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A CHARACTERIZATION OF POSITIVE CONSTRICTIVE OPERATORS

DU HONG-KE

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ABSTRACT. In this note we prove that if T is a positive operator on a real Banach lattice, then T is constrictive if and only if that T has the operator matrix decomposition

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where T_1 is a power-bounded generalized permutation matrix on a finite-dimensional space and $T_2^n \to 0$ strongly.

1. INTRODUCTION

Let X be a real Banach lattice. Throughout this paper an operator means a bounded linear operator on X; an operator $T: X \to X$ is called positive if $x \geq 0$ implies $Tx \geq 0$. We say that T is constrictive (cf. [5]) if there exists a compact set $F \subset X$ such that

$$\lim_{n\to\infty} d(T^n x, F) = 0 \quad \text{whenever } \|x\| \leq 1,$$

where $d(y, F) = \inf\{\|y - f\| : f \in F\}$. T is said to be power bounded if $\sup\{\|T^n\|, n = 1, 2, \dots\} < \infty$. It is clear that a constrictive operator is power bounded by the uniformly bounded theorem. T is said to be contractive if $\|T\| \leq 1$.

For a vector $x \in X$ and an operator T define $Q(x)$ by

$$Q(x) = \left\{ y : y = \lim_{i\to\infty} T^{n_i} x \text{ for some } n_i \to \infty \right\}.$$

We observe that $Q(x)$ is a nonempty compact T-invariant subset of X if T is constrictive. In this case, we denote $Y = \bigcup_{x\in X} Q(x)$. Clearly, Y is closed and T-invariant. Moreover, by [6, Theorem 2.1], Y is also a finite-dimensional subspace of X and an invariant subspace under T. We will call Y the constrictive subspace for T. In this note, for notation and terminology concerning the Banach lattice not explained below, we refer the reader to [7].

In recent years, constrictive operators have been studied by many authors (see [2, 5, 6]); various interesting results concerning constrictive operators have been obtained. For example, Bartoszek in [2] has proved

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Theorem B. *Let T be a positive linear contraction acting on a real Banach lattice. If T is constrictive, then there exists a sequence of normalized vectors $y_1, \dots, y_r$ in X such that $\lim_{n\to\infty} \|T^n(x - \sum_{i=1}^r \lambda_i(x)y_i)\| = 0$ $(x \in X)$. Moreover, there exists a permutation δ of the set $\{1, \dots, r\}$ such that $Ty_i = y_{\delta(i)}$.*

The following theorem is due to Miklavcic (see [6]).

Theorem M. *Suppose that T is a constriction on a real Banach lattice X, and the dimension of the constrictive subspace Y for T is equal to N. Then there exist $f_1, \dots, f_N$ in X^* such that for all i, $j \in \{1, \dots, N\}$:*

(1) $\lim_{n\to\infty} \|T^n(x - \sum_{k=1}^N (x, f_k)e_k)\| = 0$ *for all* $x \in X$;
(2) $(e_i, f_j) = \delta_{ij}$ *(Kronecker delta)*;
(3) $0 \le (x, f_k) \le M\|x\|$ *for all* $x \in X$;
(4) $\|f_i\| \le MM_0$;
(5) $T^* f_{\rho(i)} = \lambda(i) f_i$, *where* $\rho(i)$ *is a permutation of the set* $\{1, \dots, N\}$;
(6) $\lim_{n\to\infty}(x, T^{*n}(y - \sum_{k=1}^N (e_k, y)f_k)) = 0$ *for all* $x \in X$ *and all* $y \in X^*$;

here $M = \sup_{n\ge 0} \|T^n\|$ and M_0 is a constant.

The purpose of this note is to provide a geometric characterization of positive constrictive operators. We will prove

Theorem. *Suppose that T is a linear positive operator acting on a real Banach lattice X. Then T is constrictive if and only if T has the operator matrix form*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \tag{1}$$

with respect to a space decomposition $X = X_1 \dotplus X_2$, where $\dotplus$ denotes the algebraically direct sum, X_1 is a finite-dimensional vector lattice, $T_2^n \to 0$ (as $n \to \infty$) for any $x \in X_2$, and T_1 as an operator on the finite-dimensional space X_1 has the following properties:

(1) T_1 *is a generalized permutation matrix*;
(2) T_1 *is similar to a unitary matrix*;
(3) *there exists a positive integer* k *such that* $T_1^k = I$, *where* I *denotes the identity operator on* X_1.

