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A FACTORIZATION CONSTANT FOR l_p^n, $0 < p < 1$

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ABSTRACT. We prove that if PT is a factorization of the identity operator on ℓ_p^n through ℓ_∞^k, $0 < p$, then $\|P\| \, \|T\| \geq Cn^{1/p-1/2}(\log n)^{-1/2}$. This is a corollary of a more general result on factoring the identity operator on a quasi-normed space X through ℓ_∞^k.

In this paper we are concerned with factoring the identity operator on an n-dimensional quasi-normed space X through a space ℓ_∞^K:

$$\begin{tikzcd} & \ell_\infty^K \arrow[dr, "P"] & \\ X \arrow[ur, "T"] \arrow[rr, "Id"] & & X \end{tikzcd}$$

We seek a good lower bound $\underline{\lambda}(x)$ for $\| P \| \, \| T \|$ over all factorizations $\mathrm{Id}_X = PT$ as above. When X is ℓ_p^n, $1 \leq p$, the constant is known (see [5, Theorem 32.9] and the references given for that theorem).

For $p < 1$, we will obtain the lower estimate $\underline{\lambda}(l_p^n) \geq Cn^{\frac{1}{p}-\frac{1}{2}}(\log n)^{-\frac{1}{2}}$. (A T and P with $\| P \| \, \| T \| \leq Cn^{\frac{1}{p}-\frac{1}{2}}$ are easily obtained.)

Throughout, C denotes a constant, which may vary from one occurrence to the next but which is independent of n.

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Lemma 1. *Let (r_i), $1 \leq i \leq n$, be the first n Rademacher functions, and let $\alpha_1, \ldots, \alpha_n$ be real. Letting m denote Lebesque measure on $(0, 1)$, we have*

$$m\left\{\left|\sum_{i=1}^n \alpha_i r_i\right| > \alpha\sqrt{\log n \sum \alpha_i^2}\right\} \leq n^{-C\alpha^2},$$

for any positive α.

Proof. This is well known; for completeness, we sketch a proof, following a suggestion of R. Kaufman.

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We can assume $\sum_{i=1}^n \alpha_i^2 = 1$. Put $f = \sum_{i=1}^n \alpha_i r_i$. By Khintchine's inequality, for some constant C and all $p \geq 1$, $\|f\|_p \leq Cp^{\frac{1}{2}}$. From this, $m\{|f| > \lambda\} \leq (C/\lambda)^p p^{\frac{p}{2}} = K^p p^{\frac{p}{2}}$ with $K = C/\lambda$.

Now minimize $K^p p^{\frac{p}{2}}$ in p. At the minimizer, we find $p = K^{-2}e$, from which $K^p p^{\frac{p}{2}} = \exp(-\lambda^2/eC^2)$. (Note that $p > 1$ for $\lambda > C$.) Finally, put $\lambda = \alpha\sqrt{\log n}$ to get the conclusion. □

The space ℓ_∞ is not of type 2; but the conclusion of the next lemma will suffice for our purposes.

Lemma 2. *Let Y be an n-dimensional subspace of $L_1(0, 1)$, and let $f_1, \ldots, f_n$ be elements of Y^*, of norm at most 1. Then for some $\bar{s}$ in $(0, 1)$,*

$$\sup_{\|y\|\leq 1} \left| \sum_{i=1}^n r_i(\bar{s}) f_i(y) \right| \leq C\sqrt{n \log n}.$$

Proof. For any $0 < \epsilon < \frac{1}{2}$, a result of Schechtman [3] implies that there are an $N \leq C\epsilon^{-2}\log(\epsilon^{-1})n^2$ and an isomorphism $U : Y \xrightarrow{\text{into}} \ell_1^N$ such that $\|U\|\,\|U^{-1}\| \leq 1 + \epsilon$. See also the results of Bourgain, Lindenstrauss, and Milman [1] and Talagrand [4]. In particular, taking $\epsilon = \frac{1}{4}$, say, we obtain the corresponding U; we can assume $\|U\| = 1$, so that $\|U^{-1}\| \leq \frac{5}{4}$ and $N \leq Cn^2$ (after changing C) $\leq n^3$, if n is sufficiently large.

