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PERTURBATION OF SPECTRUMS OF 2×2 OPERATOR MATRICES

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ABSTRACT. In this paper, we study the perturbation of spectrums of 2×2 operator matrices such as $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ on the Hilbert space $H \oplus K$. For given A and B, we prove that

$$\bigcap_{C \in B(K,H)} \sigma(M_C) = \sigma_\pi(A) \cup \sigma_\delta(B) \cup \{\lambda \in C : n(B-\lambda) \neq d(A-\lambda)\},$$

where $\sigma(T)$, $\sigma_\pi(T)$, $\sigma_\delta(T)$, $n(T)$, and $d(T)$ denote the spectrum of T, approximation point spectrum, defect spectrum, nullity, and deficiency, respectively. Some related results are obtained.

1. INTRODUCTION

Let H and K be Hilbert spaces. Throughout this paper "operator" shall always mean "bounded linear operator". In this note, we will discuss the perturbation of spectrums of 2×2 upper triangular operator matrices on $H \oplus K$ as the following form:

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}. \tag{1}$$

Although 2×2 upper triangular operator matrices have been frequently used in many literatures (see [1]), its perturbations of spectrums have not been considered significantly thus far.

In order to state our main result, let us introduce some notation and then continue with a simple lemma. $B(H)$, $B(K)$, and $B(K, H)$ denote the set of all linear bounded operators on H, on K, and from K into H, respectively. For an operator T, $\sigma_\pi(T) = \{\lambda \in C:$ there is a sequence $\{x_n\}$ of unit vectors such that $(T-\lambda)x_n \to 0\}$ is the approximation point spectrum of T. $\sigma_\delta(T) = \{\lambda \in C : T - \lambda$ is not surjective$\}$ is the defect spectrum of $T \cdot n(T)$ is the nullity of T which is equal to $\dim N(T)$. $d(T)$ is the deficiency of T which is equal to $\dim N(T^*)$. $N(T)$ denotes the null space of an operator T. $R(T)$ is the range of T. $\sigma(T)$ is the spectrum of T.

Lemma 1. *If $A \in B(H)$, $B \in B(K)$, and $C \in B(K, H)$, then $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$.*

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Proof. Since

$$\begin{bmatrix} I & 0 \\ 0 & nI \end{bmatrix}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \frac{1}{n}I \end{bmatrix} = \begin{bmatrix} A & \frac{1}{n}C \\ 0 & B \end{bmatrix},$$

letting $n \to \infty$, we have the limit

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \lim_{n\to\infty}\begin{bmatrix} A & \frac{1}{n}C \\ 0 & B \end{bmatrix}.$$

Thus, by the upper semicontinuity of spectrum, $\sigma(M_C) \subset \sigma(\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]) = \sigma(A) \cup \sigma(B)$.

From the above lemma, some natural questions arise.

Question 1. Is there an operator $C \in B(K, H)$ such that the inclusion in Lemma 1 is proper for given A and B?

Question 2. $\bigcap_{C\in B(K,H)} \sigma(M_C) = ?$.

Question 3. Is there an operator $C_0 \in B(K, H)$ such that

$$\sigma(M_{C_0}) = \bigcap_{C\in B(K,H)} \sigma(M_C)$$

for a given pair (A, B) of operators, where $A \in B(H)$ and $B \in B(K)$?

In this note, we give complete answers for Questions 1 and 2, but Question 3 is still open.

Theorem 2. *For a given pair* (A, B) *of operators, we have*

$$\bigcap_{C\in B(K,H)} \sigma(M_C) = \sigma_\pi(A) \cup \sigma_\delta(B) \cup \{\lambda \in C : n(B-\lambda) \neq d(A-\lambda)\}.$$

Proof. We will divide the proof into two steps.

In the first step we shall prove that the left side of the above equality includes the right side.

Claim. $\sigma_\pi(A) \cup \sigma_\delta(B) \subset \bigcap_{C\in B(K,H)} \sigma(M_C)$.

For any $C \in B(K, H)$, if $\lambda \in \sigma_\pi(A)$, then there exists a sequence $\{x_n\}$ of unit vectors in H such that $(A-\lambda)x_n \to 0$ (as $n \to \infty$). For M_C, consider the sequence $\{\binom{x_n}{0}\}$ of unit vectors in $H \oplus K$. We have

$$(M_C - \lambda)\begin{pmatrix} x_n \\ 0 \end{pmatrix} = \begin{pmatrix} (A-\lambda)x_n \\ 0 \end{pmatrix} \to 0,$$

so $\lambda \in \sigma(M_C)$. If $\lambda \in \sigma_\delta(B)$ since $\overline{\lambda} \in \delta_\pi(B^*)$, by the similar argument we have $\overline{\lambda} \in \sigma(M_C^*)$; therefore, $\lambda \in \sigma(M_C)$.

