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THE SUFFICIENT CONDITION FOR A CONVEX BODY TO ENCLOSE ANOTHER IN $\mathbb{R}^4$

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ABSTRACT. We follow Hadwiger and Ren's ideas to estimate the kinematic measure of a convex body D_1 with C^2-boundary ∂D_1 moving inside another convex body D_0 with the same kind of boundary ∂D_0 under the isometry group G in $\mathbb{R}^4$. By using Chern and Yen's kinematic fundamental formula, C-S. Chen's kinematic formula for the total square mean curvature $\int_{\partial D_0 \cap g\partial D_1} H^2\,dv$, and some well-known results about the curvatures of the 2-dimensional intersection submanifold $\partial D_0 \cap g\partial D_1$, we obtain a sufficient condition to guarantee that one convex body can enclose another in $\mathbb{R}^4$.

1. INTRODUCTION

Many mathematicians have been interested in getting sufficient conditions to insure that a given domain D_1 of surface area F_1, bounded by boundary ∂D_1, of volume V_1 can be contained in a second domain D_0 of surface area F_0, bounded by boundary ∂D_0, of volume V_0 in an n-dimensional Euclidean space $\mathbb{R}^n$. The type of condition sought is meaningful if it is 'intrinsic' or 'extrinsic', i.e., if it only depends on the volumes of D_0, D_1, the areas of the boundaries ∂D_0, ∂D_1, and some curvature invariants of D_0, D_1. Hadwiger [4, 5] was first to use the method of integral geometry to obtain some sufficient conditions for one domain to contain another in the Euclidean plane $\mathbb{R}^2$. Ren [3] derived another condition in $\mathbb{R}^2$. It is natural to try to generalize Hadwiger's theorem to higher-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 3)$. For a long time (before Zhang [9] and our works [10–14]) there was no general result for $\mathbb{R}^n$ $(n \geq 3)$. Even restricted results, with some strong conditions placed on the two domains involved (for example, convexity and some topological conditions), were not available. Hadwiger and Ren used Blaschke's and Poincaré's formulas to estimate the kinematic measure of one domain moving into another under rigid motions in $\mathbb{R}^2$. However, there is no general Poincaré formula in $\mathbb{R}^n$ $(n \geq 3)$ which can be used in our discussion. There *is* a general (or generalized) Blaschke's formula in $\mathbb{R}^n$ [1], i.e., Chern and Yen's kinematic fundamental formula. Our strategy is to find another formula to substitute for Poincaré's

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formula, upon which Hadwiger and Ren's results in $\mathbb{R}^2$ are based. There are some different extensions of Poincaré's formula [1], but none of them are applicable for the situation of $\mathbb{R}^n$. In [11] the author used C-S. Chen's formula [6] to substitute for Poincaré's formula and obtained two different conditions in $\mathbb{R}^3$ by estimating the arc length of the intersection curve $\partial D_0 \cap g\partial D_1$ of two boundaries ∂D_0 and ∂D_1 of the two convex bodies involved. In [10] we obtained a sufficient condition for n-dimensional $(n \geq 4)$ Euclidean space $\mathbb{R}^n$ by using the kinematic formula for the total scalar curvature $\int_{\partial D_0 \cap g\partial D_1} R\,dv$ of the $(n-2)$-dimensional submanifold $\partial D_0 \cap g\partial D_1$ to substitute for Poincaré's formula. Later, we removed the convexity restriction on the domains and got some analogues of Hadwiger's theorem in space $\mathbb{R}^3$ (see [12–14]).

In this paper we restrict our discussion to $\mathbb{R}^4$. There the situation is very concrete, and there are some well-known results we can use for the 2-dimensional intersection manifold $\partial D_0 \cap g\partial D_1$ of two boundaries ∂D_0 and ∂D_1. We use Chern and Yen's kinematic fundamental formula and Chen's general formulas [1, 6] to substitute for Blaschke and Poincaré's formulas and obtain a sufficient condition for $\mathbb{R}^4$. Our work is a direct application of the higher-dimensional Chern and Yen's formula. For a discussion of the situation in $\mathbb{R}^3$ and $\mathbb{R}^n$ $(n \geq 4)$, see [10–14].

