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W^p-SPACES AND FOURIER TRANSFORM

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ABSTRACT. The spaces W_M^p, $W_{M,a}^p$, $W^{\Omega,p}$, $W^{\Omega,b,p}$, $W_M^{\Omega,p}$, $W_{M,a}^{\Omega,b,p}$ generalizing the spaces of type W due to Gurevich (also given by Friedman, and Gelfand and Shilov) are investigated. Here M, Ω are certain continuous increasing convex functions, a, b are positive constants and $1 \le p < \infty$. The Fourier transformation F is shown to be a continuous linear mapping as follows: $F\colon W_{M,a}^p \to W^{\Omega,1/a,r}$, $F\colon W^{\Omega,b,p} \to W_{M,1/b}^r$, $F\colon W_{M,a}^{\Omega,b,p} \to W_{M,1/b}^{\Omega,1/a,r}$. These results will be used in investigating uniqueness classes of certain Cauchy problems in future work.

1. THE SPACES W_M^p, $W_{M,a}^p$

Let $\mu(\xi)$ be a continuous increasing function on $[0, \infty)$ such that $\mu(0) = 0$, $\mu(\infty) = \infty$ for $x \ge 0$ define an increasing convex continuous function M by

$$M(x) = \int_0^x \mu(\xi)\, d\xi, \qquad M(-x) = M(x).$$

Then $M(0) = 0$, $M(\infty) = \infty$, and

$$M(x_1) + M(x_2) \le M(x_2 + x_2). \tag{1}$$

Now the space $W_M^p(\mathbb{R})$ is defined as the set of all infinitely differentiable functions $\varphi(x)$ $(-\infty < x < \infty)$ satisfying

$$\left(\int_{-\infty}^{+\infty} |e^{M(ax)}\varphi^{(q)}(x)|^p\, dx\right)^{1/p} \le C_{q,p}, \qquad 1 \le p < \infty, \tag{2}$$

for each nonnegative integer q where the positive constants a and $C_{q,p}$ depend upon φ. Clearly W_M^p is a linear space. The space W_M^p can be regarded as the union of countably normed spaces $W_{M,a}^p$ of all complex valued C^∞-functions φ, which for any $\delta > 0$ satisfy

$$\left(\int_{-\infty}^{+\infty} |e^{M[(a-\delta)x]}\varphi^{(q)}(x)|^p\, dx\right)^{1/p} \le C_{q,p}, \qquad q = 0, 1, 2, \ldots. \tag{3}$$

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The topology over $W^p_{M,a}$ is generated by

$$(4)\qquad \|\varphi\|_p = \sup_{q\le l}\left(\int_{-\infty}^{+\infty} |M_l(x)\varphi^{(q)}(x)|^p\,dx\right)^{1/p}$$

where

$$M_l(x) = \exp(M[a(1-1/l)x]), \qquad l = 2, 3, \dots .$$

From [2, p. 85] it follows that $W^p_{M,a}$ is sequentially complete countably normed linear space. A generalized function belonging to dual space $(W^p_{M,a})'$, $p \ge 1$, has the form

$$f = \sum_{s=0}^{l} D^s[M_l(x)f_s(x)]$$

for some nonnegative integer l, where the f_s are measurable functions such that

$$\sum_{s=0}^{l}\int_{-\infty}^{\infty}[|f_s(x)|^q]\,dx < \infty \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

For $a = 0$, the space $W^p_{M,a}(\mathbb{R})$ reduces to the Schwartz space $D_{L^p}(\mathbb{R})$.

The following properties can be established by using techniques similar to those employed in [2, pp. 12–13].

(1) The operation of differentiation is bounded in W^p_M and is a continuous operation.
(2) The operation of multiplication by x is bounded in W^p_M and is a continuous operation.

2. The spaces $W^{\Omega,b,p}$, $W^{\Omega,p}$

Let Ω be another increasing, continuous, convex function possessing properties similar to M. Then $W^{\Omega,p}$ is defined to be the set of all entire analytic functions $\varphi(z)$ $(z = x+iy)$ satisfying the inequalities

$$(5)\qquad \left(\int_{-\infty}^{+\infty} |e^{-\Omega(by)}z^k\varphi(z)|^p\,dx\right)^{1/p} \le C_{k,p}$$

where the constants $C_{k,p}$ and b depend on the function φ. A sequence $\{\varphi_\nu(z)\} \in W^{\Omega,p}$ is said to converge to zero if the functions $\varphi_\nu(z)$ converge to zero uniformly on any bounded domain of the z-plane and satisfy the inequalities