Claim. If $\lambda \in \{\lambda \in C : d(A-\lambda) \neq n(B-\lambda)\} \backslash (\sigma_\pi(A) \cup \sigma_\delta(B))$, then $\lambda \in \sigma(M_C)$ for any $C \in B(K, H)$.

We shall divide this claim into two cases to consider.

Case 1. Assume that $n(B-\lambda) < d(A-\lambda)$. Then $R(A-\lambda) + CN(B-\lambda) \neq H$. Take a nonzero vector $y_0 \in H \setminus (R(A-\lambda) + CN(B-\lambda))$; we will prove that $y_0 \notin R(M_C - \lambda)$. To do this it has the decomposition $x = y + z$ for any vector $x \in H \oplus K$, where $y \in H$ and $z \in K$. Thus we have

$$(M_C - \lambda)x = (A-\lambda)y + Cz + (B-\lambda)z.$$

If there exists a vector x with $(M_C - \lambda)x = y_0$, since $y_0 \in H$, we must have $(B-\lambda)z = 0$. Therefore, $(A-\lambda)y + Cz = y_0 \in R(A-\lambda) + CN(B-\lambda)$, but it contradicts the hypothesis about y_0. Hence $\lambda \in \sigma_\delta(M_C) \subset \sigma(M_C)$.

Case 2. Assume that $n(B-\lambda) > d(A-\lambda)$. Then $d(A-\lambda) < \infty$. If $N(C) \cap N(B-\lambda) \neq \{0\}$, then, for any nonzero vector $z_0 \in N(C) \cap N(B-\lambda)$, we have $M_C z_0 = 0$; therefore, $\lambda \in \sigma_p(M_C) \subseteq \sigma(M)$ ($\sigma_p(T)$ denotes the point spectrum of an operator T). If $N(C) \cap N(B-\lambda) = \{0\}$, then

$$\dim CN(B-\lambda) = \dim N(B-\lambda) = n(B-\lambda) > d(A-\lambda).$$

Therefore, $R(A-\lambda) \cap CN(B-\lambda) \neq \{0\}$. Take a nonzero vector $y_1 \in R(A-\lambda) \cap CN(B-\lambda)$. Then there exist vectors $y_2 \in H$ and $z_2 \in K$ with $(A-\lambda)y_2 = y_1 = Cz_2$ and $z_2 \in N(B-\lambda) \setminus \{0\}$ so that

$$(M_C - \lambda)(y_2 - z_2) = (A-\lambda)y_2 - Cz_2 - (B-\lambda)z_2 = 0.$$

Then $\lambda \in \sigma_p(M_C) \subseteq \sigma(M_C)$.

In the second step we will show that the converse inclusion is also true.

If $\lambda \in \bigcap_{C \in B(K,H)} \sigma(M_C)$, it means that, for any operator $C \in B(K, H)$, $\lambda \in \sigma(M_C)$. To complete the proof, it is sufficient to show that, if $\lambda \notin \sigma_\pi(A) \cup \sigma_\delta(B)$ and $d(A-\lambda) = n(B-\lambda)$, we may choose an operator $C_0 \in B(K, H)$ such that $\lambda \notin \sigma(M_{C_0})$.

Since $n(B-\lambda) = d(A-\lambda)$, there exist an orthonormal basis $\{g_i\}_{i=1}^n$ of $N(B-\lambda)$ and an orthonormal basis $\{f_i\}_{i=1}^n$ of $R(A-\lambda)^\perp$ (n is not necessarily finite). Define an operator C_0 from K into H by

$$\begin{cases} C_0 g_i = f_i, & i = 1, 2, \ldots, n, \\ C_0 g = 0, & g \in N(B-\lambda)^\perp (\subset K). \end{cases}$$

We shall prove $\lambda \notin \sigma(M_{C_0})$. To do this, we will prove that M_{C_0} is injective and surjective.

If there exists a vector $x = y + z$, $y \in H$ and $z \in K$, with $(M_{C_0} - \lambda)x = (M_{C_0} - \lambda)(y+z) = 0$, then $z \in N(B-\lambda)$ and $C_0 z = -(A-\lambda)y$. By the definition of C_0, $C_0 z \in R(A-\lambda)^\perp$; thus $C_0 z = 0$. Moreover, since C_0 is injective on $N(B-\lambda)$, we have $z = 0$. Therefore, $(A-\lambda)y = 0$, but we assume that $\lambda \notin \sigma_\pi(A)$. Hence $y = 0$, so M_{C_0} is injective.

Now we will show that M_{C_0} is surjective.