Let $e_1, \ldots, e_N$ be the unit basis vectors in ℓ_1^N. For each i let Φ_i be a Hahn-Banach extension to ℓ_1^N of $f_i U^{-1}$ on $U(Y)$, with $\|\Phi_i\| \leq \frac{5}{4}$; then for each j, $\Phi_i(e_j)\| \leq \frac{5}{4}$.

Now fix α with $C\alpha^2 > 3$, where C is the constant in the conclusion of Lemma 1; then $n^3 n^{-C\alpha^2} \leq \frac{1}{4}$ if n is sufficiently large. Since

$$m\left\{\left|\sum \phi_i(e_j) r_i\right| > \frac{5}{4}\alpha\sqrt{n\log n}\right\}$$
$$\leq m\left\{\left|\sum \phi_i(e_j) r_i\right| > \alpha\sqrt{\log n \sum \phi_i^2(e_j)}\right\} \leq n^{-C\alpha^2} \quad \text{for each} \quad j,$$

there is a set A, $m(A) > \frac{3}{4}$, such that $\left|\sum_{i=1}^n \phi_i(e_j) r_i(\bar{s})\right| \leq \frac{5}{4}\alpha\sqrt{n\log n}$ for each j and each $\bar{s}$ in A.

Now if $y \in Y$ and $\|y\| \leq 1$, then $\|Uy\| \leq 1$. Write $Uy = \sum_{j=1}^N \alpha_j e_j$, $\sum_{j=1}^N |\alpha_j| \leq 1$; applying the above inequality to each j and recalling that $f_i = f_i U^{-1} U$ on Y, we have

$$\left|\sum_{i=1}^n f_i(y) r_i(\bar{s})\right| \leq \frac{5}{4}\alpha\sqrt{n\log n} = C\sqrt{n\log n}$$

for each $\bar{s}$ in A. □

Notation. Let $\mathscr{A}$ be an algebra of measurable subsets of $(0, 1)$. For $0 < p \leq \infty$, $L_p(\mathscr{A})$ is the space of functions in $L_p(0, 1)$ which are $\mathscr{A}$-measurable. For ease of argument, we deal with an $L_\infty(\mathscr{A})$ with "homogeneous" $\mathscr{A}$ rather than ℓ_∞^K.

Theorem. *Let $\mathscr{A}$ be a finite subalgebra of measurable subsets of $(0, 1)$ containing the dyadic intervals $\left(\frac{j-1}{2^n}, \frac{j}{2^n}\right)$, $1 \le j \le 2^n$, and assume the atoms of $\mathscr{A}$ all have the same measure. Let X be an n-dimensional vector space. Let $T : X \longrightarrow L_\infty(\mathscr{A})$ be a linear map; and let $(e_i)_{i=1}^n$ be elements of X such that $\|T(e_i)\|_\infty \le 1$, $1 \le i \le n$. Let $P : L_\infty(\mathscr{A}) \longrightarrow X$ be a linear operator such that $PT = \mathrm{Id}_X$. Then there is w in $L_\infty(\mathscr{A})$ with $\|w\|_\infty \le C\sqrt{n \log n}$ such that $P(w) = \sum_{i=1}^n r_i(\bar{s})e_i$, for some $\bar{s}$ in $(0, 1)$.*

Proof. Let $(I_j)_{j=1}^k$ be the atoms of $\mathscr{A}$, and let $z_1, \dots, z_{k-n}$ be a basis for $\ker P$. Define a $(k-n) \times k$ matrix by $z_{i,j} =$ constant value of z_i on the atom I_j.