2. Preliminaries

Let M^p, M^q be a pair of closed submanifolds in Euclidean space $\mathbb{R}^n$. We assume $p+q \geq n+1$ so that generically $M^p \cap gM^q$ is a submanifold of dimension $p+q-n$ for every $g \in G$, the group of motions in $\mathbb{R}^n$. Let M be a submanifold of $\mathbb{R}^n$. The second fundamental form of M at $x \in M$ is a symmetric bilinear mapping

$$T : M_x \times M_x \longrightarrow M_x^{\perp}, \tag{1}$$

where M_x is the tangent space of M at x and $M_x^{\perp}$ is the normal space of M at x. Let ∇ be the covariant derivative of $\mathbb{R}^n$ so that the directional derivative of a vector field X along another Y is $\nabla_Y X$. If M is immersed in $\mathbb{R}^n$ and X, Y are tangent to M, then at $x \in M$

$$T_X Y = \text{normal component of } \nabla_X Y \text{ with respect to } M_x. \tag{2}$$

It can be verified that T is a symmetric bilinear tensor. We shall denote the second fundamental form of M^p by T^p, where the superscript p is the dimension of the submanifold. For a smooth curve C in $\mathbb{R}^n$, T^1 has the following interpretation. Let t be a unit vector tangent to C; $\kappa = \|T_t^1 t\|$ will be the curvature of C, and $n = T_t^1 t/\kappa$ will be the normal of C. These definitions agree with the classical theory of curves in $\mathbb{R}^3$.

From now on, we assume that G is the isometry group in $\mathbb{R}^n$ and dg the kinematic density for $\mathbb{R}^n$. Let $H(\cdot)$ and $R(\cdot)$, respectively, be the mean curvature and scalar curvature of the submanifold. Assume $\tau(M^m)$ is defined by

$$\tau(M^m) = \frac{3m}{m+2} O_{m-1} \int_{M^m} H^2(M^m)\,dv - \frac{4}{m(m+2)} O_{m-1} \int_{M^m} R(M^m)\,dv, \tag{3}$$

where O_m is the surface area of the m-dimensional unit sphere and its value is given by

$$O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)}.$$

Then C-S. Chen's theorem [6] says

$$\int_{\{g\,:\,M^p \cap gM^q \neq \varnothing\}} \tau(M^p \cap gM^q)\,dg = C_0\tau(M^p)F(M^q) + C_2\tau(M^q)F(M^p), \tag{4}$$

where $F(\cdot)$ is the volume function and

$$\begin{aligned} C_0 &= \frac{O_n \cdots O_0 O_{q-1} O_{p+q-n+1} O_{p+q-n}}{O_0 O_{p-1} O_p O_{q+1} O_q}, \\ C_2 &= \frac{O_n \cdots O_0 O_{p-1} O_{p+q-n+1} O_{p+q-n}}{O_0 O_{q-1} O_q O_{p+1} O_p}. \end{aligned} \tag{5}$$

In [2] the kinematic formula for

$$\mu_2(M^m) = \frac{1}{m(m-1)}\tilde{R}(M^m) = \frac{1}{m(m-1)}\int_{M^m} R\,dv,$$

where $m = \dim M$, reads

$$\begin{aligned} &\frac{1}{(p+q-n)(p+q-n-1)}\int_{\{g\,:\,M^p \cap gM^q \neq \varnothing\}} \tilde{R}(M^p \cap gM^q)\,dg \\ &\quad = \frac{C_0}{p(p-1)}\frac{O_{p-1}}{O_{p+q-n-1}}\tilde{R}(M^p)F(M^q) \\ &\qquad + \frac{C_2}{q(q-1)}\frac{O_{q-1}}{O_{p+q-n-1}}\tilde{R}(M^q)F(M^p). \end{aligned} \tag{6}$$

Let $\tilde{H}(M^m)$ and $\tilde{R}(M^m)$ denote the total square of mean curvature and total scalar curvature of M^m, respectively, i.e.,

$$\tilde{H}(M^m) = \int_{M^m} H^2(M^m)\,dv, \qquad \tilde{R}(M^m) = \int_{M^m} R(M^m)\,dv. \tag{7}$$