For any vector $x_0 = y_0 + z_0$, $y_0 \in H$ and $z_0 \in K$, since $\lambda \notin \sigma_\delta(B-\lambda)$, it follows that $R(B-\lambda) = K$. Then there must be a vector $z_1 \in K$ such that $(B-\lambda)z_1 = z_0$. On the other hand, since $\lambda \notin \sigma_\pi(A)$, $R(A-\lambda)$ is closed. Thus $R(A-\lambda) \oplus R(A-\lambda)^\perp = H$. Hence we can assume that $y_0 = \xi_0 + \eta_0$, where $\xi_0 \in R(A-\lambda)$ and $\eta_0 \in R(A-\lambda)^\perp$. So there exist vectors $y_1 \in H$ and $z_2 \in K$ such that $(A-\lambda)y_1 = \xi_0$ and $C_0 z_1 + \eta_0 = -C_0 z_2$. Note that since C_0 is onto $R(A-\lambda)^\perp$, the last equality is possible. Thus

$$\begin{aligned}(M_{C_0} - \lambda)(y_1 + z_1 + z_2) &= (A-\lambda)y_1 + C_0(z_1 + z_2) + (B-\lambda)(z_1 + z_2) \\ &= \xi_0 + \eta_0 + (B-\lambda)z_1 = y_0 + z_0 = x_0.\end{aligned}$$

Because x_0 is arbitrary, $M_{C_0} - \lambda$ is surjective. The proof is complete.

A simple example will show that the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ may be proper.

Example 3. If $\{g_i\}_{i=1}^{\infty}$ is an orthonormal basis of K, define an operator B_0 by

$$\begin{cases} B_0 g_1 = 0, \\ B_0 g_i = g_{i-1}, \qquad i = 2, 3, \dots . \end{cases}$$

If $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis of H, define an operator A_0 by $A_0 f_i = f_{i+1}$, $i = 1, 2, \dots$, and an operator C_0 by $C_0 = (\cdot, g_1)f_1$ from K into H. Then it is easy to see that $\sigma(A_0) = \sigma(B_0) = \{\lambda : |\lambda| \le 1\}$. But, in this case, M_{C_0} is a unitary operator, $\sigma(M_{C_0}) \subseteq \{\lambda : |\lambda|\}$, so the inclusion $\sigma(M_{C_0}) \subset \sigma(A) \cup \sigma(B)$ is proper.

The above example is an affirmative answer to Question 1.

Here we need to point out that although the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ may be proper, the spectral radius of M_C is always a constant which is independent of C and equal to $\max\{r_\sigma(A), r_\sigma(B)\}$, where $r_\sigma(T)$ denotes the spectral radius of an operator T. This is the following proposition.

Proposition 4. *For given operators A and B, $r_\sigma(M_C)$ is a constant.*

Proof. Note that since $\sigma_\pi(A) \cup \sigma_\delta(B) \subset \sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, Proposition 4 is clear.

At this point, one naturally asks which kinds of spectrums in $\sigma(A)$ and $\sigma(B)$ can be perturbed out by choosing a suitable operator $C \in B(K, H)$. Now we shall answer this question.

Theorem 5. *Assume that there exists an operator $C \in B(K, H)$ such that the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ is proper. Then for any $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$, we have $\lambda \in \sigma(A) \cap \sigma(B)$, $R(B - \lambda) = K$, $R(A^* - \bar{\lambda}) = H$, and $n(B - \lambda) = d(A - \lambda)$.*

Proof. The last part of the theorem was contained in Theorem 2, so we only need to prove that, for an operator $C \in B(K, H)$, if $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$, then $R(B - \lambda) = K$, $R(A^* - \bar{\lambda}) = H$, and $\lambda \in \sigma(A) \cap \sigma(B)$.

Without loss of generality, assume that $\lambda = 0$. Then $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ is invertible and it is clear that $R(B) = K$ and $R(A^*) = H$.

Next, we shall prove that $\lambda \in \sigma(A \cap \sigma(B)$. In the converse, assume that $0 \in \sigma(A)$ but $0 \notin \sigma(B)$. Since $\sigma_\pi(A) \subset \sigma(M_C)$ (by Lemma 1), $0 \notin \sigma_\pi(A)$. Consider the adjoint M_C^* of M_C

$$M_C^* = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix}.$$

Then $N(A^*) \ne \{0\}$. Take a nonzero vector $y \in N(A^*)$. Under the assumption $0 \notin \sigma(B)$, since B^* is invertible, $R(B^*) = K$, so we may find a vector $z \in K$ with $B^* z = -C^* y$ (it is easy to see $C \ne 0$). Therefore, we have

$$M_C^* \begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} A^* y \\ C^* y + B^* z \end{pmatrix} = 0;$$

that is, $0 \in \sigma(M_C^*)$, i.e., $0 \in \sigma(M_C)$. This contradicts the hypothesis. A similar argument will show that $0 \in \sigma(B)$ and $0 \notin \sigma(A)$ simultaneously holding is impossible too. The proof is finished.