$$\left(\int_{-\infty}^{+\infty} |e^{-\Omega(by)}z^k\varphi_\nu(z)|^p\,dx\right)^{1/p} \le C_{k,p}, \qquad k = 0, 1, 2, \dots,$$

where the constants $C_{k,p}$ and b do not depend on the index ν. The space $W^{\Omega,b,p}$ defined below consists of the set of all those functions in $W^{\Omega,p}$ which satisfy the inequalities

$$(6)\qquad \left(\int_{-\infty}^{+\infty} |\exp\{-\Omega[(b+\rho)y]\}z^k\varphi(z)|^p\,dx\right)^{1/p} \le C_{k,p}.$$

The topology over $W^{\Omega,b,p}$ is generated by the norms

$$(7)\qquad \|\varphi\|_{k,\rho,p} = \|\varphi\|_{k,\rho,b,p} = \left(\int_{-\infty}^{+\infty} |\exp\{-\Omega[(b+\rho)y]\}z^k\varphi(z)|^p\,dx\right)^{1/p}$$

with these norms $W^{\Omega,b,p}$ is a complete, perfect, countably normed linear space.

Theorem 2.1. *Let $f(z)$ be an entire analytic function satisfying the inequalities*

$$\|(1+|x|^h)^{-1}f(z)\|_p \le D_p e^{\Omega(b_0 y)}$$

and $\varphi(z) \in W^{\Omega,b,q}$. Then $\varphi(z)f(z) \in W^{\Omega,b+b_0,r}$, $1/p + 1/q = 1/r$.

Proof. Since $\varphi(z) \in W^{\Omega,b,q}$ for $r > 0$, we have

$$\begin{aligned}
&\left(\int_{-\infty}^{+\infty} |\exp\{-\Omega[(b+b_0+\rho)y]\}z^k f(z)\varphi(z)|^r\,dx\right)^{1/r} \\
&\quad\le \left(\int_{-\infty}^{+\infty} |(1+|x|^h)\exp\{-\Omega[(b+\rho)y]\}z^k\varphi(z)|^q\,dx\right)^{1/q} \\
&\quad\quad\times \left(\int_{-\infty}^{+\infty} \left|\frac{e^{-\Omega(b_0 y)}f(z)}{1+|x|^h}\right|^p dx\right)^{1/p} \\
&\quad\le D_p\left\{\left(\int_{-\infty}^{+\infty} |\exp\{-\Omega[(b+\rho)y]\}z^k\varphi(z)|^q\,dx\right)^{1/q}\right. \\
&\quad\quad\left.+\left(\int_{-\infty}^{+\infty} |\exp\{-\Omega[(b+\rho)y]\}|x|^h z^k\varphi(z)^q\right)^{1/q}\right\} \\
&\quad\le D_p\{C_{k,q} + C_{k,q+h}\} \le D_{k,q}.
\end{aligned}$$

Therefore $\varphi(z)f(z) \in W^{\Omega,b+b_0,r}$.

3. The spaces $W_{M,a}^{\Omega,p}$, $W_M^{\Omega,p}$

Let M, Ω be the same functions as defined in §1 and §2 respectively and let $a, b > 0$. We denote by $W_M^{\Omega,p}$ the set of all entire analytic functions $\varphi(z)$ $(z = x+iy)$, which satisfy the inequalities

$$(8)\qquad \left(\int_{-\infty}^{+\infty} |\exp[M(ax) - \Omega(by)]\varphi(z)|^p\,dx\right)^{1/p} \le C_p$$

where the constants a, b, and C_p depend upon the function $\varphi(z)$. Obviously the space $W_M^{\Omega,p}$ is a vector space over $\mathbb{C}$ under usual operations. A sequence $\{\varphi_\nu(z)\} \in W_M^{\Omega,p}$ is said to converge to zero if the function $\varphi_\nu(z)$ converges uniformly to zero in any bounded domain of the z-plane and in addition the following inequalities:

$$\left(\int_{-\infty}^{+\infty} |\exp\{M(ax) - \Omega(by)\}\varphi_\nu(z)|^p\,dx\right)^{1/p} \le C_p$$

hold with constants C_p, a, b, which do not depend upon the index ν.