After a little calculation, we obtain the kinematic formula

$$\begin{aligned} &\int_{\{g\,:\,M^p \cap gM^q \neq \varnothing\}} \tilde{H}(M^p \cap gM^q)\,dg \\ &\quad = \frac{C_0}{(p+2)(p+q-n)}\frac{O_{p-1}}{O_{p+q-n-1}} \\ &\qquad \times \left[p(p+q-n+2)\tilde{H}(M^p) - \frac{4(n-q)}{p(p-1)}\tilde{R}(M^p)\right]F(M^q) \\ &\qquad + \frac{C_2}{(q+2)(p+q-n)}\frac{O_{q-1}}{O_{p+q-n-1}} \\ &\qquad \times \left[q(p+q-n+2)\tilde{H}(M^q) - \frac{4(n-p)}{q(q-1)}\tilde{R}(M^q)\right]F(M^p). \end{aligned} \tag{8}$$

Let the boundary ∂D of a convex body D be a hypersurface of class C^2. It is known that at each point of a hypersurface Σ in $\mathbb{R}^n$ there are $n-1$ principal directions and $n-1$ principal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. If dv denotes the area element of Σ, then the rth integral of mean curvature is defined by

$$M_r(\Sigma) = \binom{n-1}{r}^{-1} \int_\Sigma \{\kappa_{i_1}, \kappa_{i_2}, \ldots, \kappa_{i_r}\}\, dv, \tag{9}$$

where $\{\kappa_{i_1}, \kappa_{i_2}, \ldots, \kappa_{i_r}\}$ denotes the rth elementary symmetric function of the principal curvatures. In particular, M_0 is the area, and M_{n-1} is a numerical multiple of the degree of mapping of Σ into the unit hypersphere defined by the field of normals. Let D_0 and D_1 be two convex bodies in $\mathbb{R}^n$ bounded by the hypersurfaces ∂D_0 and ∂D_1, which we assume to be of class C^2. $(M^0)_i$, $(M^1)_i$ are the ith integrals of mean curvature of ∂D_0 and ∂D_1, respectively. For simplicity we will denote these by M_i^0 and M_i^1. Chern and Yen's kinematic fundamental formula [1] is

$$\begin{aligned} &\int_{\{g\,:\,D_0\cap gD_1\neq\varnothing\}} dg \\ &\quad = O_{n-2}\cdots O_1\left[O_{n-1}(V_0+V_1) + \frac{1}{n}\sum_{h=0}^{n-2}\binom{n}{h+1} M_h^0 M_{n-2-h}^1\right]. \end{aligned} \tag{10}$$

Let M be an n-dimensional closed submanifold in Euclidean space $\mathbb{R}^m$, H the mean curvature of M. Then we have B-Y. Chen's formula [7]

$$\int_M |H|^n\, dv \geq O_n. \tag{11}$$

Equality holds in (11) precisely when M is embedded as an n-sphere of $\mathbb{R}^m$.

3. Main theorem

Theorem 1. *Let D_i $(i = 0, 1)$ be convex bodies in 4-dimensional Euclidean space $\mathbb{R}^4$, with C^2 boundaries ∂D_i, and let V_i, F_i, M_r^i, $\tilde{H}_i$, $\tilde{R}_i$ be the volume of D_i, the surface area of D_i, the rth integral of mean curvature of ∂D_i, the total square mean curvature, and integral of scalar curvature of ∂D_i, respectively. H_i and R_i represent, respectively, the mean curvature and scalar curvature of ∂D_i. Then a sufficient condition for D_0 to contain D_1 or for D_1 to contain D_0 is*

$$\begin{aligned} &[2\pi^2(V_0+V_1) + F_0M_2^1 + F_1M_2^0 + \tfrac{3}{2}M_1^0M_1^1] \\ &\quad - \tfrac{2}{15}[(18\tilde{H}_0 - \tilde{R}_0)F_1 + (18\tilde{H}_1 - \tilde{R}_1)F_0] > 0. \end{aligned} \tag{12}$$

