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A NOTE ON THE M_{23}- AND Fi_{23}-MINIMAL PARABOLIC GEOMETRIES

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ABSTRACT. Characterizations of the minimal parabolic geometries for the sporadic simple groups M_{23} and Fi_{23} are presented.

1. INTRODUCTION

In this note we study geometries that are related to the minimal parabolic geometries of the sporadic simple groups M_{23} (Mathieu group of degree 23) and Fi_{23} (Fischer group). These geometries, along with many others, are itemized in [7]. There it is noted that there are two nonisomorphic minimal parabolic geometries for each of M_{23} and Fi_{23} which are "locally isomorphic". Our two main results give "local characterizations" of these geometries.

Theorem A. *Let Γ be a residually connected string geometry with type set $\{0, 1, 2\}$. Suppose that G is a flag transitive subgroup of* $\operatorname{Aut}\Gamma$ *and that the following hold*:

(i) *Γ_a is the geometry of duads and triduads on the Steiner system* $S(22, 3, 6)$ *and $G_a/Q(a) \cong M_{22}$ for each $a \in \Gamma_0$;*
(ii) *Γ_X is the geometry of points and lines of a projective plane of order* 2 *for each $X \in \Gamma_2$; and*
(iii) *for a maximal flag $\{a, l, X\}$ G_{alX} is finite.*

Then Γ is isomorphic to one of the M_{23}-minimal parabolic geometries and $G \cong M_{23}$.

Theorem B. *Let Γ be a residually connected string geometry with type set $\{0, 1, 2, 3\}$. Suppose that G is a flag transitive subgroup of* $\operatorname{Aut}\Gamma$ *and that the following hold*:

(i) *for each flag of F of type $\{0, 3\}$ Γ_F is the geometry of duads and triduads on the Steiner system* $S(22, 3, 6)$ *and $G_F/Q(F) \cong M_{22}$;*

(ii) *for each flag F of type $\{0, 1\}$ Γ_F is the geometry of nonzero isotropic vectors and isotropic 2-spaces in a 4-dimensional* GF(4)*-unitary space and $G_F/Q(F) \cong U_4(2){:}\,2$;*

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(iii) *for each flag* F *of type* $\{2, 3\}$ Γ_F *is the geometry of points and lines of a projective plane of order* 2; *and*

(iv) *for a maximal flag* $\{a, l, \pi, X\}$ $G_{al\pi X}$ *is finite.*

Then Γ *is isomorphic to one of the* Fi_{23}*-minimal parabolic geometries and* $G \cong \mathrm{Fi}_{23}$.

The arguments used here to establish Theorems A and B are for the most part group theoretic with only a smattering of geometric considerations. However, the proof of the main result in [8] is dominated by geometric methods, and this result is employed in the proof of Theorem B. Together, [8] and Theorem A enable us, in the proof of Theorem B, to identify $G_a/Q(a)$ for $a \in \Gamma_0$, and then using a recent result of Meixner [6] we may deduce that $G \cong \mathrm{Fi}_{23}$, from which Theorem B follows. As will be seen, a key part in proving both Theorems A and B is the pinpointing of G_{ab} for collinear points a and b in Γ. We remark that Theorem A has also been obtained by Ivanov and Shpectorov [3] (using different arguments).

We now review our notation. Let $\Gamma = (\Gamma, \tau, *)$ be a geometry over $\{0, 1, \dots, n\}$ with type map τ and an incidence relation $*$. For $i \in \{0, 1, \dots, n\}$, F a flag in Γ, and $\Sigma \subseteq \Gamma$,

Γ_i denotes the set of all objects of type i in Γ,
Γ_F denotes the residue geometry of F, and
$\Gamma_i(\Sigma)$ denotes the set of all objects of type i in Γ incident with all objects in Σ.

Recall that a flag F of Γ is a set of pairwise incident elements of Γ, and the type of F is the set $\tau(F)$. When $F = \{x\}$ we shall write Γ_x instead of $\Gamma_{\{x\}}$. Now assume that $i = 0, n$. The collinearity graph $\mathscr{G}_i(\Gamma)$ has Γ_i as its vertex set, and $a, b \in \Gamma_i$ are adjacent if $\Gamma_j(a, b) \neq \varnothing$ where $j = 1$ if $i = 0$ and $j = n - 1$ if $i = n$. We shall say that $a, b \in \Gamma_i$ are collinear if they are adjacent in $\mathscr{G}_i(\Gamma)$. For $a \in \Gamma_i$, $\Delta_j(a)$ denotes the set of all objects in Γ_i which are distance j from a in $\mathscr{G}_i(\Gamma)$.