Now row-reduce the matrix $(z_{i,j})$. In $k-n$ of the columns there will be one 1 with all other entries 0; denote the atoms corresponding to the n remaining "distinguished" columns by $I_{s_1}, \dots, I_{s_n}$. Enlarge the matrix $(z_{i,j})$ to a $k \times k$ matrix $(y_{i,j})$ by adding n rows of zeros in each of rows s_1 through s_n.

We can obviously regard $y_{i,j}$ as an $(\mathscr{A} \times \mathscr{A})$-measurable function $y(s, t)$ on $(0, 1) \times (0, 1)$, which satisfies these properties:

(1) if $s \notin \bigcup_j I_{s_j}$ and if s and t are in the same atom, $y(s, t) = 1$;
(2) if $s \notin \bigcup_j I_{s_j}$ and if s and t are in different atoms, $y(s, t) = 0$;
(3) if $t \notin \bigcup_j I_{s_j}$, $y(s, t) = 0$ for all s;
(4) if we define y_t on $(0, 1)$ by $y_t(s) = y(s, t)$, then $y_t \in \ker P$;
(5) a function f in $L_\infty(\mathscr{A})$ is in $\ker P$ if and only if there is a function $\beta(t)$ in $L_\infty(\mathscr{A})$ so that $f(s) = \int y_t(s)\beta(t)\,dt$ for all s.

Properties (1) – (5) are evident from the description of $\ker P$ and the properties of a row-reduced matrix.

Let s_i^* be a point of the atom I_{s_i}, $1 \le i \le n$, and let $g_i = T(e_i)$, $1 \le i \le n$; then $\|g_i\|_\infty \le 1$.

Now define

$$R(s, t) = \sum_{i=1}^n r_i(s) g_i(t),$$

and for a function $y(t)$, define

$$\psi_s(y) = \int R(s, t) y(t)\,dt = \sum_{i=1}^n \left(\int y(t) g_i(t)\,dt\right) r_i(s).$$

Let Y be the span of the functions $y_t(s_i^*)$, $1 \le i \le n$, regarded as functions of t. Since $y \longrightarrow \int y(t) g_i(t)\,dt$ is of norm at most 1, Lemma 2 implies that there is a set A of measure $> \frac{3}{4}$ such that for any $\bar{s}$ in A,

$$|\psi_{\bar{s}}(y)| \le C\sqrt{n \log n}\,\|y\|_1$$

for all y in Y.

Take any norm-preserving extension of ψ_s to all of $L_1(\ , dt)$; then there is a function $h_{\bar{s}}(t)$ in $L_\infty(., dt)$ with $\|h_{\bar{s}}\|_\infty \le C\sqrt{n \log n}$ and such that

$$\psi_{\bar{s}}(y) = \int y(t) h_{\bar{s}}(t)\,dt$$

for all y in Y. Restating this,

$$\int (R_{\bar{s}}(t) - h_{\bar{s}}(t)) y(t)\,dt = 0 \tag{6}$$

for all y in Y. Now put

$$\alpha_{\bar{s}}(s) = \frac{1}{m}\int (R_{\bar{s}}(t) - h_{\bar{s}}(t))y_t(s)\,dt,$$

where m is the measure of an atom of $\mathscr{A}$. Then $\alpha_{\bar{s}} \in \ker P$ by property (5), $\alpha_{\bar{s}}(s) = R_{\bar{s}}(s) - h_{\bar{s}}(s)$, for s not in $\bigcup_{i=1}^n I_{s_i}$, by (1) and (2), and $\alpha_{\bar{s}}(s_i) = 0$, $1 \le i \le n$, by (6).

The set A in the proof of Lemma 2 has measure $\ge \frac{3}{4}$. Applying Lemma 1 again, we can require that

$$|R_{\bar{s}}(s_i^*)| \le C\sqrt{n\log n}$$

for each i, $1 \le i \le n$, for all $\bar{s}$ in a set B with $m(B) \ge \frac{3}{4}$. Thus $A \cap B$ has positive measure, so choose $\bar{s}$ in $A \cap B$.

