\colon X \to (M-A)\cup\delta(A)$, is a Z-embedding and $\delta(A) \subseteq \varphi(X) \equiv Y$. Since Y is a compact μ^{n+1}-manifold there exists [7] a UV^n-retraction $s\colon Y \to K$ onto a finite $(n+1)$-dimensional polyhedron K. Similarly fix a proper UV^n-retraction $r\colon (M-A)\cup\delta(A) \to T$, where T is a polyhedron. Let $i\colon Y \to (M-A)\cup\delta(A)$ denote the inclusion map and $j\colon K \to Y$ be a section of s (i.e., $sj = \mathrm{id}_K$). Note that i is an n-homotopy equivalence. Let $p\colon K \to T$ be a PL-map homotopic to the compostion rij. Form the mapping cylinder $M(p) \equiv P$ of the map p. Let us recall [4] that this is a space formed from the disjoint union $(K\times[0, 1])\oplus T$, by identifying $(k, 1)$ with $p(k)$, $k \in K$. At the same time we identify K with $K\times\{0\}$. Since p is a PL-map, $K\times\{0\}$ and T are subpolyhedra of the polyhedron P. Let $c\colon P \to T$ be the collapse to the base, i.e., the natural retraction defined by sending (k, t) to $p(k)$ for each $(k, t) \in K \times [0, 1]$. Obviously c is a proper CE-map that is a proper homotopy equivalence. Fix a proper UV^n-surjection $f\colon N \to P$ of some μ^{n+1}-manifold onto P satisyfying the conditions (a) and (b) in the proof of Lemma 2.3. Compact μ^{n+1}-manifolds Y and $N_1 = f^{-1}(K\times\{0\})$ admit UV^n-surjections $s\colon Y \to K\times\{0\}$ and $f/N_1\colon N_1 \to K \times \{0\}$ onto the same polyhedron. Consequently, by [1], there exists a homeomorphism $g_1\colon Y \to N_1$ such that $fg_1 \overset{n}{\simeq} s$. Similarly, we have two proper UV^n-surjections $r\colon (M-A)\cup\delta(A)\to T$ and $cf\colon N\to T$. As above there is a homeomorphism $g_2\colon (M-A)\cup\delta(A)\to N$ such that $cfg_2 \overset{n}{\underset{p}{\simeq}} r$. By the construction and the corresponding properties of UV^n-surjections [12], we have $cfg_1 \overset{n}{\simeq} cs = ps \simeq rijs \overset{n}{\simeq} ri \overset{n}{\simeq} cfg_2 i$. Since cf is a proper n-homotopy equivalence, we conclude that $g_1\colon Y\to N$ and $g_2/Y\colon Y\to N$ are n-homotopic. Consider the homeomorphism $\alpha = g_1g_2^{-1}/g_2(Y)\colon g_2(Y)\to N_1$. Clearly, $\alpha \overset{n}{\simeq} g_2g_2^{-1}/g_2(Y) = \mathrm{id}_{g_2(Y)}$. By the properties of the map f, N_1 is a Z-set in N. Note also that, by our construction, $g_2(Y)$ is a Z-set in N as well. Using the Z-set unknotting theorem [1] we can find a homeomorphism $G\colon N\to N$ extending α. Let $H = Gg_2$. Note that $H(Y) = Gg_2(Y) = \alpha g_2(Y) = N_1$. Finally, let $B = A\cup H^{-1}(f^{-1}(K\times[0, 2^{-1}]))$ and $\delta(B) = H^{-1}(f^{-1}(K\times\{2^{-1}\}))$. It follows from the properties of the map f and Proposition 2.4 that B is a compact and n-clean submanifold of M, $A\subseteq B-\delta(B)$ and the inclusion $\delta(B)\to(M-B)\cup\delta(B)$ is an n-homotopy equivalence (note that the map p and consequently the inclusion $K\times\{2^{-1}\}\to P-(K\times[0, 2^{-1}])$ are n-homotopy equivalences). This finishes the proof.