Remark 6. By Theorem 5 and its proof, we see that the part of $\sigma(A) \cup \sigma(B)$ perturbed out by choosing a suitable operator $C \in B(K, H)$ is not only in the intersection of $\sigma(A)$ and $\sigma(B)$ but also in the intersection of semi-Fredholm domains of A and B, whose index is not zero.

Using Theorem 5, we may immediately give an example that, for a given pair (A, B) of operators, $\sigma(M_C)$ is invariant for any $C \in B(K, H)$.

Example 7. If $A \in B(H)$ and $B \in B(K)$ are normal operators, then, for any $C \in B(K, H)$, $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.

We need to point out that using results obtained in this note may lead to simpler proofs of propositions in [4]. In [4] some properties of the generalized deriviation were considered. Recall that the generalized derivation δ_{AB} induced by operators A and B is defined by

$$\delta_{AB}\colon X \to AX - XB, \qquad X \in B(K, H).$$

We have the following inclusions:

$$\begin{aligned}\{C : C \in R(\delta_{AB})\} &\subset \left\{C : \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\} \\ &\subset \left\{C : \sigma\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right\}.\end{aligned}$$

By the discussion in this note, we easily see that, in general, all of these inclusions are proper. Under which conditions with operators A and B will these inclusions instead be of equalities? The answer to this question is not clear now.

Finally, we turn to the so-called four block operator matrices. This subject was suggested by Professor C. Foias.

For an operator G_X defined on $H \oplus K$ by

$$G_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix}, \tag{2}$$

if operators A, B, and C are given and X is taken over $B(H, K)$, (2) is called a four block operator matrix. For the minimal norm of G_X we have known a famous theorem for a long time (see [2] and [3]).

Theorem P (Parott [5]). *If given A, B, and C, then*

$$\min\left\{\left\|\begin{bmatrix} A & C \\ X & B \end{bmatrix}\right\|, X \in B(H, K)\right\} = \max\left\{\|(AC)\|, \left\|\begin{pmatrix} C \\ B \end{pmatrix}\right\|\right\}.$$

But, for spectrum of G_X, what can we say? As in Proposition 4, for 2×2 upper triangular operator matrices, when A and B are given, then $r_\sigma(M_C)$ is a constant. However, for four block operator matrices, when A, B, and C are given and $C \neq 0$, in general, the spectral radius $r_\sigma(G_X)$ of G_X may be large enough. We have

Theorem 8. *For a four block operator matrix* (2), *if A, B, and C are given and $C \neq 0$, then, for any $\lambda \in \rho(A)$ ($\rho(A)$ is the resolvent of A), there exists a one-rank operator $X \in B(H, K)$ such that $\lambda \in \sigma_p(G_X)$.*

Proof. Since $C \neq 0$, there exists a vector $x_2 \in K$ with $Cx_2 \neq 0$. By the assumption of $\lambda \in \rho(A)$ and by putting

$$x_1 = -(A - \lambda)^{-1} C x_2,$$

considering the one-rank operator

$$X = \frac{1}{\|(A-\lambda)^{-1}/Cx_2\|^2}(\cdot\,,(A-\lambda)^{-1}Cx_2)(B-\lambda)x_2$$

and letting $x = -(A-\lambda)^{-1}Cx_2 + x_2$, one obtains

$$\begin{aligned}(G_X-\lambda)x &= \begin{bmatrix} A-\lambda & C \\ X & B-\lambda \end{bmatrix} x \\ &= \begin{bmatrix} A-\lambda & C \\ \frac{(\cdot\,,(A-\lambda)^{-1}Cx_2)}{\|(A-\lambda)^{-1}Cx_2\|^2}(B-\lambda)x_2 & B-\lambda \end{bmatrix} \begin{pmatrix} -(A-\lambda)^{-1}Cx_2 \\ x_2 \end{pmatrix} = 0.\end{aligned}$$

So $\lambda \in \sigma_p(G_X)$. The proof is completed.

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References

1. J. B. Conway and B. B. Morrel, *Surveys of some resent results in operator theory*, Vol. 1, Pitman Res. Notes Math. Ser., vol. 171, Longman Sci. Tech., Harlow, 1988.
2. C. Foias, *Commutant lifting techniques for computing optimal H^∞ controller*, Indiana University at Bloomington, 1990.
3. C. Gu, *On the four block problem and simultaneous H^∞ and L^2 suboptimization*, preprint.
4. A. Schweinsberg, *Similarity orbits and the range of the generalized derivation $X \to MX - XN$*, Trans. Amer. Math. Soc. **324** (1991), 201–211.
5. S. C. Power, *Hankel operators on Hilbert space*, Pitman Res. Notes Math. Ser., vol. 64, Longman Sci. Tech., Harlow, 1982.

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