Moreover,

(1) *if $V_1 \geq V_0$, then D_1 can contain D_0;*
(2) *if $V_1 \leq V_0$, then D_1 can be contained in D_0.*

This formula comes from estimating the kinematic measure of the set of rigid motions which move one convex body inside another in $\mathbb{R}^4$, i.e.,

$$
\begin{aligned}
&m\{g \in G : gD_1 \subseteq D_0 \text{ or } gD_0 \subseteq D_1\} \\
&\quad = \int_{\{g\,:\,D_0 \cap gD_1 \neq \varnothing\}} dg - \int_{\{g\,:\,\partial D_0 \cap g\partial D_1 \neq \varnothing\}} dg \\
&\quad \geq 8\pi^2 \left[2\pi^2(V_0 + V_1) + F_0 M_2^1 + F_1 M_2^0 + \frac{3}{2} M_1^0 M_1^1\right] \\
&\qquad - \frac{16\pi^2}{15}[(18\tilde{H}_0 - \tilde{R}_0)F_1 + (18\tilde{H}_1 - \tilde{R}_1)F_0].
\end{aligned} \tag{13}
$$

The equality holds if and only if the two convex bodies are balls.

4 THE PROOF OF THE MAIN RESULT

First, we estimate the integral

$$
\Phi = \int_{\{g\,:\,\partial D_0 \cap g\partial D_1 \neq \varnothing\}} dg. \tag{14}
$$

For each $g \in G$, the intersection $\partial D_0 \cap g\partial D_1$ may be composed of several connected components, i.e., $\partial D_0 \cap g\partial D_1 = \sum_{i=1}^{N_g} M_i$, where each M_i is a connected closed surface and N_g is always finite and only depends on g. By applying B-Y. Chen's formula (11) to our generic 2-dimensional submanifold $\partial D_0 \cap g\partial D_1$ we have

$$
\int_{\partial D_0 \cap g\partial D_1} H^2\, dv \geq 4\pi. \tag{15}
$$

For fixed g, equality holds in (15) if and only if $\partial D_0 \cap g\partial D_1$ is a 2-sphere in $\mathbb{R}^4$ (B-Y. Chen [7]). If equality holds in (15) for *every* g then the two convex bodies D_0 and D_1 must be balls (a result of P. Goodey [15, 16]). By (15) and (8) we have

$$
\begin{aligned}
4\pi \int_{\{g\,:\,\partial D_0 \cap g\partial D_1 \neq \varnothing\}} dg &\leq \int_{\{g\,:\,\partial D_0 \cap g\partial D_1 \neq \varnothing\}} \left(\int_{\partial D_0 \cap g\partial D_1} H^2\, dv\right) dg \\
&= \frac{64\pi^3}{15}[(18\tilde{H}_0 - \tilde{R}_0)F_1 + (18\tilde{H}_1 - \tilde{R}_1)F_0],
\end{aligned} \tag{16}
$$

i.e.,

$$
\Phi = \int_{\{g\,:\,\partial D_0 \cap g\partial D_1 \neq \varnothing\}} dg \leq \frac{16\pi^2}{15}[(18\tilde{H}_0 - \tilde{R}_0)F_1 + (18\tilde{H}_1 - \tilde{R}_1)F_0]. \tag{17}
$$

Using Chern and Yen's formula (10) and (17) we obtain

$$
\begin{aligned}
&m\{g \in G : gD_1 \subseteq D_0 \text{ or } gD_0 \subseteq D_1\} \\
&\quad = \int_{\{g\,:\,D_0 \cap gD_1 \neq \varnothing\}} dg - \int_{\{g\,:\,\partial D_0 \cap g\partial D_1 \neq \varnothing\}} dg \\
&\quad \geq 8\pi^2 \left[2\pi^2(V_0 + V_1) + F_0 M_2^1 + F_1 M_2^0 + \frac{3}{2} M_1^0 M_1^1\right] \\
&\qquad - \frac{16\pi^2}{15}[(18\tilde{H}_0 - \tilde{R}_0)F_1 + (18\tilde{H}_1 - \tilde{R}_1)F_0].
\end{aligned} \tag{18}
$$

As before, the case of equality in (16), i.e., (17) and (18), hold if the 2-dimensional manifold $\partial D_0 \cap g\partial D_1$ is a 2-sphere in $\mathbb{R}^4$ (in particular, is connected), and if this holds for almost *every* g then the convex bodies D_0 and D_1 are balls. Equality in (18) has the same consequences.