Let G be a subgroup of the automorphism group of Γ and $\{x_1, \dots, x_k\} = \Sigma \subseteq \Gamma$. Then G_Σ or $G_{x_1 \cdots x_k}$ is the subgroup of G, fixing every object in Σ and $Q(\Sigma) = \{g \in G_\Sigma | g$ fixes every object which is incident with all objects in $\Sigma\}$. In the case when $\Sigma = \{x\}$ we shall write $Q(x)$ for $Q(\{x\})$.

We follow the conventions of the Atlas [2, §5] in describing groups.

2. Proof of Theorem A

In this section we assume the hypothesis of Theorem A and name the objects of Γ as follows:

type 0 1 2
o———o———o
points lines

Note that for $a \in \Gamma_0$, in Γ_a objects of type 1 and 2 correspond, respectively, to the duads and triduads of the Steiner system $\mathrm{S}(22, 3, 6)$.

Lemma 2.1. *Let* $\{a, l, X\}$ *be a maximal flag of* Γ, *with* $a \in \Gamma_0$, $l \in \Gamma_1$, $X \in \Gamma_2$, *and put* $P_1 = G_{al}$ *and* $P_2 = G_{aX}$.

(i) *We have* $G_a \cong \mathrm{M}_{22}$, $G_l \cong 2^4(\mathbb{Z}_3 \times A_5) \cdot 2$, *and* $G_X \cong 2^4{:}\, L_3(2)$ (*so* $P_1 \cong 2^4{:}\, S_5$ *and* $P_2 \cong 2^6{:}\, S_3$).

(ii) *Let* $\widetilde{P}_1$ *denote the* (*unique*) *maximal subgroup of* P_1 *containing* G_{alX} (*so* $\widetilde{P}_1 \cong 2^4\colon S_4$). *Then* $\langle \widetilde{P}_1, P_2\rangle \cong 2^4\colon A_6$, *a hexad stabilizer, and* $\langle \widetilde{P}_1, P_2\rangle$ *normalizes* $O_2(G_X)$.

(iii) $|\Gamma_0(l)| = 3$.

(iv) *Let* $b \in \Gamma_0(l)\backslash\{a\}$. *Then* $[G_{al}\colon G_{abl}] = 2$ *and* $G_{abl} \cong 2^4\colon A_5$. *In particular,* G_a *is transitive on* $\Delta_1(a)$.

Proof. (i) By hypothesis we have $G_F/Q(F) \cong S_5$ for F a flag of type $\{0, 1\}$ and $G_F/Q(F) \cong S_3$ for F a flag of type $\{0, 2\}$ or $\{1, 2\}$. Also by hypothesis we have that $G_F/Q(F) \cong M_{22}$ for a flag F of type $\{0\}$ and $G_F/Q(F) \cong L_3(2)$ for a flag F of type $\{2\}$. We now show that for a flag F of type $\{1\}$, $G_F/Q(F) \cong S_3 \times S_5$ or $(\mathbb{Z}_3 \times A_5)\cdot 2$. Since Γ is a string geometry and G is flag transitive, we have that $G_l = G_{al}G_{lX}$. Let N_0 (respectively, N_2) be the subgroup of G_{lX} (respectively, G_{al}) fixing all the objects in $\Gamma_0(l)$ (respectively, $\Gamma_2(l)$). The assumed structures of residue geometries give $G_{lX}/N_0 \cong S_3$ and $G_{al}/N_2 \cong S_5$. Clearly $N_0 \cap N_2 = Q(l)$, from which it follows that $G_l/Q(l) \cong S_3 \times S_5$ or $(\mathbb{Z}_3 \times A_5)\cdot 2$, as claimed. Using the same arguments as in Lemmas 2.4 and 2.6 of [9], but noting that our hypotheses obviate the necessity of invoking the results of Ivanov and Shpecterov, yields the stated structures for G_a, G_l, and G_X.

(ii) By [9, Proposition 2.1(i)(c)] $\langle \widetilde{P}_1, P_2\rangle$ normalizes $O_2(G_X)$. That $\langle \widetilde{P}_1, P_2\rangle \cong 2^4\colon A_6$ is a consequence of the structure of M_{22}.

(iii) This follows from Γ being a string geometry and Γ_X being the points and lines of a projective plane of order 2.

(iv) Since, by hypothesis, G_{lX} induces S_3 upon the 3 points of the line l, $[G_{al}\colon G_{abl}] = 2$. Hence, as $G_{al} \cong 2^4\colon S_5$, we obtain $G_{abl} \cong 2^4\colon A_5$.

