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AN INTRINSIC CHARACTERIZATION FOR ZERO-DIAGONAL OPERATORS

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ABSTRACT. The purpose of this paper is to present the following intrinsic characterization for zero-diagonal operators.

Theorem. *An operator* T *has a zero diagonal if and only if* $\operatorname{tr}\operatorname{Re}(e^{i\theta})_+ = \operatorname{tr}\operatorname{Re}(e^{i\theta}T)_-$ *for all* θ, $0 \leq \theta < 2\pi$.

A bounded linear operator T on a complex separable Hilbert space H is said to have a zero diagonal if there is an orthonormal basis $\{b_n\}$ such that $(Tb_n, b_n) = 0$ for all n. It is well known that if the dimension of H is finite, then an operator on H has a zero diagonal if and only if its trace is zero; see, e.g., [3, p. 109]. This was generalized to operators on Hilbert spaces in the following form.

Theorem A [1, Theorem 1]. *An operator* T *has a zero diagonal if and only if there exists an orthonormal basis* $\{b_n\}$ *such that the sequence* $\{s_n\}$ *of partial sums of the diagonal entries*

$$s_n = \sum_{k=1}^{n}(Tb_k, b_k)$$

has a subsequence converging to zero.

The above characterization is very sensitive to a base change. However, it can be reformulated into base free descriptions in special cases. For instance, it reduces to $\operatorname{tr} T = 0$ if T is a trace class operator. When T is hermitian, it can be converted into the following.

Theorem B [2, Theorem 2]. *A hermitian operator* H *has a zero diagonal if and only if* $\operatorname{tr} H_+ = \operatorname{tr} H_-$.

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Theorem. *An operator* T *has a zero diagonal if and only if*

$$\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ = \operatorname{tr}\operatorname{Re}(e^{i\theta}T)_-$$

for all θ, $0 \le \theta < 2\pi$.

Throughout this paper, $\operatorname{Re} T$ denotes the real part of T; H_+ and H_- stand for $(|H| + H)/2$ and $|H| - H_+$ respectively where H is hermitian; and $\operatorname{tr} P$ means the trace of a positive operator P. In addition, we denote by $R\{\operatorname{tr} T\}$ the set of all sums $\sum(Tb_n, b_n)$ whenever the series converges with respect to some orthonormal basis $\{b_n\}$.

The proof of our Theorem relies primarily on the following result about the "shape" of $R\{\operatorname{tr} T\}$.

Theorem C [2, Theorem 4]. *For an operator* T, $R\{\operatorname{tr} T\}$ *is either empty, or a point, or a line, or the complex plane. Precisely*:

(i) $R\{\operatorname{tr} T\} = \varnothing$ *iff* $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_- < +\infty$ *but* $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ = +\infty$ *for some* θ.

(ii) $R\{\operatorname{tr} T\}$ *is a point iff* $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ < +\infty$ *for all* θ.

(iii) $R\{\operatorname{tr} T\}$ *is a line iff* $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_\pm < +\infty$ *and* $\operatorname{tr}\operatorname{Im}(e^{i\theta}T)_\pm = +\infty$ *for some* θ.

(iv) $R\{\operatorname{tr} T\}$ *is the complex plane iff* $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ = +\infty$ *for all* θ.

Proof of Theorem. Necessity. Suppose T has a zero diagonal. Then obviously $0 \in R\{\operatorname{tr} T\}$ and thus $R\{\operatorname{tr} T\}$ takes only three shapes by Theorem C. When $R\{\operatorname{tr} T\} = \{0\}$, $T \in C_1$, the trace class; hence $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_\pm = 0$. When $R\{\operatorname{tr} T\}$ is a line containing 0, by (iii) of Theorem C, there is a θ such that $\operatorname{Re}(e^{i\theta}T) \in C_1$ and $\operatorname{tr}\operatorname{Im}(e^{i\theta}T)_\pm = +\infty$. This implies $\operatorname{tr}\operatorname{Re}(e^{i\phi}T)_\pm = +\infty$ for $e^{i\phi} \ne e^{i\theta}$ and $e^{i(\theta+\pi)}$. Lastly when $R\{\operatorname{tr} T\}$ is the plane, $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ = +\infty$ for all θ.

Sufficiency. If $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ = \operatorname{tr}\operatorname{Re}(e^{i\theta}T)_-$ for all θ, $R\{\operatorname{tr} T\}$ acquires the same three shapes. It is enough to show $0 \in R\{\operatorname{tr} T\}$ in each case because this proves that T has a zero diagonal, by Theorem A. When $R\{\operatorname{tr} T\}$ is the complex plane, obviously it contains 0. When $R\{\operatorname{tr} T\}$ is a point, the hypothesis implies that $\operatorname{tr} T = 0$. Finally when $R\{\operatorname{tr} T\}$ is a line, by (iii) we can write $e^{i\theta}T = H + iK$ such that $\operatorname{tr} H_\pm = +\infty$ and $\operatorname{tr} K_+ = \operatorname{tr} K_- < +\infty$. Now according to Theorem B, H has a zero diagonal with respect to some orthonormal basis $\{b_n\}$. Thus

$$\sum(Tb_n, b_n) = e^{-i\theta}\sum(iKb_n, b_n) = 0.$$

This shows $0 \in R\{\operatorname{tr} T\}$ and completes the proof.

Remarks. (i) We point out here that not a single θ can be omitted from the hypothesis in the proof of the sufficiency part above. Indeed, define $T = \operatorname{diag}\{1, -1, i, \frac{1}{2}, -\frac{1}{2}, \frac{i}{2}, \dots, \frac{1}{n}, -\frac{1}{n}, \frac{i}{n}, \dots\}$. Observe that $\operatorname{tr}\operatorname{Re}(e^{i\theta}T)_+ = +\infty$ for all θ except for $\theta = \frac{\pi}{2}$ (in fact, $\operatorname{tr}\operatorname{Re}(iT)_+ = 0$). But this operator does not have a zero diagonal because the imaginary part is positive.

(ii) We provide here an alternative proof for the necessity part. Suppose $(Tb_n, b_n) = 0$, for all n, with respect to an orthonormal basis $\{b_n\}$. Write H for $\operatorname{Re}(e^{i\theta}T)$. Obviously $(Hb_n, b_n) = 0$ for all n. Hence

$$\operatorname{tr} H_+ = \sum(H_+b_n, b_n) = \sum((H + H_-)b_n, b_n) = \sum(H_-b_n, b_n) = \operatorname{tr} H_-.$$

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