In the sequel, for convenience, we do not distinguish among identity operators acting on different spaces and denote them by I in the same way.

An $n \times n$ matrix is said to be a generalized permutation matrix if it contains exactly one entry > 0 in each row and column and the rest of the entries are zero. An $m \times m$ matrix J is called an m-order irreducible block (cf. [7]) if J has the form

$$J = \begin{pmatrix} 0 & \lambda_1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & \lambda_{m-1} \\ \lambda_m & & & & 0 \end{pmatrix},$$

$\lambda_i \ne 0$, $i = 1, \dots, m$. Define $\Pi(J) := \prod_{i=1}^m \lambda_i$; then $J^m = \Pi(J)I$.

It is easy to see that the Theorem improves the main results in [2] and [6]. Clearly, using the Theorem we can make almost all of the properties concerning positive constrictive operators in [2] and [6] clearer.

2. Proof of the Theorem

We begin with two lemmas due to Miklavcic (see [6]).

Lemma 1 [6, Lemma 3.8]. *If T is constrictive and Y is its constrictive subspace, then, for each $x \in X$, there exists a unique $x_0 \in Y$ such that*

$$\lim_{n\to\infty} T^n(x - x_0) = 0.$$

Moreover, $x_0 \in Q(x)$.

Lemma 2 [6, Lemma 3.7]. *Suppose that T is constrictive and Y is its constrictive subspace. If $x \in Y$, $y \in Y$, $n \geq 0$, and $T^n x = T^n y$, then $x = y$.*

Proof of the Theorem. $\Leftarrow$ It is evident.

$\Rightarrow$ For each vector $x \in X$ we know, by Lemma 1, that there exists a unique vector $x_0 \in Y$ (the constrictive subspace of T) such that

$$\lim_{n\to\infty} T^n(x - x_0) = 0.$$

Define the operator P by $Px = x_0$. We will first prove that P is a positive projection acting on X and $PX = Y$.

For the sake of convenience, we shall divide the proof into four steps.

(1) *Claim.* P is linear.

For α, $\beta \in R$ (the real number field) and x, $y \in X$, by Lemma 1, there exists a unique vector $z_0 \in Y$ such that

$$\lim_{n\to\infty} T^n((\alpha x + \beta y) - z_0) = 0.$$

Denote $x_0 = Px$ and $y_0 = Py$. Since x_0, $y_0 \in Y$ and

$$T^n((\alpha x + \beta y) - (\alpha x_0 + \beta y_0)) = \alpha T^n(x - x_0) + \beta T^n(y - y_0) \to 0$$

(as $n \to \infty$), by the uniqueness of z_0, $z_0 = \alpha x_0 + \beta y_0$. This shows that P is linear.

(2) *Claim.* P is positive.

For $x \in X$ and $x \geq 0$, using Lemma 1 again, there exists a unique vector $x_0 \in Y$ such that $T^n(x - x_0) \to 0$. From [6, Theorem 3.11], there exist $x_+^0 \in X^+ \cap Y$ and $x_-^0 \in X^+ \cap Y$ such that $x_0 = x_+^0 - x_-^0$. Hence, to prove that P is positive, it is enough to show that $x_-^0 = 0$. By [5, Theorem 1], it is easy to see that there exists a sequence $1 < n_1 < n_2 < \cdots$ such that

$$x_0 = \lim_{i\to\infty} T^{n_i} x_0 = x_+^0 - x_-^0.$$

Moreover, since T is constrictive, there exists a subsequence $1 < n_{i_1} < n_{i_2} < \cdots$ such that the $\lim_{j\to\infty} T^{n_{i_j}} x$ exists. Denote $z = \lim_{j\to\infty} T^{n_{i_j}} x$. Since $T \geq 0$, it follows that $z \geq 0$, $z \in Y$, and $z = x_+^0 - x_-^0$. By the second assertion of [6, Theorem 3.11], $-x_-^0 = z - x_+^0 \in X^+$; therefore, $x_-^0 = 0$. By the way, from [1, Theorem 12.3], P is continuous. Hence $N(P)$, the nullspace of P, is a closed subspace.