5 Remarks

1. Let D_0 and D_1 be two domains in $\mathbb{R}^4$ bounded by the hypersurfaces ∂D_0 and ∂D_1, which we assume to be of class C^2. Denote by $\chi(\cdot)$ the *Euler-Poincaré* characteristic. We have Chern and Yen's kinematic formula [1]

$$\begin{aligned}\int_{\{g:\, D_0\cap gD_1\neq\varnothing\}} &\chi(D_0\cap gD_1)\,dg \\ &= O_{n-2}\cdots O_1\left[O_{n-1}(\chi(D_0)V_1+\chi(D_1)V_0)\right. \\ &\qquad\left.+\frac{1}{n}\sum_{h=0}^{n-2}\binom{n}{h+1}M_h^0M_{n-h-2}^1\right].\end{aligned} \tag{19}$$

Then we have

Theorem 2. *Let D_i $(i=0,1)$ be two domains in 4-dimensional Euclidean space $\mathbb{R}^4$ bounded by the C^2-smooth boundaries ∂D_i. Suppose that V_i, F_i M_r^i, H_i, R_i, $\tilde{H}_i$, $\tilde{R}_i$ are as in Theorem 1. Denote by $\chi(\cdot)$ the Euler-Poincaré characteristic. Moreover, assume that for all rigid motion $g\in G$ in $\mathbb{R}^4$, $\chi(D_0\cap gD_1)\le n_0$, a finite integer. Then a sufficient condition for D_1 to enclose, or to be enclosed in, D_0 is*

$$\begin{aligned}&2\pi^2(\chi(D_0)V_1+\chi(D_1)V_0)+F_0M_2^1+F_1M_2^0+\frac{3}{2}M_1^0M_1^1 \\ &\quad-\frac{2n_0}{15}[(18\tilde{H}_0-\tilde{R}_0)F_1+(18\tilde{H}_1-\tilde{R}_1)F_0]>0.\end{aligned} \tag{20}$$

Moreover,

(1) *if $V_1\ge V_0$, then D_1 can enclose D_0;*
(2) *if $V_1\le V_0$, then D_1 can be enclosed in D_0.*

(20) comes from estimating the kinematic measure of one domain moving into another under the rigid motions in $\mathbb{R}^4$, i.e.,

$$\begin{aligned}m\{g\in G:\ &gD_1\subset D_0 \text{ or } gD_0\subset D_1\} \\ &=\int_{\{g:\, D_0\cap gD_1\neq\varnothing\}}dg-\int_{\{g:\, \partial D_0\cap g\partial D_1\neq\varnothing\}}dg \\ &\ge\frac{8\pi^2}{n_0}\left[2\pi^2(\chi(D_0)V_1+\chi(D_1)V_0)+F_0M_2^1+F_1M_2^0+\frac{3}{2}M_1^0M_1^1\right] \\ &\quad-\frac{16\pi^2}{15}[(18\tilde{H}_0-\tilde{R}_0)F_1+(18\tilde{H}_1-\tilde{R}_1)F_0].\end{aligned} \tag{21}$$

If D_0 and D_1 are convex bodies, we have $\chi(D_0)=\chi(D_1)=\chi(D_0\cap gD_1)=n_0=1$ and Theorem 2 becomes Theorem 1.

2. It would be interesting to remove the "smooth" restriction from the convex bodies involved in Theorem 1. All the notions except $\tilde{H}$ here are well defined

for nonsmooth convex bodies. In fact, we can use *Quermassintegrals* to substitute for M_r^i. If we could find other substitutions for $\tilde{H}$, the results in this paper can be interpreted for arbitrary convex bodies. This is definitely worth thinking about.

3. Of course, the sufficient conditions (12) and (20) are not necessary.

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