A CHARACTERIZATION OF POSITIVE CONSTRUCTIVE OPERATORS

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Abstract. In this note we prove that if \( T \) is a positive operator on a real Banach lattice, then \( T \) is constrictive if and only if that \( T \) has the operator matrix decomposition
\[
T = \begin{pmatrix}
T_1 & 0 \\
0 & T_2
\end{pmatrix},
\]
where \( T_1 \) is a power-bounded generalized permutation matrix on a finite-dimensional space and \( T_2^n \to 0 \) strongly.

1. Introduction

Let \( X \) be a real Banach lattice. Throughout this paper an operator means a bounded linear operator on \( X \); an operator \( T: X \to X \) is called positive if \( x \geq 0 \) implies \( Tx \geq 0 \). We say that \( T \) is constrictive (cf. [5]) if there exists a compact set \( F \subseteq X \) such that
\[
\lim_{n \to \infty} d(T^n x, F) = 0 \quad \text{whenever } \|x\| \leq 1,
\]
where \( d(y, F) = \inf\{\|y - f\| : f \in F\} \). \( T \) is said to be power bounded if \( \sup\{\|T^n\|, n = 1, 2, \ldots\} < \infty \). It is clear that a constrictive operator is power bounded by the uniformly bounded theorem. \( T \) is said to be contractive if \( \|T\| \leq 1 \).

For a vector \( x \in X \) and an operator \( T \) define \( Q(x) \) by
\[
Q(x) = \left\{ y : y = \lim_{i \to \infty} T^{n_i}x \text{ for some } n_i \to \infty \right\}.
\]
We observe that \( Q(x) \) is a nonempty compact \( T \)-invariant subset of \( X \) if \( T \) is constrictive. In this case, we denote \( Y = \bigcup_{x \in X} Q(x) \). Clearly, \( Y \) is closed and \( T \)-invariant. Moreover, by [6, Theorem 2.1], \( Y \) is also a finite-dimensional subspace of \( X \) and an invariant subspace under \( T \). We will call \( Y \) the constrictive subspace for \( T \). In this note, for notation and terminology concerning the Banach lattice not explained below, we refer the reader to [7].

In recent years, constrictive operators have been studied by many authors (see [2, 5, 6]); various interesting results concerning constrictive operators have been obtained. For example, Bartoszek in [2] has proved

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Theorem B. Let $T$ be a positive linear contraction acting on a real Banach lattice. If $T$ is constrictive, then there exists a sequence of normalized vectors $y_1, \ldots, y_r$ in $X$ such that $\lim_{n \to \infty} \|T^n(x - \sum_{i=1}^r \lambda_i(x)y_i)\| = 0$ ($x \in X$). Moreover, there exists a permutation $\delta$ of the set $\{1, \ldots, r\}$ such that $Ty_i = y_{\delta(i)}$.

The following theorem is due to Miklavcic (see [6]).

Theorem M. Suppose that $T$ is a constrictive on a real Banach lattice $X$, and the dimension of the constrictive subspace $Y$ for $T$ is equal to $N$. Then there exist $f_1, \ldots, f_N$ in $X^*$ such that for all $i, j \in \{1, \ldots, N\}$:

1. $\lim_{n \to \infty} \|T^n(x - \sum_{k=1}^N (x, f_k)e_k)\| = 0$ for all $x \in X$;
2. $(e_i, f_j) = \delta_{ij}$ (Kronecker delta);
3. $0 \leq (x, f_k) \leq M\|x\|$ for all $x \in X$;
4. $\|f_i\| \leq M M_0$;
5. $T^* f_{\rho(i)} = \lambda(i)f_i$, where $\rho(i)$ is a permutation of the set $\{1, \ldots, N\}$;
6. $\lim_{n \to \infty} (x, T^* T^n (y - \sum_{k=1}^N (e_k, y)f_k)) = 0$ for all $x \in X$ and all $y \in X^*$;

here $M = \sup_{n \geq 0} \|T^n\|$ and $M_0$ is a constant.

The purpose of this note is to provide a geometric characterization of positive constrictive operators. We will prove

Theorem. Suppose that $T$ is a linear positive operator acting on a real Banach lattice $X$. Then $T$ is constrictive if and only if $T$ has the operator matrix form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

with respect to a space decomposition $X = X_1 + X_2$, where $+$ denotes the algebraically direct sum, $X_1$ is a finite-dimensional vector lattice, $T_2^n \to 0$ (as $n \to \infty$) for any $x \in X_2$, and $T_1$ as an operator on the finite-dimensional space $X_1$ has the following properties:

1. $T_1$ is a generalized permutation matrix;
2. $T_1$ is similar to a unitary matrix;
3. there exists a positive integer $k$ such that $T_1^k = I$, where $I$ denotes the identity operator on $X_1$.

