A FACTORIZATION CONSTANT FOR $l^p_n$, $0 < p < 1$

N. T. PECK

(Communicated by Dale Alspach)

Abstract. We prove that if $PT$ is a factorization of the identity operator on $l^p_n$ through $l^p_0$, $0 < p$, then $\|P\| \|T\| \geq Cn^{1/p-1/2}(\log n)^{-1/2}$. This is a corollary of a more general result on factoring the identity operator on a quasi-normed space $X$ through $\ell^k_\infty$.

In this paper we are concerned with factoring the identity operator on an $n$-dimensional quasi-normed space $X$ through a space $\ell^k_\infty$:

$$
\begin{array}{ccc}
\ell^K_\infty & \xrightarrow{P} & X \\
\downarrow T & & \downarrow \text{Id} \\
X & & \ell^k_\infty
\end{array}
$$

We seek a good lower bound $\lambda(x)$ for $\|P\| \|T\|$ over all factorizations $\text{Id}_X = PT$ as above. When $X$ is $l^p_n$, $1 \leq p$, the constant is known (see [5, Theorem 32.9] and the references given for that theorem).

For $p < 1$, we will obtain the lower estimate $\lambda(l^p_n) \geq Cn^{1/p-1/4}(\log n)^{-1/4}$. (A $T$ and $P$ with $\|P\| \|T\| \leq Cn^{1/p-1/4}$ are easily obtained.)

Throughout, $C$ denotes a constant, which may vary from one occurrence to the next but which is independent of $n$.

We thank Y. Gordon for valuable conversations.

Lemma 1. Let $(r_i), 1 \leq i \leq n$, be the first $n$ Rademacher functions, and let $\alpha_1, \ldots, \alpha_n$ be real. Letting $m$ denote Lebesgue measure on $(0,1)$, we have

$$m\left\{\sum_{i=1}^n \alpha_i r_i > \alpha \sqrt{\log n} \sum \alpha_i^2 \right\} \leq n^{-C\alpha^2},$$

for any positive $\alpha$.

Proof. This is well known; for completeness, we sketch a proof, following a suggestion of R. Kaufman.

Received by the editors June 8, 1992.

1991 Mathematics Subject Classification. Primary 46A15, 46E30.

Key words and phrases. Factorization of the identity, $l^p_n$, $0 < p < 1$, factorization constant, Rademacher functions, Hahn-Banach extension.
We can assume $\sum_{i=1}^{n} \alpha_i^2 = 1$. Put $f = \sum_{i=1}^{n} \alpha_i r_i$. By Khintchine's inequality, for some constant $C$ and all $p \geq 1$, $\|f\|_p \leq Cp^{\frac{1}{p}}$. From this, $m\{|f| > \lambda\} \leq (C/\lambda)^p p^\frac{p}{2} = K^p P^\frac{p}{2}$ with $K = C/\lambda$.

Now minimize $K^p P^\frac{p}{2}$ in $p$. At the minimizer, we find $p = K^{-2} e$, from which $K^p P^\frac{p}{2} = \exp(-\lambda^2/eC^2)$. (Note that $p > 1$ for $\lambda > C$.) Finally, put $\lambda = \sqrt{\log n}$ to get the conclusion. \(\square\)

The space $\ell_\infty$ is not of type 2; but the conclusion of the next lemma will suffice for our purposes.

**Lemma 2.** Let $Y$ be an $n$-dimensional subspace of $L_1(0, 1)$, and let $f_1, \ldots, f_n$ be elements of $Y^*$, of norm at most 1. Then for some $\overline{s}$ in $\ell_1$, $n$,

$$\sup_{\|y\| \leq 1} \left| \sum_{i=1}^{n} r_i(\overline{s}) f_i(y) \right| \leq C \sqrt{n \log n}.$$  

**Proof.** For any $0 < \epsilon < \frac{1}{2}$, a result of Schechtman [3] implies that there are an $N \leq C e^{-2\log(\epsilon^{-1})} n^2$ and an isomorphism $U : Y \to \ell_1^n$ such that $\|U\| \|U^{-1}\| \leq 1 + \epsilon$. See also the results of Bourgain, Lindenstrauss, and Milman [1] and Talagrand [4]. In particular, taking $\epsilon = \frac{1}{4}$, say, we obtain the corresponding $U$; we can assume $\|U\| = 1$, so that $\|U^{-1}\| \leq \frac{5}{4}$ and $N \leq C n^2$ (after changing $C$) is $\leq n^3$, if $n$ is sufficiently large.

