PERTURBATION OF SPECTRUMS OF $2 \times 2$ OPERATOR MATRICES

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Abstract. In this paper, we study the perturbation of spectra of $2 \times 2$ operator matrices such as $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ on the Hilbert space $H \oplus K$. For given $A$ and $B$, we prove that

$$\bigcap_{C \in B(K, H)} \sigma(M_C) = \sigma_\pi(A) \cup \sigma_\delta(B) \cup \{ \lambda \in \mathbb{C} : n(B - \lambda) \neq d(A - \lambda) \},$$

where $\sigma(T)$, $\sigma_\pi(T)$, $\sigma_\delta(T)$, $n(T)$, and $d(T)$ denote the spectrum of $T$, approximation point spectrum, defect spectrum, nullity, and deficiency, respectively. Some related results are obtained.

1. Introduction

Let $H$ and $K$ be Hilbert spaces. Throughout this paper "operator" shall always mean "bounded linear operator". In this note, we will discuss the perturbation of spectra of $2 \times 2$ upper triangular operator matrices on $H \oplus K$ as the following form:

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$  

Although $2 \times 2$ upper triangular operator matrices have been frequently used in many literatures (see [1]), its perturbations of spectra have not been considered significantly thus far.

In order to state our main result, let us introduce some notation and then continue with a simple lemma. $B(H)$, $B(K)$, and $B(K, H)$ denote the set of all linear bounded operators on $H$, on $K$, and from $K$ into $H$, respectively. For an operator $T$, $\sigma_\pi(T) = \{ \lambda \in \mathbb{C} : \text{there is a sequence } \{ x_n \} \text{ of unit vectors } \text{such that } (T - \lambda)x_n \to 0 \}$ is the approximation point spectrum of $T$. $\sigma_\delta(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not surjective} \}$ is the defect spectrum of $T$. $n(T)$ is the nullity of $T$ which is equal to $\dim N(T)$. $d(T)$ is the deficiency of $T$ which is equal to $\dim N(T^*)$. $N(T)$ denotes the null space of an operator $T$. $R(T)$ is the range of $T$. $\sigma(T)$ is the spectrum of $T$.

Lemma 1. If $A \in B(H)$, $B \in B(K)$, and $C \in B(K, H)$, then $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$.

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Proof. Since
\[
\begin{bmatrix}
I & 0 \\
0 & nI
\end{bmatrix}
\begin{bmatrix}
A & C \\
0 & B
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & \frac{1}{n}I
\end{bmatrix} = \begin{bmatrix}
A & \frac{1}{n}C \\
0 & B
\end{bmatrix},
\]
letting \( n \to \infty \), we have the limit
\[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix} = \lim_{n \to \infty} \begin{bmatrix}
A & \frac{1}{n}C \\
0 & B
\end{bmatrix}.
\]
Thus, by the upper semicontinuity of spectrum, \( \sigma(M_C) \subseteq \sigma([A \ 0]_B) = \sigma(A) \cup \sigma(B) \).

From the above lemma, some natural questions arise.

**Question 1.** Is there an operator \( C \in B(K, H) \) such that the inclusion in Lemma 1 is proper for given \( A \) and \( B \)?

**Question 2.** \( \bigcap_{C \in B(K, H)} \sigma(M_C) = \).?

**Question 3.** Is there an operator \( C_0 \in B(K, H) \) such that
\[
\sigma(M_{C_0}) = \bigcap_{C \in B(K, H)} \sigma(M_C)
\]
for a given pair \((A, B)\) of operators, where \( A \in B(H) \) and \( B \in B(K) \)?

In this note, we give complete answers for Questions 1 and 2, but Question 3 is still open.

**Theorem 2.** For a given pair \((A, B)\) of operators, we have
\[
\bigcap_{C \in B(K, H)} \sigma(M_C) = \sigma(A) \cup \sigma(B) \cup \{\lambda \in C : n(B - \lambda) \neq d(A - \lambda)\}.
\]

**Proof.** We will divide the proof into two steps.

In the first step we shall prove that the left side of the above equality includes the right side.