In the sequel, for convenience, we do not distinguish among identity operators acting on different spaces and denote them by $I$ in the same way.

An $n \times n$ matrix is said to be a generalized permutation matrix if it contains exactly one entry $> 0$ in each row and column and the rest of the entries are zero. An $m \times m$ matrix $J$ is called an $m$-order irreducible block (cf. [7]) if $J$ has the form

$$J = \begin{pmatrix} 0 & \lambda_1 & 0 & \ldots & 0 \\ \lambda_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m-1} & \lambda_{m-2} & \ldots & 0 \\ \lambda_m & 0 & \ldots & \lambda_1 \end{pmatrix},$$

$\lambda_i \neq 0$, $i = 1, \ldots, m$. Define $\Pi(J) := \prod_{i=1}^m \lambda_i$; then $J^m = \Pi(J)I$. 

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It is easy to see that the Theorem improves the main results in [2] and [6]. Clearly, using the Theorem we can make almost all of the properties concerning positive constrictive operators in [2] and [6] clearer.

2. Proof of the Theorem

We begin with two lemmas due to Miklavcic (see [6]).

**Lemma 1** [6, Lemma 3.8]. If $T$ is constrictive and $Y$ is its constrictive subspace, then, for each $x \in X$, there exists a unique $x_0 \in Y$ such that

$$\lim_{n \to \infty} T^n(x - x_0) = 0.$$ 

Moreover, $x_0 \in Q(x)$.

**Lemma 2** [6, Lemma 3.7]. Suppose that $T$ is constrictive and $Y$ is its constrictive subspace. If $x \in Y$, $y \in Y$, $n \geq 0$, and $T^n x = T^n y$, then $x = y$.

**Proof of the Theorem.** $\Leftarrow$ It is evident.

$\Rightarrow$ For each vector $x \in X$ we know, by Lemma 1, that there exists a unique vector $x_0 \in Y$ (the constrictive subspace of $T$) such that

$$\lim_{n \to \infty} T^n(x - x_0) = 0.$$ 

Define the operator $P$ by $Px = x_0$. We will first prove that $P$ is a positive projection acting on $X$ and $PX = Y$.

For the sake of convenience, we shall divide the proof into four steps.

(1) **Claim.** $P$ is linear.

For $\alpha, \beta \in \mathbb{R}$ (the real number field) and $x, y \in X$, by Lemma 1, there exists a unique vector $z_0 \in Y$ such that

$$\lim_{n \to \infty} T^n((\alpha x + \beta y) - z_0) = 0.$$ 

Denote $x_0 = Px$ and $y_0 = Py$. Since $x_0, y_0 \in Y$ and

$$T^n((\alpha x + \beta y) - (\alpha x_0 + \beta y_0)) = \alpha T^n(x - x_0) + \beta T^n(y - y_0) \to 0$$

(as $n \to \infty$), by the uniqueness of $z_0$, $z_0 = \alpha x_0 + \beta y_0$. This shows that $P$ is linear.

(2) **Claim.** $P$ is positive.

For $x \in X$ and $x \geq 0$, using Lemma 1 again, there exists a unique vector $x_0 \in Y$ such that $T^n(x - x_0) \to 0$. From [6, Theorem 3.11], there exist $x_0^+ \in X^+ \cap Y$ and $x_0^0 \in X^+ \cap Y$ such that $x_0 = x_0^+ - x_0^0$. Hence, to prove that $P$ is positive, it is enough to show that $x_0^+ = 0$. By [5, Theorem 1], it is easy to see that there exists a sequence $1 < n_1 < n_2 < \cdots$ such that

$$x_0 = \lim_{i \to \infty} T^{n_i} x_0 = x_0^+ - x_0^0.$$ 

Moreover, since $T$ is constrictive, there exists a subsequence $1 < n_{i_1} < n_{i_2} < \cdots$ such that the $\lim_{j \to \infty} T^{n_{i_j}} x$ exists. Denote $z = \lim_{j \to \infty} T^{n_{i_j}} x$. Since $T \geq 0$, it follows that $z \geq 0$, $z \in Y$, and $z = x_0^+ - x_0^0$. By the second assertion of [6, Theorem 3.11], $-x_0^0 = z - x_0^0 \in X^+$; therefore, $x_0^0 = 0$. By the way, from [1, Theorem 12.3], $P$ is continuous. Hence $N(P)$, the nullspace of $P$, is a closed subspace.