Let $e_1, \ldots, e_N$ be the unit basis vectors in $\ell_1^n$. For each $i$ let $\Phi_i$ be a Hahn-Banach extension to $\ell_1^n$ of $f_i U^{-1}$ on $U(Y)$, with $\|\Phi_i\| \leq \frac{3}{4}$; then for each $j$, $\Phi_i(e_j) \leq \frac{3}{4}$.

Now fix $\alpha$ with $C \alpha^2 > 3$, where $C$ is the constant in the conclusion of Lemma 1; then $n \alpha^2 n^{-C \alpha^2} \leq \frac{1}{4}$ if $n$ is sufficiently large. Since

$$m \left\{ \left| \sum_{i=1}^{n} \phi_i(e_j) r_i(\overline{s}) \right| > \frac{5}{4} \alpha \sqrt{n \log n} \right\} \leq n^{-C \alpha^2} \quad \text{for each } j,$$

there is a set $A$, $m(A) > \frac{3}{4}$, such that $\left| \sum_{i=1}^{n} \phi_i(e_j) r_i(\overline{s}) \right| \leq \frac{5}{4} \alpha \sqrt{n \log n}$ for each $j$ and each $\overline{s}$ in $A$.

Now if $y \in Y$ and $\|y\| \leq 1$, then $\|U y\| \leq 1$. Write $U y = \sum_{j=1}^{N} \alpha_j e_j$, $\sum_{j=1}^{N} |\alpha_j| \leq 1$; applying the above inequality to each $j$ and recalling that $f_i = f_i U^{-1} U$ on $Y$, we have

$$\left| \sum_{i=1}^{n} f_i(y) r_i(\overline{s}) \right| \leq \frac{5}{4} \alpha \sqrt{n \log n} = C \sqrt{n \log n}$$

for each $\overline{s}$ in $A$. \(\square\)

**Notation.** Let $\mathcal{A}$ be an algebra of measurable subsets of $\mathcal{B}$. For $0 < p \leq \infty$, $L_p(\mathcal{A})$ is the space of functions in $L_p(0, 1)$ which are $\mathcal{A}$-measurable. For ease of argument, we deal with an $L_\infty(\mathcal{A})$ with "homogeneous" $\mathcal{A}$ rather than $\ell^\infty$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem. Let $\mathcal{A}$ be a finite subalgebra of measurable subsets of $(0, 1)$ containing the dyadic intervals $\left(\frac{j-1}{2^n}, \frac{j}{2^n}\right), 1 \leq j \leq 2^n$, and assume the atoms of $\mathcal{A}$ all have the same measure. Let $X$ be an $n$-dimensional vector space. Let $T : X \to L_\infty(\mathcal{A})$ be a linear map; and let $(e_i)_{i=1}^n$ be elements of $X$ such that $\|T(e_i)\|_\infty \leq 1, 1 \leq i \leq n$. Let $P : L_\infty(\mathcal{A}) \to X$ be a linear operator such that $PT = Id_X$. Then there is $w$ in $L_\infty(\mathcal{A})$ with $\|w\|_\infty \leq C\sqrt{n\log n}$ such that $P(w) = \sum_{i=1}^n r_i(s) e_i$, for some $s$ in $(0, 1)$.

Proof. Let $(I_j)_{j=1}^k$ be the atoms of $\mathcal{A}$, and let $z_1, \ldots, z_{k-n}$ be a basis for $\ker P$. Define a $(k-n) \times k$ matrix by $z_{i,j} = \text{constant value of } z_i$ on the atom $I_j$.