Lemma 2.6. *Let a μ^{n+1}-manifold M be a Z-set in a compact μ^{n+1}-manifold N. If the inclusion $i\colon M\to N$ is an n-homotopy equivalence, then there exists a UV^n-surjection of N onto M.*

Proof. Let $j\colon N\to M$ be an n-homotopy inverse of i. By [1, Theorem 2.8.6], there is a homeomorphism $h\colon N\to M$ such that $h\overset{n}{\simeq} j$. Then $hi \overset{n}{\simeq} ji \overset{n}{\simeq} \mathrm{id}_M$. Consequently, by [1, Proposition 5.1.2], there is a UV^n-retraction $r\colon N\to M$ such that $ri = \mathrm{id}_M$. The proof is finished.

The following theorem is the main result of this paper and gives a characterization of μ^{n+1}-manifolds with boundaries.

Theorem 2.7. *A μ^{n+1}-manifold admits a boundary iff it is n-tame at ∞.*

Proof. Let M be a μ^{n+1}-manifold which is n-tame at ∞. By Lemma 2.5, we can represent M as a union $M = \bigcup\{M_i : i\in\omega\}$ such that all the M_i's are compact and n-clean, $M_i\subseteq M_{i+1}-\delta(M_{i+1})$ and the inclusion $\delta(M_i)\to(M_{i+1}-M_i)\cup\delta(M_i)$ is n-homotopy equivalent for each $i\in\omega$. By Lemma 2.6, for each $i\in\omega$ there exists a UV^n-retraction $f_i\colon (M_{i+1}-M_i)\cup\delta(M_i)\to\delta(M_i)$. Let a UV^n-retraction $r_i\colon M_{i+1}\to M_i$ coincide with f_i on $M_{i+1}-M_i$ and with identity on M_i, $i\in\omega$. Then we have an inverse sequence $S = \{M_i, r_i\}$ consisting of compact μ^{n+1}-manifolds and UV^n-retractions. By [1, Corollary 4.3.2], r_i is a near-homeomorphism for each $i\in\omega$. By [3, Theorem 4], each limit projection of the spectrum S is a near-homeomorphism as well. Consequently, $N = \varprojlim S$, being homeomorphic to M_0, is a compact μ^{n+1}-manifold. Since $\delta(M_i)$ is a Z-set in M_i for each $i\in\omega$, we conclude that the subset $Z = \varprojlim\{\delta(M_{i+1}),\ r_i/\delta(M_{i+1})\}$ is a Z-set in N. It only remains to note that $N-Z$ is naturally homeomorphic to M.

Conversely, suppose that a μ^{n+1}-manifold M admits a boundary. This means that there are a compact μ^{n+1}-manifold N and a Z-set Z in N such that $M = N-Z$. Let us show that M is n-tame at ∞. Let A be a compact subspace of M. By the proof of Lemma 2.3, there exists a compact and n-clean submanifold B of M such that $A\subseteq B-\delta(B)$. It suffices to show that

$(M - B) \cup \delta(B)$ is n-homotopy equivalent to an at most $(n+1)$-dimensional finite poyhedron. Indeed, it is easy to see that $(M - B) \cup \delta(B)$ is n-homotopy equivalent to a compact μ^{n+1}-manifold $(N - B) \cup \delta(B)$. It only remains to note that, by [8, Proposition 1.5], each compact μ^{n+1}-manifold is n-homotopy equivalent to an at most $(n+1)$-dimensional finite polyhedron. The proof is finished.

In [5] Whitehead's example of contractible open subspace W of R^3 (which is not tame at ∞) was used to construct a Q-manifold without boundary. The same example can be used in our case as well. Indeed, consider any μ^4-manifold M admitting a proper UV^3-retraction onto W. Then M is not 3-tame at ∞ and, by Theorem 2.7, does not admit a boundary.

Theorem 2.7 can be used in somewhat different direction as well. Consider the problem of topological characterization of the space $\mu^{n+1} - \{\text{pt}\}$. This space is stable in the sense of [9] and there are many other reasons indicating that it deserves special attention. Of course, $\mu^{n+1} - \{\text{pt}\}$ is a μ^{n+1}-manifold, and hence, in the light of Bestvina's results [1], we have to characterize this space only among μ^{n+1}-manifolds.

Corollary 2.8. *Let X be a μ^{n+1}-manifold satisfying the following conditions*:

(i) *X is n-tame at ∞*;
(ii) *X is LC^n at ∞; and*
(iii) *$X \in C^n$.*

Then X is homeomorphic to $\mu^{n+1} - \{\text{pt}\}$.