**Claim.** \( \sigma(A) \cup \sigma(B) \subseteq \bigcap_{C \in B(K, H)} \sigma(M_C) \).

For any \( C \in B(K, H) \), if \( \lambda \in \sigma(A) \), then there exists a sequence \( \{x_n\} \) of unit vectors in \( H \) such that \((A - \lambda)x_n \to 0 \) (as \( n \to \infty \)). For \( M_C \), consider the sequence \( \{(x_n^0)\} \) of unit vectors in \( H \oplus K \). We have
\[
(M_C - \lambda)\begin{pmatrix} x_n \\ 0 \end{pmatrix} = \begin{pmatrix} (A - \lambda)x_n \\ 0 \end{pmatrix} \to 0,
\]
so \( \lambda \in \sigma(M_C) \). If \( \lambda \in \sigma(B) \) since \( \bar{\lambda} \in \delta_\pi(B^*) \), by the similar argument we have \( \bar{\lambda} \in \sigma(M_C^*) \); therefore, \( \lambda \in \sigma(M_C) \).

**Claim.** If \( \lambda \in \{\lambda \in C : d(A - \lambda) \neq n(B - \lambda)\} \setminus \sigma(A) \cup \sigma(B) \), then \( \lambda \in \sigma(M_C) \) for any \( C \in B(K, H) \).

We shall divide this claim into two cases to consider.

**Case 1.** Assume that \( n(B - \lambda) < d(A - \lambda) \). Then \( R(A - \lambda) + CN(B - \lambda) \neq H \).

Take a nonzero vector \( y_0 \in H \setminus (R(A - \lambda) + CN(B - \lambda)) \); we will prove that \( y_0 \notin R(M_C - \lambda) \). To do this it has the decomposition \( x = y + z \) for any vector \( x \in H \oplus K \), where \( y \in H \) and \( z \in K \). Thus we have
\[
(M_C - \lambda)x = (A - \lambda)y + Cz + (B - \lambda)z.
\]
If there exists a vector \( x \) with \((M_C - \lambda)x = y_0\), since \( y_0 \in H \), we must have \((B - \lambda)z = 0\). Therefore, \((A - \lambda)y + Cz = y_0 \in R(A - \lambda) + CN(B - \lambda)\), but it contradicts the hypothesis about \( y_0 \). Hence \( \lambda \in \sigma_\delta(M_C) \subset \sigma(M_C)\).

Case 2. Assume that \( n(B - \lambda) > d(A - \lambda)\). Then \( d(A - \lambda) < \infty \). If \( N(C) \cap N(B - \lambda) \neq \{0\} \), then for any nonzero vector \( z_0 \in N(C) \cap N(B - \lambda)\), we have \( M_Cz_0 = 0\); therefore, \( \lambda \in \sigma_p(M_C) \subseteq \sigma(M) \) (\( \sigma_p(T) \) denotes the point spectrum of an operator \( T \)). If \( N(C) \cap N(B - \lambda) = \{0\} \), then

\[
\dim CN(B - \lambda) = \dim N(B - \lambda) = n(B - \lambda) > d(A - \lambda).
\]

Therefore, \( R(A - \lambda) \cap CN(B - \lambda) \neq \{0\} \). Take a nonzero vector \( y_1 \in R(A - \lambda) \cap CN(B - \lambda)\). Then there exist vectors \( y_2 \in H \) and \( z_2 \in K \) with \((A - \lambda)y_2 = y_1 = Cz_2\) and \( z_2 \in N(B - \lambda) \setminus \{0\}\) so that

\[
(M_C - \lambda)(y_2 - z_2) = (A - \lambda)y_2 - Cz_2 - (B - \lambda)z_2 = 0.
\]

Then \( \lambda \in \sigma_p(M_C) \subseteq \sigma(M_C)\).