Lemma 2.2. *Let* $a \in \Gamma_0$ *and* $b \in \Delta_1(a)$. *Then* $G_{ab} \cong L_3(4)$.

Proof. Let $l \in \Gamma_1(a, b)$. From Lemma 2.1(iv) we have that $G_{abl} \cong 2^4\colon A_5$. We claim that $G_{ab} \neq G_{abl}$. Assuming that $G_{ab} = G_{abl}$ we derive a contradiction. By Lemma 2.1(iv) G_a is transitive on $\Delta_1(a)$, and therefore $|\Delta_1(a)| = [G_a\colon G_{ab}] = 2.3.7.11$ with the set $\Delta_1(a)$ in one-to-one correspondence with the cosets of G_{ab} in G_a. Since $G_{ab} \cong 2^4\colon A_5$ is contained in a subgroup of M_{22} isomorphic to $L_3(4)$, we can partition $\Delta_1(a)$ into 22 blocks (of imprimitivity) each containing 21 points and such that the stabilizer (in $G_a \cong M_{22}$) of each block is isomorphic to $L_3(4)$. Let $\mathscr{P}$ denote this partition of $\Delta_1(a)$.

Choose $X \in \Gamma_2(a)$ such that $l \in \Gamma_1(X)$. By hypothesis $\Gamma_0(X)$ consists of 7 points which are pairwise collinear. Let $S = \Gamma_0(X)\backslash\{a\}$ denote the 6 point subset of $\Delta_1(a)$, and put $N = O_2(G_X)$. Note that N fixes each point in S. Also, each of the points in S lie in separate blocks of $\mathscr{P}$. For suppose $c, d \in S$, $c \neq d$ with $c, d \in B$ where $B \in \mathscr{P}$. Let $k \in \Gamma_1(c, d)$. Then, since G is transitive on $\{0, 1\}$-flags, Lemma 2.1(iv) implies that $G_{cdk} \cong 2^4\colon A_5$. By properties of blocks of imprimitivity, G_{cdk} is contained in the stabilizer of B. But the stabilizer of 2 points in a block is isomorphic to the stabilizer of two distinct 1-spaces in the action of $L_3(4)$ on a projective plane of order 4 which is isomorphic to a proper subgroup of $2^4\colon A_5$. Thus points of S lie in separate blocks of $\mathscr{P}$ and consequently N fixes at least 6 blocks of $\mathscr{P}$. Because of Lemma 2.1(ii) N must fix exactly 6 blocks of $\mathscr{P}$, and these define a hexad of the 22 element set $\mathscr{P}$. Letting $G_a \cong M_{22}$ act upon $\mathscr{P}$ we get all 77 hexads of

$\mathscr{P}$. For $c \in S$ with $c \in B$ where $B \in \mathscr{P}$, under the action of the stabilizer of B on c we get every point of B. In this way we obtain 77.21 6-element sets of $\Delta_1(a)$ by letting G_a act upon S. Each of these 6-elements sets together with a is the set of points incident with some plane of $\Gamma_2(a)$. Therefore, we have found 77.21 images of X under G_a; in other words there are at least 77.21 triduads in $\Gamma_2(a)$. However, $|\Gamma_2(a)| = 77.15$, so giving the required contradiction.

Since, by Lemma 2.1(iv), $G_a \neq G_{ab}$ and $G_{ab} \neq G_{al}$ $(\cong 2^4{:}\,S_5)$, looking at the subgroups of M_{22} we see that the only possibility is $G_{ab} \cong L_3(4)$.

Proof of Theorem A. Let $a \in \Gamma_0$ and $b \in \Delta_1(a)$. By Lemmas 2.1(iv) and 2.2 we see that $|\Delta_1(a)| = 22$ with $G_a \cong \mathrm{M}_{22}$ acting in its 3-transitive representation on $\Delta_1(a)$. In particular, G_{ab} acts transitively on $\Delta_1(a)\backslash\{b\}$ and so, in view of Lemma 2.1(iii), b must be collinear with all points in $\Delta_1(a)$. Thus $\Delta_1(a) \cup \{a\} = \Delta_1(b) \cup \{b\}$, whence, by [1, 2.10], $\Gamma_0 = \{a\} \cup \Delta_1(a)$. So G is a transitive extension of M_{22} and hence $G \cong \mathrm{M}_{23}$ by [5, (13.2)].