Proof. By Theorem 2.7 and (i) we can write $X = N - Z$, where N is a compact μ^{n+1}-manifold and Z is a Z-set in N. By (iii) and [1], N is a copy of μ^{n+1}. By (ii), Z is a UV^n-compactum. But UV^n-compacta have trivial n-shape [6]. Consequently, by [6], $X = \mu^{n+1} - Z \approx \mu^{n+1} - \{\text{pt}\}$.

3. Boundaries and compactifications

In this section we present two propositions which give classifications of boundaries and compactifications of μ^{n+1}-manifolds.

Proposition 3.1. *If a compactum X is a boundary for a μ^{n+1}-manifold M, then a compactum Y is also a boundary for M iff* $\dim Y \leq n+1$ *and* $n - Sh(Y) = n - Sh(X)$.

Proof. Let X and Y be boundaries for M. We wish to show that $n - Sh(Y) = n - Sh(X)$. Let $N = M \cup X$ and $T = M \cup Y$ be corresponding compactifcations of M that are μ^{n+1}-manifolds. Fix a Z-embedding $f\colon N \to N$ such that $f \overset{n}{\simeq} \text{id}_N$ and $f(N) \cap X = \varnothing$. Embed $f(N)$ in μ^{n+1} as a Z-set (see [1]). Identifying $f(N)$ with the copy of $f(N)$ in μ^{n+1} we obtain a compactum $N_1 = N \cup \mu^{n+1}$. By Proposition 2.4, N_1 is a μ^{n+1}-manifold. It is easy to verify that in our situation $N_1 \in C^n$ and, hence, by [1], N_1 is homeomorphic to μ^{n+1}.

By our assumption, there is a homeomorphism $h\colon N - X \to T - Y$. Embed $h(f(N))$ into μ^{n+1} as a Z-set and identify $h(f(N))$ with the copy of $h(f(N))$ in μ^{n+1}. As above a μ^{n+1}-manifold $T_1 = T \cup \mu^{n+1}$ is homeomorphic to μ^{n+1}. Evidently there is a homeomorphism $H\colon N_1 \to T_1$ such that $H(N_1 - X) = T_1 - Y$. Consequently, by the main result of [6], $n - Sh(Y) = n - Sh(X)$.

Conversely, let X be a boundary for M and Y be an at most $(n+1)$-dimensional compactum such that $n-Sh(Y)=n-Sh(X)$. Let $N=M\cup X$ be a compactification of M corresponding to X. Form $N_1 = N\cup\mu^{n+1}$ as above. Since $\dim Y\leq n+1$, we can assume that Y is a Z-set in N_1. Since $n-Sh(Y)=n-Sh(X)$, it follows from [6] that there is a homeomorphism $h\colon N_1-X\to N_1-Y$. Let us show that $T=h(N-X)\cup Y$ is a compact μ^{n+1}-manifold and Y is a Z-set in T. Since N is an LC^n-compactum, so is $h(f(N))$. Hence, there is a retraction $s\colon S\to h(f(N))$, where S is an open neighbourhood of $h(f(N))$ in $h(\mu^{n+1})$. Let $G=S\cup T$ and the map $r\colon G\to T$ coincide with s on S with id_T on T. Clearly G is an open neighbourhood of T in N_1 and r is a retraction. This shows that T is an LC^n-compactum. In order to prove that Y is a Z-set in T, fix an arbitrary open cover $\mathscr{U}\in\mathrm{cov}(T)$ and let $\mathscr{V}=\{r^{-1}(U) : U\in\mathscr{U}\}\cup\{N_1-T\}$. Clearly $\mathscr{V}$ is an open cover of N_1. Since Y is a Z-set in N_1, there is a map $g\colon N_1\to N_1$ such that $g(N_1)\cap Y=\varnothing$ and which is $\mathscr{V}$-close to the identy map of N_1. Note that $g(T)\subseteq G$ and consequently the composition $rg/T\colon T\to T$ is well defined. An easy verification shows that rg/T is $\mathscr{U}$-close to id_T and $rg(T)\cap Y=\varnothing$. Thus, Y is a Z-set in T. Finally note that T is an $(n+1)$-dimensional LC^n-compactum and contains a Z-set Y complement $T-Y=h(N-X)$ of which is a μ^{n+1}-manifold. This allows us to conclude that T is a μ^{n+1}-manifold itself. The proof is finished.