In the second step we will show that the converse inclusion is also true.

If \( \lambda \in \bigcap_{C \in B(K, H)} \sigma(M_C) \), it means that, for any operator \( C \in B(K, H) \), \( \lambda \in \sigma(M_C) \). To complete the proof, it is sufficient to show that, if \( \lambda \notin \sigma_p(A) \cup \sigma(B) \) and \( d(A - \lambda) = n(B - \lambda) \), we may choose an operator \( C_0 \in B(K, H) \) such that \( \lambda \notin \sigma(M_{C_0}) \).

Since \( n(B - \lambda) = d(A - \lambda) \), there exist an orthonormal basis \( \{g_i\}_{i=1}^n \) of \( N(B - \lambda) \) and an orthonormal basis \( \{f_i\}_{i=1}^n \) of \( R(A - \lambda)^\perp \) (\( n \) is not necessarily finite). Define an operator \( C_0 \) from \( K \) into \( H \) by

\[
\begin{cases}
C_0g_i = f_i, & i = 1, 2, \ldots, n, \\
C_0g = 0, & g \in N(B - \lambda)^\perp (\subset K).
\end{cases}
\]

We shall prove \( \lambda \notin \sigma(M_{C_0}) \). To do this, we will prove that \( M_{C_0} \) is injective and surjective.

If there exists a vector \( x = y + z \), \( y \in H \) and \( z \in K \), with \((M_{C_0} - \lambda)x = (M_{C_0} - \lambda)(y + z) = 0\), then \( z \in N(B - \lambda) \) and \( C_0z = -(A - \lambda)y \). By the definition of \( C_0 \), \( C_0z \in R(A - \lambda)^\perp \); thus \( C_0z = 0 \). Moreover, since \( C_0 \) is injective on \( N(B - \lambda) \), we have \( z = 0 \). Therefore, \( A - \lambda)y = 0 \), but we assume that \( \lambda \notin \sigma_p(A) \). Hence \( y = 0 \), so \( M_{C_0} \) is injective.

Now we will show that \( M_{C_0} \) is surjective.

For any vector \( x_0 = y_0 + z_0 \), \( y_0 \in H \) and \( z_0 \in K \), since \( \lambda \notin \sigma_\delta(B - \lambda) \), it follows that \( R(B - \lambda) = K \). Then there must be a vector \( z_1 \in K \) such that \((B - \lambda)z_1 = z_0\). On the other hand, since \( \lambda \notin \sigma_\delta(A) \), \( R(A - \lambda) \) is closed. Thus \( R(A - \lambda) \oplus R(A - \lambda)^\perp = H \). Hence we can assume that \( y_0 = \xi_0 + \eta_0 \), where \( \xi_0 \in R(A - \lambda) \) and \( \eta_0 \in R(A - \lambda)^\perp \). So there exist vectors \( y_1 \in H \) and \( z_2 \in K \) such that \((A - \lambda)y_1 = \xi_0 \) and \( C_0z_1 + \eta_0 = -C_0z_2 \). Note that since \( C_0 \) is onto \( R(A - \lambda)^\perp \), the last equality is possible. Thus

\[
(M_{C_0} - \lambda)(y_1 + z_1 + z_2) = (A - \lambda)y_1 + C_0(z_1 + z_2) + (B - \lambda)(z_1 + z_2) = \xi_0 + \eta_0 + (B - \lambda)z_1 = y_0 + z_0 = x_0.
\]

Because \( x_0 \) is arbitrary, \( M_{C_0} - \lambda \) is surjective. The proof is complete.