Now row-reduce the matrix $(z_{i,j})$. In $k-n$ of the columns there will be one 1 with all other entries 0; denote the atoms corresponding to the $n$ remaining “distinguished” columns by $I_{s_1}, \ldots, I_{s_n}$. Enlarge the matrix $(z_{i,j})$ to a $k \times k$ matrix $(y_{i,j})$ by adding $n$ rows of zeros in each of rows $s_1$ through $s_n$.

We can obviously regard $y_{i,j}$ as an $(\mathcal{A} \times \mathcal{A})$-measurable function $y(s, t)$ on $(0, 1) \times (0, 1)$, which satisfies these properties:

1. if $s \notin \bigcup_j I_{s_j}$ and if $s$ and $t$ are in the same atom, $y(s, t) = 1$;
2. if $s \notin \bigcup_j I_{s_j}$ and if $s$ and $t$ are in different atoms, $y(s, t) = 0$;
3. if $t \notin \bigcup_j I_{s_j}$, $y(s, t) = 0$ for all $s$;
4. if we define $y_t$ on $(0, 1)$ by $y_t(s) = y(s, t)$, then $y_t \in \ker P$;
5. a function $f$ in $L_\infty(\mathcal{A})$ is in $\ker P$ if and only if there is a function $\beta(t)$ in $L_\infty(\mathcal{A})$ so that $f(s) = \int y_t(s)\beta(t) dt$ for all $s$.

Properties (1) - (5) are evident from the description of $\ker P$ and the properties of a row-reduced matrix.

Let $s_i^*$ be a point of the atom $I_{s_i}, 1 \leq i \leq n$, and let $g_i = T(e_i), 1 \leq i \leq n$; then $\|g_i\|_\infty \leq 1$.

Now define

$$R(s, t) = \sum_{i=1}^n r_i(s) g_i(t),$$

and for a function $y(t)$, define

$$\psi_\delta(y) = \int R(s, t) y(t) dt = \sum_{i=1}^n \left( \int y(t) g_i(t) dt \right) r_i(s).$$

Let $Y$ be the span of the functions $y_t(s_i^*), 1 \leq i \leq n$, regarded as functions of $t$. Since $y \to \int y(t) g_i(t) dt$ is of norm at most 1, Lemma 2 implies that there is a set $A$ of measure $> \frac{3}{4}$ such that for any $s$ in $A$,

$$|\psi_\delta(y)| \leq C\sqrt{n \log n} \|y\|_1$$

for all $y$ in $Y$.

Take any norm-preserving extension of $\psi_\delta$ to all of $L_1(., dt)$; then there is a function $h_\delta(t)$ in $L_\infty(., dt)$ with $\|h_\delta\|_\infty \leq C\sqrt{n \log n}$ and such that

$$\psi_\delta(y) = \int y(t) h_\delta(t) dt$$

for all $y$ in $Y$. Restating this,

$$\int (R_\delta(t) - h_\delta(t)) y(t) dt = 0$$

(6)
for all \( y \) in \( Y \). Now put
\[
\alpha_{\tilde{f}}(s) = \frac{1}{m} \int (R_\xi(t) - h_\xi(t))y_t(s) \, dt,
\]
where \( m \) is the measure of an atom of \( \mathcal{A} \). Then \( \alpha_{\tilde{f}} \in \ker P \) by property (5), \( \alpha_{\tilde{f}}(s) = R_\xi(s) - h_\xi(s) \), for \( s \) not in \( \bigcup_{i=1}^n I_{s_i} \), by (1) and (2), and \( \alpha_{\tilde{f}}(s_i) = 0 \), \( 1 \leq i \leq n \), by (6).

The set \( A \) in the proof of Lemma 2 has measure \( \geq \frac{3}{4} \). Applying Lemma 1 again, we can require that
\[
|R_\xi(s_i^*)| \leq C \sqrt{n \log n}
\]
for each \( i, 1 \leq i \leq n \), for all \( \xi \) in a set \( B \) with \( m(B) \geq \frac{3}{4} \). Thus \( A \cap B \) has positive measure, so choose \( \xi \) in \( A \cap B \).