The space $W^{\Omega,p}$ can also be represented as a union of countably normed linear spaces. We denote by $W_{M,a}^{\Omega,b,p}$ the set of all those functions belonging to the space $W_M^{\Omega,p}$ which satisfy the inequalities

$$\left(\int_{-\infty}^{+\infty} |\exp\{M[(a-\delta)x] - \Omega[(b+\rho)y]\}\varphi(z)|^p\,dx\right)^{1/p} \le C_p. \tag{9}$$

Define the following norms in the space $W_{M,a}^{\Omega,b,p}$:

$$\|\varphi\|_{\delta,\rho,p} = \left(\int_{-\infty}^{+\infty} |\exp\{M[(a-\delta)x] - \Omega[(b+\rho)y]\}\varphi(z)|^p\,dx\right)^{1/p}. \tag{10}$$

Example. Let us take $M(x) = x^\alpha$, $\Omega(y) = y^\beta$, $\alpha > 1$, $\beta > 1$. Then $W^{\Omega,p}$ consists of entire functions $\varphi(x+iy)$ satisfying

$$\|\exp[a|x|^\alpha - b|y|^\beta]\varphi(x+iy)\|_p \le C_p, \qquad a, b, C_p > 0.$$

Theorem 3.1. *Let $f(z)$ be an entire analytic function satisfying the inequalities*

$$\|\exp[-M(a_0x)]f(z)\|_p \le D_p \exp\{\Omega(b_0y)\}$$

and $\varphi(z) \in W^{\Omega,b,q}$. Then $\varphi(z)f(z) \in W_{M,a-a_0}^{\Omega,b+b_0,r}$, $1/p + 1/q = 1/r$.

Proof. Let $\varphi(z) \in W_{M,a}^{\Omega,b,q}$ then for $r > 0$ we have

$$\begin{aligned}
&\left(\int_{-\infty}^{+\infty} |\exp\{M[(a-a_0-\delta)x] - \Omega[(b+b_0+\rho)y]\}f(z)\varphi(z)|^r\,dx\right)^{1/r} \\
&\quad\le \left(\int_{-\infty}^{+\infty} |\exp\{M[(a-\delta)x] - \Omega[(b+\rho)y]\}\varphi(z)|^q\,dx\right)^{1/q} \\
&\quad\times \left(\int_{-\infty}^{+\infty} |\exp[-M(a_0x) - \Omega(by)]f(z)|^p\,dx\right)^{1/p} \\
&\quad< \infty.
\end{aligned}$$

Therefore, $f(z)\varphi(z) \in W_{M,a-a_0}^{\Omega,b+b_0,r}$.

4. The Fourier transformation

Theorem 4.1. *Let $M(x)$, $\Omega(y)$ be the pair of functions which are dual in the sense of Young. Then $F(W_{M,a}^p) \subset W^{\Omega,1/a,r}$, $p, r \ge 1$.*

Proof. Let $\varphi \in W_{M,a}^p$. Then the Fourier transformation exists in the L_1-sense. Let ψ be its Fourier transformation. Then by [2, pp. 21–22] is an entire function of $s = \sigma + i\tau$, and we have

$$\|(is)^k \psi(s)\|_r = \left(\int_{-\infty}^{+\infty} |(is)^k \psi(s)|^r \, d\sigma\right)^{1/r}$$

$$= \left(\int_{-\infty}^{+\infty} \left(\frac{|s|^{k+2} + |s|^k}{\sigma^2+1}\right)^r |\psi(s)|^r \, d\sigma\right)^{1/r}$$

$$\leq \left(\int_{-\infty}^{+\infty} \left(\frac{|s|^{k+2}|\psi(s)|}{\sigma^2+1}\right)^r d\sigma\right)^{1/r} + \left(\int_{-\infty}^{+\infty} \left(\frac{|s|^{k}|\psi(s)|}{\sigma^2+1}\right)^r d\sigma\right)^{1/r}$$

$$\leq \left(\int_{-\infty}^{+\infty} \frac{d\sigma}{(\sigma^2+1)^r} \left(\int_{-\infty}^{+\infty} |e^{-x\tau}\phi^{(k+2)}(x)|\, dx\right)^r\right)^{1/r}$$

$$+ \left(\int_{-\infty}^{+\infty} \frac{d\sigma}{(\sigma^2+1)^r} \left(\int_{-\infty}^{+\infty} |e^{-x\tau}\phi^{(k)}(x)|\, dx\right)^r\right)^{1/r}$$