Now, to finish the proof, set $w(s) = R_{\bar{s}}(s) - \alpha_{\bar{s}}(s)$. If $s \notin \bigcup_{i=1}^n I_{s_i}$, $w(s) = R_{\bar{s}}(s) - \alpha_{\bar{s}}(s) = h_{\bar{s}}(s)$, so $|w(s)| \le C\sqrt{n\log n}$. Also, $|w(s_i^*)| = |R_{\bar{s}}(s_i^*)| \le C\sqrt{n\log n}$; thus $\|w\|_\infty \le C\sqrt{n\log n}$. Finally, $Pw = \sum_{i=1}^n r_i(\bar{s})e_i$ since $\alpha_{\bar{s}} \in \ker P$. This completes the proof. □

Corollary 1. *Let X be n-dimensional, and let $X \xrightarrow{T} \ell_\infty^K \xrightarrow{P} X$ be a factorization of Id_X through ℓ_∞^K. Then $\|P\|\,\|T\| \ge Cs_n(n\log n)^{\frac{1}{2}}$, where*

$$s_n = \sup_{\substack{\|Te_i\|\le 1\\ 1\le i\le n}} \inf_{\pm 1} \left\| \sum_{i=1}^n \pm e_i \right\|.$$

In particular, if $\|T\| \le 1$, then $\|P\| \ge Cb_n(n\log n)^{-\frac{1}{2}}$, where $b_n = \sup_{\|e_i\|\le 1,\ 1\le i\le n} \inf_{\pm 1} \|\sum_{i=1}^n \pm e_i\|$. (See [2].)

Proof. For $K \ge n$, this is an immediate consequence of the theorem. Assume now that $K < n$. Let $j = n - K$, and define $\tilde{T} : X \longrightarrow \ell_\infty^n = \ell_\infty^K \otimes \ell_\infty^j$ by $\tilde{T}(x) = (T(x), 0)$. Define $\tilde{P} : \ell_\infty^n \longrightarrow X$ by $\tilde{P}(w, z) = P(w)$. Note that $\tilde{P}\tilde{T} = \mathrm{Id}_X$ and that $\|\tilde{P}\| = \|P\|$, $\|\tilde{T}\| = \|T\|$. The result now follows from the theorem. □

Corollary 2. *Suppose $\ell_p^n \xrightarrow{T} \ell_\infty^K \xrightarrow{P} \ell_p^n$ is any factorization of the identity on ℓ_p^n through ℓ_∞^K, $0 < p < 1$, with $\|T\| = 1$. Then $\|P\| \ge Cn^{\frac{1}{p}-\frac{1}{2}}(\log n)^{-\frac{1}{2}}$.*

Proof. Take $X = \ell_p^n$, e_i the usual ith basis vector in ℓ_p^n, $1 \le i \le n$, for $0 < p < 1$. Now apply Corollary 1. □

Remark. For $0 < p < 1$, define $T : \ell_p^n \longrightarrow L_\infty(\mathscr{A})$ by defining $T(e_i) = r_i$, $1 \le i \le n$, and extending linearly. Then $\|T\| = 1$. Define $P : L_\infty(\mathscr{A}) \longrightarrow \ell_p^n$ by $P(x) = \sum_{i=1}^n (\int xr_i)e_i$. It is easily checked that $\|P : L_\infty(\mathscr{A}) \longrightarrow \ell_1^n\| \le \sqrt{n}$; since $\|\mathrm{Id} : \ell_1^n \longrightarrow \ell_p^n\| = n^{\frac{1}{p}-1}$, it follows that $\|P\| \le n^{\frac{1}{p}-\frac{1}{2}}$. Obviously $PT = \mathrm{Id}$, so up to a logarithmic factor, the order of $\underline{\lambda}(\ell_p^n)$ is correct.

Added in proof

Y. Gordon has informed the author that the factor $\sqrt{\log n}$ in the statement of the theorem can be removed.

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