Now, to finish the proof, set \( w(s) = R_\xi(s) - \alpha_{\tilde{f}}(s) \). If \( s \notin \bigcup_{i=1}^n I_{s_i} \), \( w(s) = R_\xi(s) - h_\xi(s) \), so \( |w(s)| \leq C \sqrt{n \log n} \). Also, \( |w(s_i^*)| = |R_\xi(s_i^*)| \leq C \sqrt{n \log n} \); thus \( \|w\|_\infty \leq C \sqrt{n \log n} \). Finally, \( Pw = \sum_{i=1}^n r_i(\xi)e_i \) since \( \alpha_{\tilde{f}} \in \ker P \). This completes the proof. \( \square \)

**Corollary 1.** Let \( X \) be \( n \)-dimensional, and let \( X \xrightarrow{T} \ell^K_{\infty} \xrightarrow{P} X \) be a factorization of \( \text{Id}_X \) through \( \ell^K_{\infty} \). Then \( \|P\| \|T\| \geq C s_n(n \log n)^{-\frac{1}{2}} \), where
\[
s_n = \sup_{\|T e_i\| \leq 1} \inf_{1 \leq i \leq n} \left| \sum_{i=1}^n \pm e_i \right|.
\]
In particular, if \( \|T\| \leq 1 \), then \( \|P\| \geq C b_n(n \log n)^{-\frac{1}{2}} \), where \( b_n = \sup_{\|e_i\| \leq 1} \inf_{1 \leq i \leq n} \|\sum_{i=1}^n \pm e_i\| \). (See [2].)

**Proof.** For \( K \geq n \), this is an immediate consequence of the theorem. Assume now that \( K < n \). Let \( j = n - K \), and define \( \hat{T} : X \rightarrow \ell^j_{\infty} = \ell^j_{\infty} \otimes \ell^j_{\infty} \) by \( \hat{T}(x) = (T(x), 0) \). Define \( \hat{P} : \ell^j_{\infty} \rightarrow X \) by \( \hat{P}(w, z) = P(w) \). Note that \( \hat{P} \hat{T} = \text{Id}_X \) and that \( \|\hat{P}\| = \|P\|, \|\hat{T}\| = \|T\| \). The result now follows from the theorem. \( \square \)

**Corollary 2.** Suppose \( \ell_p^n \xrightarrow{T} \ell^K_{\infty} \xrightarrow{P} \ell_p^n \) is any factorization of the identity on \( \ell_p^n \) through \( \ell^K_{\infty} \), \( 0 < p < 1 \), with \( \|T\| = 1 \). Then \( \|P\| \geq C n^{\frac{1}{2} - \frac{1}{p}} \), \( 0 < p < 1 \). Now apply Corollary 1. \( \square \)

**Remark.** For \( 0 < p < 1 \), define \( T : \ell_p^n \rightarrow L_\infty(\mathcal{A}) \) by defining \( T(e_i) = r_i \), \( 1 \leq i \leq n \), and extending linearly. Then \( \|T\| = 1 \). Define \( P : L_\infty(\mathcal{A}) \rightarrow \ell_p^n \) by \( P(x) = \sum_{i=1}^n (\int x r_i) e_i \). It is easily checked that \( \|P : L_\infty(\mathcal{A}) \rightarrow \ell_p^n \| \leq \sqrt{n} \); since \( \|\text{Id} : \ell_p^n \rightarrow \ell_p^n\| = n^{\frac{1}{2} - \frac{1}{p}} \), it follows that \( \|P\| \leq n^{\frac{1}{2} - \frac{1}{p}} \). Obviously \( PT = \text{Id} \), so up to a logarithmic factor, the order of \( \frac{1}{2}(\ell_p^n) \) is correct.

**ADDED IN PROOF**

Y. Gordon has informed the author that the factor \( \sqrt{\log n} \) in the statement of the theorem can be removed.

---

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
A FACTORIZATION CONSTANT FOR $l^n_p$, $0 < p < 1$

References

1. J. Bourgain, J. Lindenstrauss, and V. Milman, Approximation of zonoids by zonotopes, Acta
   363–369.
5. N. Tomczak-Jaegermann, Banach-Mazur distances and finite-dimensional operator ideals,

Department of Mathematics, University of Illinois, Urbana, Illinois 61801
E-mail address: peck@math.uiuc.edu