We next identify Γ. Let $\mathscr{C}(\Gamma)$ denote the chamber system associated to Γ. Then $\mathscr{C}(\Gamma)$ is isomorphic to the chamber system $\mathscr{C} = (G; G_{alX}, \{G_{al}, G_{aX}, G_{lX}\})$ where $\{a, l, X\}$ is a maximal flag of Γ. Since G_{al}, G_{aX}, G_{lX} are minimal parabolic subgroups of $G \cong \mathrm{M}_{23}$, by [7], $\mathscr{C}$ is one of two possible chamber systems (which are the chamber systems of the two minimal parabolic geometries). Hence, as both Γ and the minimal parabolic geometries for M_{23} are residually connected geometries, we conclude that Γ is isomorphic to one of the M_{23}-minimal parabolic geometries (see [10, (2.2) and (2.3)] for further details). This completes the proof of Theorem A.

3. Proof of Theorem B

Throughout this section we suppose the hypothesis of Theorem B to hold. So this time the diagram for Γ is:

type 0 1 2 3

points lines

Let $\{a, l, \pi, X\}$ be a maximal flag of Γ with $a \in \Gamma_0$, $l \in \Gamma_1$, $\pi \in \Gamma_2$, and $X \in \Gamma_3$. Then in the residue $\Gamma_{\{a,X\}}$ objects of type 1 and 2 correspond, respectively, to duads and triduads of the Steiner system S(22, 3, 6), while in the residue $\Gamma_{\{a,l\}}$ objects of type 2 and 3 correspond, respectively, to nonzero isotropic vectors and isotropic 2-spaces in a 4-dimensional GF(4)-unitary space.

Lemma 3.1. *Let* $\{a, l, \pi, X\}$ *be a maximal flag in* Γ *with* $a \in \Gamma_0$, $l \in \Gamma_1$, $\pi \in \Gamma_2$, *and* $X \in \Gamma_2$.

(i) $G_X/Q(X) \cong \mathrm{M}_{23}$ *with* Γ_X *a* M_{23}*-minimal parabolic geometry,* $G_a/Q(a) \cong \mathrm{Fi}_{22}$, *with* Γ_a *the* Fi_{22}*-minimal parabolic geometry,* $G_{al} \cong 2^2 \times 2^{1+8}U_4(2){:}\,2$ *and* $|Q(a)| = 2$ *with* $Q(a) \leq 2^2 \times 2^{1+8}U_4(2)$.

(ii) *Let* $b \in \Gamma_0(l)\backslash\{a\}$. *Then* $[G_{al}{:}\, G_{abl}] = 2$ *and* $G_{abl} \cong 2^2 \times 2^{1+8}U_4(2)$. *In particular,* G_a *in transitive on* $\Delta_1(a)$.

(iii) $|\Gamma_1(a)| = 3^5.5.7.11.13 = 1, 216, 215$.

Proof. (i) By our hypotheses, the residue Γ_X satisfies the conditions of Theorem A and thus $G_X/Q(X) \cong \mathrm{M}_{23}$. By applying the main theorem of [8] to the residue Γ_a we also obtain $G_a/Q(a) \cong \mathrm{Fi}_{22}$. These results also give that Γ_X

and Γ_a are as stated. Now we see that $G_F/Q(F) \cong S_5$ for F a flag of type $\{0, 1, 3\}$, while $G_F/Q(F) \cong S_3$ for the remaining rank 3 flags F. Also we observe that $G_F/Q(F) \cong S_3 \times S_3$ (respectively, $(\mathbb{Z}_3 \times A_5)2$) for F a flag of type $\{0, 2\}$ (respectively, $\{1, 3\}$). Just as in Lemma 2.1(i) we may check that, for F a flag of type $\{1, 2\}$, $G_F/Q(F) \cong S_3 \times S_3$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3)2$. Now using parts (2) and (4) of [9, Theorem] we see that $|Q(a)| = 2$ and that $G_{al} \cong 2^2 \times 2^{1+8}U_4(2){:}\, 2$ with $Q(a) \leq 2^2 \times 2^{1+8}U_4(2)$.

(ii) Let $X \in \Gamma_3(\{a, l\})$, and put $\overline{G}_{alX} = G_{alX}/Q(X)$. Then $\overline{G}_{alX} \cong 2^4{:}\, S_5$ by (i). Bearing in mind that in Γ_X a may be identified with a point of a 23-set Ω_X and l with a 3-element subset of Ω_X, we see that there exists $x \in G_{alX}$ such that $b^x \neq b$. Because Γ is a string geometry, every line is incident with 3 points, and therefore $[G_{al}{:}\, G_{abl}] = 2$ with G_a acting transitively on $\Delta_1(a)$. Further, $O_2(G_{alX}) \leq G_{abl}$. Since $2^2 \times 2^{1+8} = Q(\{a, l\}) \leq G_{alX}$, this then gives $Q(\{a, l\}) \leq O_2(G_{alX}) \leq G_{abl}$ and consequently $G_{abl} \cong 2^2 \times 2^{1+8}U_4(2)$.