(3) *Claim.* P is idempotent.

For $x \in X$, since $Px \in Y$ and $T^n(Px - Px) = 0$, by Lemma 1 and the definition of P, we have $P(Px) = Px$, so $P^2 = P$.

(4) *Claim.* $PX = Y$.

Observing $T^n(x_0 - x_0) = 0$ for $x_0 \in Y$ and the definition of P, $PX = Y$ is clear. It shows that P is the projection $X \to Y$.

Next, we shall prove $PT = TP$. For $x \in X$, by the definition of P, $T^n(x - Px) \to 0$. By Lemma 1 and the definition of P, $PTx = TPx$, so $PT = TP$. In this case, we see that $(I - P)X$, the null space of P, is a complemented subspace for Y and $T(I-P) = (I-P)T$. Therefore, both PX and $(I - P)X$ are T-invariant subspaces, so T has the operator matrix form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \tag{2}$$

with respect to the space decomposition $X = PX \dot{+} (I-P)X$. By form (2), for any $z \in (I-P)X$, if $T_2^n z \to z_0$, then $z_o \in PX \cap (I-P)X$. Hence $z_0 = 0$, so $T_2^n \to 0$.

It remains to show the properties of T_1.

(1) Since P is a positive projection, by [7, Proposition III.11.5], $PX = Y$ is a vector lattice under the order induced by X and a Banach lattice under a norm equivalent to the norm induced by X. It is clear that T_1 is a positive operator on Y $(T_1 = T|_Y = TP = PTP)$. Therefore, to show that T_1 is a generalized permutation matrix, by [4, Theorem 9.1] or [8, Lemma], it suffices to prove that T_1^{-1} is also a positive operator on Y. For any pair x_1 and x_2 of vectors in Y, if $Tx_1 = Tx_2$, by Lemma 2, then $x_1 = x_2$. Since Y is finite dimensional, T_1^{-1} exists. Moreover, for $y \in Y$ and $y \geq 0$, by [5, Theorem 1], there exists a sequence $1 < n_1 < n_2 < \cdots$ such that $y = \lim_{i\to\infty} T^{n_i}y = \lim_{i\to\infty} T_1^{n_i}y$. In this case,

$$T_1^{-1}y = \lim_{i\to\infty} T^{n_i - 1}y \geq 0, \tag{3}$$

so T_1^{-1} is positive. Hence T_1 is a generalized permutation matrix according to a basis consisting of positive normalized vectors Y.

(2) To prove that T is similar to a unitary, it is enough to prove that both T_1 and T_1^{-1} are power bounded. From the constrictiveness of T, by the principle of uniform boundedness, T_1 is power bounded; so is T_1^{-1} by formula (3). From (1), since T is a generalized permutation, there must be a permutation matrix S such that

$$ST_1S^{-1} = \begin{pmatrix} J & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

where J_i, $i = 1, \ldots, k$, is an m_i-order irreducible block. Because both T_1 and T_1^{-1} are power bounded, it is not difficult to see that $\Pi(J_i) = 1$, $i = 1, \ldots, k$. If m_0 is the least common multiple of $\{m_1, \ldots, m_k\}$, then $T_1^{m_0} = 1$ (see [6, Lemma 4.7]). This completes the proof.

Remarks. (1) Suppose that T is a constriction. Define a new norm for X by

$$|||x||| = \sup\{\|T^n x\| : n = 1, 2, \ldots\}.$$

Directly checking, we know that $|||\cdot|||$ is a lattice norm equivalent to the norm $\|\cdot\|$ and T is also a positive operator on $(X, |||\cdot|||)$. In this case, T is a contractive operator on $(X, |||\cdot|||)$. If T is a positive contraction, then T_1 in the Theorem is exactly a permutation matrix (cf. [2]).

(2) Suppose that T is a positive constrictive operator on a Banach space. From the Theorem, it is easy to see that its constrictive subspace $Y \neq \{0\}$ if and only if there exists a nonzero vector $x_0 \geq 0$ such that $Tx_0 = x_0$.

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DEPARTMENT OF MATHEMATICS, SHAANXI NORMAL UNIVERSITY, XI'AN, 710062 PEOPLE'S REPUBLIC OF CHINA

Current address: Department of Mathematics, Indiana University, Swain East, Bloomington, Indiana 47405