(3) **Claim.** $P$ is idempotent.
For $x \in X$, since $Px \in Y$ and $T^n(Px - Px) = 0$, by Lemma 1 and the definition of $P$, we have $P(Px) = Px$, so $P^2 = P$.

(4) **Claim.** $PX = Y$.

Observing $T^n(x_0 - x_0) = 0$ for $x_0 \in Y$ and the definition of $P$, $PX = Y$ is clear. It shows that $P$ is the projection $X \rightarrow Y$.

Next, we shall prove $PT = TP$. For $x \in X$, by the definition of $P$, $T^n(x - Px) \rightarrow 0$. By Lemma 1 and the definition of $P$, $PTx = TPx$, so $PT = TP$. In this case, we see that $(I - P)X$, the null space of $P$, is a complemented subspace for $Y$ and $T(I - P) = (I - P)T$. Therefore, both $PX$ and $(I - P)X$ are $T$-invariant subspaces, so $T$ has the operator matrix form

$$
T = \begin{pmatrix}
T_1 & 0 \\
0 & T_2
\end{pmatrix}
$$

with respect to the space decomposition $X = PX + (I - P)X$. By form (2), for any $z \in (I - P)X$, if $T^n_2 z \rightarrow z_0$, then $z_0 \in PX \cap (I - P)X$. Hence $z_0 = 0$, so $T^n_2 \rightarrow 0$.

It remains to show the properties of $T_1$.

(1) Since $P$ is a positive projection, by [7, Proposition III.11.5], $PX = Y$ is a vector lattice under the order induced by $X$ and a Banach lattice under a norm equivalent to the norm induced by $X$. It is clear that $T_1$ is a positive operator on $Y$ ($T_1 = T|_Y = TP = PTP$). Therefore, to show that $T_1$ is a generalized permutation matrix, by [4, Theorem 9.1] or [8, Lemma], it suffices to prove that $T_1^{-1}$ is also a positive operator on $Y$. For any pair $x_1$ and $x_2$ of vectors in $Y$, if $Tx_1 = Tx_2$, by Lemma 2, then $x_1 = x_2$. Since $Y$ is finite dimensional, $T_1^{-1}$ exists. Moreover, for $y \in Y$ and $y \geq 0$, by [5, Theorem 1], there exists a sequence $1 < n_1 < n_2 < \cdots$ such that $y = \lim_{i \rightarrow \infty} T^n_i y = \lim_{i \rightarrow \infty} T_1^n y$. In this case,

$$
T_1^{-1} y = \lim_{i \rightarrow \infty} T_i^{-1} y \geq 0,
$$

so $T_1^{-1}$ is positive. Hence $T_1$ is a generalized permutation matrix according to a basis consisting of positive normalized vectors $Y$.

(2) To prove that $T$ is similar to a unitary, it is enough to prove that both $T_1$ and $T_1^{-1}$ are power bounded. From the constrictiveness of $T$, by the principle of uniform boundedness, $T_1$ is power bounded; so is $T_1^{-1}$ by formula (3). From (1), since $T$ is a generalized permutation, there must be a permutation matrix $S$ such that

$$
ST_1S^{-1} = \begin{pmatrix}
J_1 & & \\
& J_2 & \\
& & \ddots \\
& & & J_k
\end{pmatrix}
$$

where $J_i$, $i = 1, \ldots, k$, is an $m_i$-order irreducible block. Because both $T_1$ and $T_1^{-1}$ are power bounded, it is not difficult to see that $\Pi(J_i) = 1$, $i = 1, \ldots, k$. If $m_0$ is the least common multiple of $\{m_1, \ldots, m_k\}$, then $T_1^{m_0} = 1$ (see [6, Lemma 4.7]). This completes the proof.

**Remarks.** (1) Suppose that $T$ is a constriction. Define a new norm for $X$ by

$$
\|\|x\|\| = \sup\{\|T^n x\| : n = 1, 2, \ldots\}.
$$
Directly checking, we know that $||| \cdot |||$ is a lattice norm equivalent to the norm $\| \cdot \|$ and $T$ is also a positive operator on $(X, ||| \cdot |||)$. In this case, $T$ is a contractive operator on $(X, ||| \cdot |||)$. If $T$ is a positive contraction, then $T_1$ in the Theorem is exactly a permutation matrix (cf. [2]).

(2) Suppose that $T$ is a positive constrictive operator on a Banach space. From the Theorem, it is easy to see that its constrictive subspace $Y \neq \{0\}$ if and only if there exists a nonzero vector $x_0 \geq 0$ such that $Tx_0 = x_0$.

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