$$\leq \left(\int_{-\infty}^{+\infty} \frac{d\sigma}{(\sigma^2+1)^r} \left(\left(\int_{-\infty}^{+\infty} |\exp M[(a-\delta)x]\varphi^{(k+2)}(x)|^p \, dx\right)^{1/p}\right.\right.$$
$$\left.\left. \times \left(\int_{-\infty}^{+\infty} |e^{x\tau} \exp -[M(a-\delta)x]|^{p'} \, dx\right)^{1/p'}\right)^r\right)^{1/r}$$

$$+ \left(\int_{-\infty}^{+\infty} \frac{d\sigma}{(\sigma^2+1)^r} \left(\left(\int_{-\infty}^{+\infty} |\exp M[(a-\delta)x]\varphi^{(k)}(x)|^p \, dx\right)^{1/p}\right.\right.$$
$$\left.\left. \times \left(\int_{-\infty}^{+\infty} |e^{x\tau} \exp -[M(a-\delta)x]|^{p'} \, dx\right)^{1/p'}\right)^r\right)^{1/r}$$

$$\leq C_{k+2,p}^r e^{\Omega(\tau/\gamma)} + C_{k,p}^r e^{\Omega(\tau/\gamma)}$$
$$\left(\frac{1}{\gamma} = \left(\frac{1}{a} + \rho\right) \text{ since } \gamma = a - 2\delta \text{ and } \rho > 0 \text{ is arbitrarily small}\right)$$
$$\leq D_{k,p}^r e^{\Omega[\tau(1/a+\rho)]}.$$

Theorem 4.2. *Let* $M(x)$ *and* $\Omega(y)$ *be the same functions as in Theorem* 4.1. *Then*

$$F(W^{\Omega,b,p}) \subset W_{M,1/b}^r, \qquad p, r \geq 1.$$

Proof. Since $\varphi(x+iy) \in W^{\Omega,b,p}$, it follows that

$$\varphi(x+iy) = O(|x|^{-\delta}) \quad \text{as } x \to \infty$$

for $\delta > 0$ then following arguments given in [2, pp. 22–23] we obtain

$$|\psi^{(q)}(\sigma)| \leq C_{q,\delta,p} e^{-\sigma y}(\|z^{q+2}\varphi(z)\|_p + \|z^q\varphi(z)\|_p).$$

Now using characteristic inequality and following the arguments [2, p. 23] we can estimate

$$|\psi^{(q)}(\sigma)| \leq C_{q,\delta,p} \exp\{-M[(1/b-\delta)\sigma] - M[\rho^2\sigma/b^3]\}.$$

Hence

$$|\exp\{M[(1/b-\delta)\sigma]\}\psi^{(q)}(\sigma)|^r \le C^r_{q,\delta,p}e^{-rM(\rho^2\sigma/b^3)}$$

so that

$$\|\exp\{M[(1/b-\delta)\sigma]\}\psi^{(q)}(\sigma)\|_r \le D_{q,\delta,p}, r.$$

From Theorems 4.1 and 4.2 we conclude that

Corollary. $F(W^p_{M,a}) = W^{\Omega,1/a,p}$, $F(W^{\Omega,b,p}) = W^p_{M,1/b}$.

Theorem 4.3. *Let* $\Omega_1(y)$ *and* $M_1(x)$ *be the functions which are dual in the sense of Young to the functions* $M(x)$ *and* $\Omega(y)$. *Then*

$$F(W^{\Omega,b,p}_{M,a}) \subset W^{\Omega_1,1/a,r}_{M_1,1/b}, \qquad p, r \ge 1.$$

Proof. Following the arguments given in [2, pp. 24–25] we can show that

$$|\psi(\sigma+i\tau)| \le C'_{\rho_1,\delta_1,p}\exp\{[-M_1[(1/b-\delta)\sigma] - M_1\{(\rho^2\sigma/b^3)\}] + \Omega_1[(1/a+\rho_1)\tau]\}.$$

Hence

$$|\exp\{M_1[(1/b-\delta)\sigma]\}\psi(\sigma+i\tau)| \le C'_{\rho_1,\delta,p}\exp\{-M_1(\rho^2\sigma/b^3) + \Omega_1\tau(1/a+\rho_1)\}$$

so that

$$\|e^{M_1[(1/b-\delta)\sigma]-\Omega_1[(1/a+\rho_1)\tau]}\psi(\sigma+i\tau)\|_r \le D^{1,r}_{\rho_1,\delta_1,p}.$$

Corollary. $F[W^{\Omega,b,p}_{M,a}] = W^{\Omega_1,1/a,p}_{M_1,1/b}$.

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