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HOW FAR CAN ONE MOVE FROM A POTENTIAL PEAK WITH SMALL INITIAL SPEED?

ANGELO BARONE-NETTO AND GAETANO ZAMPIERI

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ABSTRACT. We consider a natural Lagrangian system and show that from a point q_0 in n-space, where the potential energy V has a (weak) maximum, one can go near the boundary of any compact ball where $V(q) \leq V(q_0)$ with (arbitrarily small) nonvanishing initial speeds. The result holds true for sets which are C^2-diffeomorphic to a compact ball. This property is found as a simple consequence of the Hopf-Rinow theorem and of a theorem of Gordon. As a corollary we deduce a well-known local result, namely, a 'converse' of the Lagrange-Dirichlet theorem, thus obtained via geometric arguments.

The present note deals with properties of the solutions to the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{1}$$

where the Lagrangian function is

$$L \in C^2(\Omega \times \mathbb{R}^n ; \mathbb{R}), \quad \Omega = \Omega^\circ \subseteq \mathbb{R}^n, \quad L(q, \dot{q}) = \tfrac{1}{2}\dot{q} \cdot g(q)\dot{q} - V(q) \tag{2}$$

with symmetric matrix $g(q) > 0$ at every $q \in \Omega$ and the centered dot is the Euclidean scalar product. One says that such a Lagrangian system is natural, and the function V is called the *potential energy*. The function $\frac{1}{2}\dot{q} \cdot g(q)\dot{q} + V(q)$ is a first integral called the *energy*.

Assume that $q_0 \in \Omega$ is a point of (weak) maximum, the ball $B[q_0, r_0] = \{q : |q - q_0| \leq r_0 < \infty\} \subset \Omega$, and

$$V(q) \leq V(q_0) \quad \forall q \in B[q_0, r_0]. \tag{3}$$

It is well known that (in an open set where $E - V(q) > 0$) the geodesics of the *Jacobi metric* $(E - V(q))g(q)$ give all the solutions to the Lagrange equations with energy equal to the constant E, up to reparametrization.

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Fix $E > V(q_0)$ and $r \in (0, r_0)$; moreover, consider a real-valued C^3 function $k > 0$ defined on the open ball $B(q_0, r_0) = B[q_0, r_0]^\circ$ and such that

(4) $\quad k(q) = 1 \quad \text{for } q \in B[q_0, r] \qquad \text{and} \qquad k(q) \to +\infty \quad \text{as } |q - q_0| \to r_0 .$

So k is a *proper* function; namely, the inverse image of any compact set is compact. By a theorem of Gordon (see the Corollary in Gordon [1]) we have completeness for the manifold $B(q_0, r_0)$ endowed with the Riemannian metric

$$\exp\left(\frac{k'(q) \cdot g(q)^{-1} k'(q)}{E - V(q)}\right)(E - V(q))g(q) . \tag{5}$$

That is the geodesics are all defined on the whole of $\mathbb{R}$. In formula (5), $k'(q)$ is the gradient of k at q and $g(q)^{-1}$ is the inverse matrix of $g(q)$. Now, let us invoke the Hopf-Rinow theorem to say that any two points of the aforementioned complete manifold are connected by a geodesic (see, e.g., Hermann [3]). The complete metric in formula (5) coincides with the Jacobi metric on $B[q_0, r]$; therefore, for each $\bar{q} \in B(q_0, r)$, there is a solution $q(\cdot)$ to the Lagrange equations such that

(i) $q(0) = \bar{q}$,
(ii) $E = \frac{1}{2}\dot{q}(0) \cdot g(\bar{q})\, \dot{q}(0) + V(\bar{q}) > V(q_0)$,
(iii) $q(t_r) \in \partial B[q_0, r]$ (the boundary of the ball) for some $t_r > 0$, and
(iv) $q(t) \in B(q_0, r)$ for all $t < t_r$.

Since r can be chosen arbitrarily near r_0, for every $\epsilon > 0$ there is a solution $q(\cdot) : [0, \hat{t}] \to B(q_0, r_0)$ with energy E such that

$$q(0) = \bar{q}, \qquad \operatorname{dist}\big(q(\hat{t}); \partial B[q_0, r_0]\big) < \epsilon ,$$

where dist is the distance.

To get the previous result, we have exploited condition (3) (to have a Riemannian metric on the whole ball), and that condition does not depend on the kinetic energy function $\frac{1}{2}\dot{q} \cdot g(q)\dot{q}$. We can perform a C^2-diffeomorphism $Q(q)$ (defined on Ω), and we can consider the new potential energy $V \circ Q^{-1}$. This yields the following statement.

Theorem. *Let us consider a Lagrangian system as in* (1) *and* (2), *let* $K \subset \Omega$ *be a compact set* C^2*-diffeomorphic to a compact ball, and let* $q_0 \in K^\circ$. *If* $V|K$ *has a global* (*weak*) *maximum at* q_0, *then for every choice of* $E > V(q_0)$ *and* $\bar{q} \in K$ *there exists a sequence of solutions* $\{q_n(\cdot) : [0, t_n] \to K\}$ *with energy* E *such that*

$$q(0) = \bar{q}, \qquad \operatorname{dist}\big(q_n(t_n); \partial K\big) \to 0 \quad \textit{as } n \to \infty . \tag{6}$$

The abstract shows a direct consequence, easier to be stated than the theorem itself.

We conclude by finding a known result in Stability as a corollary of the (global) theorem above.

According to the celebrated Lagrange-Dirichlet theorem, if at a certain point q_0 the potential energy has a strict minimum, then the equilibrium solution $q(t) = q_0$ is stable. This sufficient condition for stability is not necessary, and some research is still devoted to finding sufficient conditions for instability (see Kozlov [4], Maffei, Mauro, and Negrini [5], and the references contained therein). Hagedorn [2] proved that instability occurs whenever q_0 is a strict maximum via a variational theorem of Caratheodory. In a second time, Taliaferro [6] generalized Hagedorn's result to a weak maximum (and to weaker regularity hypotheses). As a corollary of the theorem above, we have the instability of a weak maximum via purely geometric arguments.

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UNIVERSIDADE DE SÃO PAULO, INSTITUTO DE MATEMATICA E ESTATISTICA, C. P. 20570, 01452-990 SÃO PAULO, BRAZIL
E-mail address: barone@ime.usp.br

UNIVERSITÀ DI PADOVA, DIPARTIMENTO DI MATEMATICA PURA E APPLICATA, VIA BELZONI 7, 35131 PADOVA, ITALY
E-mail address: gaetano@pdmat1.unipd.it