Let us recall [5] that two compactifications N and T of the same space M are said to be equivalent if for every compactum $A\subseteq M$ there is a homeomorphism of N onto T fixing A pointwise.

Of course, if μ^{n+1}-manifolds N and T are compactifications of a μ^{n+1}-manifold M, then the inclusions $M\to N$ and $M\to T$ are n-homotopy equivalences (because, $N-M$ and $T-M$ are Z-sets in N and T, respectively). Consequently, N and T are homeomorphic as n-homotopy equivalent compact μ^{n+1}-manifolds [1]. We show now that N and T are equivalent even in the sense of Chapman-Siebenmann.

Proposition 3.2. *Every two compactifications of a μ^{n+1}-manifold are equivalent in the sense of Chapman-Siebenmann.*

Proof. We keep the above notation. Let A be a compact subspace of M. Choose a compact and n-clean submanifold K of M such that $A\subseteq K$ (Lemma 2.3). It follows from the definition that $N_1=(N-K)\cup\delta(K)$ and $T_1=(T-K)\cup\delta(K)$ are compact μ^{n+1}-manifolds. Let $X=N-M$ and $Y=T-M$. Then X and Y are Z-sets in N_1 and T_1, respectively, and $N_1-X=T_1-Y=(M-K)\cup\delta(K)$. Consequently, the inclusions $i\colon N_1-X\to N_1$ and $j\colon T_1-Y\to T_1$ are n-homotopy equivalences. Let $s\colon N_1\to N_1-X$ be an n-homotopy inverse of i. Then $js\colon N_1\to T_1$ is an n-homotopy equivalence. Consequently, by [1], there is a homeomorphism $h\colon N_1\to T_1$ n-homotopic to the composition js. Using the Z-set unknotting theorem we can assume without loss of generality that $h/\delta(K)=\mathrm{id}$. The desired homeomorphism $H\colon N\to T$ can be defined as one which coincides with h on N_1 and with identity on K. The proof is finished.

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References

1. M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Ph.D. thesis, Univ. of Tennessee, Knoxville, 1984.
2. W. Browder, J. Levine, and G. R. Livesay, *Finding a boundary for an open manifold*, J. Math. **87** (1965), 1017–1028.
3. M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478–481.
4. T. A. Chapman, *Lectures on Hilbert cube manifolds*, CBMS Regional Conf. Ser. in Math., vol. 28, Amer. Math. Soc., Providence, RI, 1976.
5. T. A. Chapman and L. C. Siebenmann, *Finding a boundary for a Hilbert cube manifold*, Acta Math. **137** (1976), 171–208.
6. A. Chigogidze, *Compacta lying in the n-dimensional universal Menger compactum and having homeomorphic complements in it*, Mat. Sb. **133** (1987), 481–496.
7. ——, *Theory of n-shape*, Uspekhi Mat. Nauk **44** (1989), 117–140.
8. ——, *n-shapes and n-cohomotopy groups of compacta*, Mat. Sb. **180** (1989), 322–335.
9. ——, *Classification theorem for Menger manifolds*, Proc. Amer. Math. Soc. **116** (1992), 825–832.
10. ——, *UV^n-equivalence and n-equivalence*, Topology Appl. **45** (1992), 283–291.
11. R. H. Fox, *On the Lusternik-Schnirelman category*, Ann. of Math. **42** (1941), 333–370.
12. R. C. Lacher, *Cell-like mappings and their generalizations*, Bull. Amer. Math. Soc. **83** (1977), 495–552.
13. S. Metcalf, *Finding a boundary for a Hilbert cube manifold bundle*, Pacific J. Math. **120** (1985), 153–178.
14. L. C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Doctoral dissertation, Princeton Univ., Princeton, NJ, 1965.
15. C. T. C. Wall, *Fitness conditions for CW-complexes*, Ann. of Math. **81** (1965), 56–69.
16. J. H. C. Whitehead, *Combinatorial homotopy*. I, Bull. Amer. Math. Soc. **55** (1949), 213–245.

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