A simple example will show that the inclusion \( \sigma(M_C) \subset \sigma(A) \cup \sigma(B) \) may be proper.
Example 3. If \( \{g_i\}_{i=1}^{\infty} \) is an orthonormal basis of \( K \), define an operator \( B_0 \) by
\[
\begin{align*}
B_0 g_1 &= 0, \\
B_0 g_i &= g_{i-1}, & i = 2, 3, \ldots.
\end{align*}
\]
If \( \{f_i\}_{i=1}^{\infty} \) is an orthonormal basis of \( H \), define an operator \( A_0 \) by \( A_0 f_i = f_{i+1} \), \( i = 1, 2, \ldots \), and an operator \( C_0 \) by \( C_0 = (\cdot, g_1)f_1 \) from \( K \) into \( H \). Then it is easy to see that \( \sigma(A_0) = \sigma(B_0) = \{ \lambda : |\lambda| \leq 1 \} \). But, in this case, \( M_{C_0} \) is a unitary operator, \( \sigma(M_{C_0}) \subseteq \{ \lambda : |\lambda| \leq 1 \} \), so the inclusion \( \sigma(M_{C_0}) \subseteq \sigma(A) \cup \sigma(B) \) is proper.

The above example is an affirmative answer to Question 1.

Here we need to point out that although the inclusion \( \sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) \) may be proper, the spectral radius of \( M_C \) is always a constant which is independent of \( C \) and equal to \( \max \{ r_0(A), r_0(B) \} \), where \( r_0(T) \) denotes the spectral radius of an operator \( T \). This is the following proposition.

**Proposition 4.** For given operators \( A \) and \( B \), \( r_0(M_C) \) is a constant.

**Proof.** Note that since \( \sigma(A) \cup \sigma(B) \subseteq \sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) \), Proposition 4 is clear.

At this point, one naturally asks which kinds of spectrums in \( \sigma(A) \) and \( \sigma(B) \) can be perturbed out by choosing a suitable operator \( C \in B(K, H) \). Now we shall answer this question.

**Theorem 5.** Assume that there exists an operator \( C \in B(K, H) \) such that the inclusion \( \sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) \) is proper. Then for any \( \lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C) \), we have \( \lambda \in \sigma(A) \cap \sigma(B) \), \( R(B - \lambda) = K \), \( R(A^* - \lambda) = H \), and \( n(B - \lambda) = d(A - \lambda) \).

**Proof.** The last part of the theorem was contained in Theorem 2, so we only need to prove that, for an operator \( C \in B(K, H) \), if \( \lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C) \), then \( R(B - \lambda) = K \), \( R(A^* - \lambda) = H \), and \( \lambda \in \sigma(A) \cap \sigma(B) \).

Without loss of generality, assume that \( \lambda = 0 \). Then \( M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \) is invertible and it is clear that \( R(B) = K \) and \( R(A^*) = H \).

Next, we shall prove that \( 0 \in \sigma(A) \cap \sigma(B) \). In the converse, assume that \( 0 \in \sigma(A) \) but \( 0 \notin \sigma(B) \). Since \( \sigma(A) \subseteq \sigma(M_C) \) (by Lemma 1), \( 0 \notin \sigma(A) \). Consider the adjoint \( M_C^* \) of \( M_C \)
\[ M_C^* = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix} \]
Then \( N(A^*) \neq \{0\} \). Take a nonzero vector \( y \in N(A^*) \). Under the assumption \( 0 \notin \sigma(B) \), since \( B^* \) is invertible, \( R(B^*) = K \), so we may find a vector \( z \in K \) with \( B^*z = -C*y \) (it is easy to see \( C \neq 0 \)). Therefore, we have
\[
M_C^* \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} A^*y \\ C^*y + B^*z \end{bmatrix} = 0;
\]
that is, \( 0 \in \sigma(M_C^*) \), i.e., \( 0 \in \sigma(M_C) \). This contradicts the hypothesis. A similar argument will show that \( 0 \in \sigma(B) \) and \( 0 \notin \sigma(A) \) simultaneously holding is impossible too. The proof is finished.

**Remark 6.** By Theorem 5 and its proof, we see that the part of \( \sigma(A) \cup \sigma(B) \) perturbed out by choosing a suitable operator \( C \in B(K, H) \) is not only in the
intersection of $\sigma(A)$ and $\sigma(B)$ but also in the intersection of semi-Fredholm domains of $A$ and $B$, whose index is not zero.