(iii) Since G acts flag transitively on Γ, $|\Gamma_1(a)| = [G_a{:}\, G_{al}]$, and then (iii) follows using (i).

Lemma 3.2. *Let* $a \in \Gamma_0$ *and* $b \in \Delta_1(a)$. *Then* $G_{ab} \cong 2^2{\cdot}U_6(2)$, $|\Delta_1(a)| = 3510$, *and* $|\Gamma_1(a, b)| = 693$.

Proof. Let $l \in \Gamma_1(a, b)$. We first show that $G_{abl} \neq G_{ab}$. Assume that $G_{abl} = G_{ab}$. Then by Lemma 3.1(i), (ii) $[G_a{:}\, G_{ab}] = 2.3^5.5.7.11.13$. Since $\Delta_1(a)$ is a G_a-orbit by Lemma 3.1(ii),

$$|\Delta_1(a)| = [G_a{:}\, G_{ab}] = 2|\Gamma_1(a)|,$$

using Lemma 3.1(iii). This means that for any point $x \in \Delta_1(a)$ there is a unique line in $\Gamma_1(a, x)$. Choose $X \in \Gamma_3(a, b, l)$. In Γ_X, a and b may be identified with two elements of the 23-set Ω_X by Lemma 3.1(i). However, there are 21 triads (3-element subsets of Ω_X) containing a and b, and each one corresponds to a line in $\Gamma_1(a, b, X)$, contrary to the uniqueness of l. Thus we have shown that $G_{abl} \neq G_{ab}$. From [4] (or [2]) the only proper subgroup of Fi_{22} properly containing $2 \times 2^{1+8}U_4(2)$ (apart from $2 \times 2^{1+8}U_4(2){:}\, 2$) is isomorphic to $2 \cdot U_6(2)$. So, since $G_{abl} \cong 2^2 \times 2^{1+8}U_4(2)$ by Lemma 3.1(ii), we conclude that $G_{ab} \cong 2^2 \cdot U_6(2)$. Hence $|\Delta_1(a)| = [G_a{:}\, G_{ab}] = 3510$, and by counting the set of pairs $(\{a, c\}, k)$ where $c \in \Delta_1(a)$ and $k \in \Gamma_1(a, c)$ in two different ways we also obtain $|\Gamma_1(a, b)| = 693$.

Proof of Theorem B. Let $a \in \Gamma_0$ and $b \in \Delta_1(a)$ be fixed. By Lemma 3.1 $Q(a) \leq G_{ab}$ and hence $Q(a)$ acts trivially on $\Delta_1(a)$, as G_a is transitive on $\Delta_1(a)$. So $\overline{G}_a := G_a/Q(a) \cong Fi_{22}$ acts on $\Delta_1(a)$. From Lemma 3.2 $\overline{G}_{ab} \cong 2 \cdot U_6(2)$, and thus $\overline{G}_a$ acts as a rank-3 permutation group on $\Delta_1(a)$ with the $\overline{G}_{ab}$-orbit sizes being 1, 693, and 2816. Set $\Sigma = \Delta_1(a) \cap \Delta_1(b)$. Observe that $|\Sigma| > 0$ since a line is incident with three points and so $|\Sigma| = 693$, 2816, or 3509. If the latter possibility holds, then by [1, (2.5)] $\Gamma_0 = \{a\} \cup \Delta_1(a)$. So $|\Gamma_0| = 3511$. Since $23 \mid |G_X|$ (for $X \in \Gamma_3$), we have $23 \mid |G|$ and so, as $23 \nmid |G_a|$, we get $23 \mid [G{:}\, G_a] = |\Gamma_0|$. However, $23 \nmid 3511$, and therefore $|\Sigma| \neq 3509$. Suppose $|\Sigma| = 2816$. So Σ is a G_{ab}-orbit and hence $|\Gamma_1(a, b)| = 2816$, contradicting Lemma 3.2. Therefore, $|\Sigma| = 693$. Since G_a acts transitively on $\Delta_1(a)$, this uniquely specifies the graph $\Delta_1(a)$ from which we deduce that $\Delta_1(a)$

is isomorphic to the commuting transposition graph for Fi_{22}. Employing [6, (6.2)] then yields that $G \cong \mathrm{Fi}_{23}$. Now proceeding as in Theorem A we infer that Γ is isomorphic to one of the Fi_{23}-minimal parabolic geometries, so completing the proof of Theorem B.

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