Using Theorem 5, we may immediately give an example that, for a given pair $(A, B)$ of operators, $\sigma(M_C)$ is invariant for any $C \in B(K, H)$.

**Example 7.** If $A \in B(H)$ and $B \in B(K)$ are normal operators, then, for any $C \in B(K, H)$, $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.

We need to point out that using results obtained in this note may lead to simpler proofs of propositions in [4]. In [4] some properties of the generalized deriviation were considered. Recall that the generalized derivation $\delta_{AB}$ induced by operators $A$ and $B$ is defined by

$$\delta_{AB} : X \rightarrow AX - XB, \quad X \in B(K, H).$$

We have the following inclusions:

$$\{C : C \in R(\delta_{AB})\} \subset \left\{ C : \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\} \subset \left\{ C : \sigma \left( \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \right) = \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) \right\}.$$

By the discussion in this note, we easily see that, in general, all of these inclusions are proper. Under which conditions with operators $A$ and $B$ will these inclusions instead be of equalities? The answer to this question is not clear now.

Finally, we turn to the so-called four block operator matrices. This subject was suggested by Professor C. Foias.

For an operator $G_X$ defined on $H \oplus K$ by

$$(2) \quad G_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix},$$

if operators $A, B$, and $C$ are given and $X$ is taken over $B(H, K)$, (2) is called a four block operator matrix. For the minimal norm of $G_X$ we have known a famous theorem for a long time (see [2] and [3]).

**Theorem P** (Parott [5]). If given $A, B$, and $C$, then

$$\min \left\{ \left\| \begin{bmatrix} A & C \\ X & B \end{bmatrix} \right\|, \quad X \in B(H, K) \right\} = \max \left\{ \left\| (AC) \right\|, \quad \left\| C \right\|_B \right\}.$$

But, for spectrum of $G_X$, what can we say? As in Proposition 4, for $2 \times 2$ upper triangular operator matrices, when $A$ and $B$ are given, then $r_\sigma(M_C)$ is a constant. However, for four block operator matrices, when $A, B$, and $C$ are given and $C \neq 0$, in general, the spectral radius $r_\sigma(G_X)$ of $G_X$ may be large enough. We have

**Theorem 8.** For a four block operator matrix (2), if $A, B$, and $C$ are given and $C \neq 0$, then, for any $\lambda \in \rho(A)$ ($\rho(A)$ is the resolvent of $A$), there exists a one-rank operator $X \in B(H, K)$ such that $\lambda \in \sigma_p(G_X)$.

**Proof.** Since $C \neq 0$, there exists a vector $x_2 \in K$ with $Cx_2 \neq 0$. By the assumption of $\lambda \in \rho(A)$ and by putting

$$x_1 = -(A - \lambda)^{-1}Cx_2,$$
considering the one-rank operator
\[ X = \frac{1}{\| (A - \lambda)^{-1}Cx_2 \|^2} (\cdot, (A - \lambda)^{-1}Cx_2)(B - \lambda)x_2 \]
and letting \( x = -(A - \lambda)^{-1}Cx_2 + x_2 \), one obtains
\[
(Gx - \lambda)x = \left[ \begin{array}{cc} A - \lambda & C \\ X & B - \lambda \end{array} \right] \begin{array}{c} x \\ A - \lambda \end{array} \frac{\| (A - \lambda)^{-1}Cx_2 \|^2}{\| (A - \lambda)^{-1}Cx_2 \|^2} (B - \lambda)x_2 \\
= \begin{array}{c} \frac{\| (A - \lambda)^{-1}Cx_2 \|^2}{\| (A - \lambda)^{-1}Cx_2 \|^2} (B - \lambda)x_2 \\ (A - \lambda)^{-1}Cx_2 \end{array} \end{array} = 0.
\]
So \( \lambda \in \sigma_p(G_X) \). The proof is completed.

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