PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 3, July 1994

LOCALLY FINITE AND LOCALLY NILPOTENT DERIVATIONS WITH APPLICATIONS TO POLYNOMIAL FLOWS, MORPHISMS, AND $\mathcal{G}_a$-ACTIONS. II

ARNO VAN DEN ESSEN

(Communicated by Lance W. Small)

ABSTRACT. This paper describes several applications of locally finite and locally nilpotent derivations; we recover the main results from the theory of polynomial flows, give a new inversion formula for polynomial automorphisms, reformulate the Eulerian conjecture for ordinary systems of differential equations in terms of locally nilpotent derivations, and give an algorithm to compute the invariant ring of a $\mathcal{G}_a$-action on affine n-space (if this ring is finitely generated).

INTRODUCTION

In [13] we tried to sell the following slogan "Locally nilpotent and locally finite derivations are hidden in many problems. However once their presence is revealed, information on these derivations can give valuable information on the problem under consideration." To illustrate that statement we discussed several examples. Using properties of locally finite and locally nilpotent derivations we gave new very simple proofs of the fact that the Lorenz equations and the Maxwell-Bloch equations do not have a polynomial flow (results first obtained by Coomes in [7, 8] and Coomes and Zurkowski in [10]). Also a very short proof of the Bass-Meisters classification theorem of two dimensional polynomial flows (cf. [4]) was presented and finally we described an algorithm to decide if a two dimensional vector field is a poly-flow vector field.

In this paper we have collected several new applications of the use of locally finite and locally nilpotent derivations in the study of various problems. We have tried to make the paper self-contained by starting each section with a short introduction to the subject treated therein.

The contents are arranged as follows. In §1 we recall some well-known facts concerning derivations and give a useful formula describing the solutions of a system of ordinary differential equations of the form $\dot{y} = P(y)$. In §2 we apply the results of section one to give simplified proofs of some results of Coomes and Zurkowski [9, 10] concerning polynomial flows. At the end of that section, inspired by the papers of Meisters [18], Olech and Meisters [19], and Adjamagbo

Received by the editors June 26, 1992 and, in revised form, October 12, 1992.
1991 *Mathematics Subject Classification*. Primary 13B10, 14E07, 14E05, 34A34; Secondary 32A05, 13B25.
Key words and phrases. Derivations, polynomial automorphisms, flows, Eulerian systems.

0002-9939/94 \$1.00 + \$.25 per page

[1], we give a new inversion formula for polynomial automorphisms in terms of a locally nilpotent derivation. In §3 we present a new characterisation of locally nilpotent derivations on polynomial rings over $\mathbb{C}$, in terms of the behaviour of solutions of a corresponding system of ordinary differential equations. Using this criterion we describe a connection between Eulerian systems, as introduced in [2], and locally nilpotent derivations and give a reformulation of the Eulerian conjecture in terms of locally nilpotent derivations. Finally, in §4 we show how locally nilpotent derivations can be used to give an algorithm which describes the invariant ring of a $\mathscr{G}_a$-action on affine n-space, in case the invariant ring is a $\mathbb{C}$-algebra of finite type. This result is not new; in fact in [14] the more general case of $\mathscr{G}_a$-actions on affine varieties is treated. However for the sake of clarity of presentation we have restricted our description here to the case of the affine n-space.

Throughout the text we have formulated some interesting questions, which solution could be helpful to get a better understanding of the problems considered in this paper.

1. The exponent of a derivation

Let k be a field of characteristic zero, A a commutative k-algebra, and $A[[t]]$ the ring of formal power series in one variable over A. Let D be a k-derivation on A. Then the map $\exp tD \to A[[t]]$ defined by

$$\exp tD(a) = \sum_{p=0}^{\infty} \frac{1}{p!} D^p(a) t^p$$

is a ring homomorphism which satisfies

$$\frac{d}{dt} \circ \exp tD = \exp tD \circ D. \tag{1.1}$$

Since the constant term of $\exp tD(a)$ equals a, the map $\exp tD$ is injective. Instead of $\exp tD$ we will write φ.

In this paper we mainly consider two special kind of derivations; a derivation D on A is called *locally finite* if for each a in A the k-vectorspace generated by the elements $D^i a$, $i \in \mathbb{N}$, is finite dimensional. The derivation D is called *locally nilpotent* if for every $a \in A$ there exists an integer $n \geq 1$ such that $D^n(a) = 0$. In terms of φ a derivation D is locally nilpotent if and only if $\varphi(A) \subset A[t]$.

In the remainder of this section we describe two applications of the ring-homomorphism φ. First we have

Proposition 1.2 [24]. *Let A be a k-algebra without zero-divisors and suppose that D is locally nilpotent.*

(1) *If $f \neq 0$ and $Df = af$, then $a = 0$.*
(2) *If $0 \neq ab \in \ker D$, then both a and b belong to* $\ker D$. *In particular if A is a U.F.D., then so is* $\ker D$.

Proof. By the hypothesis $A[t]$ has no zero-divisors and $\varphi(A) \subset A[t]$.

(1) If $D(f) = af$ then $\frac{d}{dt}\varphi(f) = \varphi D(f) = \varphi(a)\varphi(f)$. So looking at the t-degrees of the polynomials $\varphi(f)$ and $\varphi(a)$ in $A[t]$, it follows that $\frac{d}{dt}\varphi(f) = 0$, hence $\varphi(a) = 0$, so $a = 0$ (φ is injective).

(2) If $ab \in \ker D$, then $D(ab) = 0$, so by definition of φ we get $ab = \varphi(ab) = \varphi(a)\varphi(b)$. Consequently both polynomials $\varphi(a)$ and $\varphi(b)$ must be constant, i.e., belong to A. So again by the definition of φ it follows that $D(a) = D(b) = 0$, as desired. □

To describe the second application, which was pointed out to me by Harm Derksen, we introduce some notations. Let $\mathscr{O} := k[[t]]$ and $P_1, \dots, P_n$ be elements of the polynomial ring $k[X] := k[X_1, \dots, X_n]$ and denote by D the derivation $D = \sum_{i=1}^{n} P_i\partial_i$ on $k[X]$. Consider the system of ordinary differential equations

$$\begin{aligned} \dot{y}_1(t) &= P_1(y_1(t), \dots, y_n(t)), \\ &\vdots \\ \dot{y}_n(t) &= P_n(y_1(t), \dots, y_n(t)). \end{aligned} \tag{1.3}$$

Proposition 1.4. *The formal power series solutions of* (1.3) *are exactly the series* $y(t) = (y_1(t), \dots, y_n(t)) \in \mathscr{O}^n$ *with*

$$y_i(t) = \exp tD(X_i)|_{X_1=c_1, \dots, X_n=c_n}$$

where $c = (c_1, \dots, c_n)$ *runs through* k^n.

Proof. (i) Let $c = (c_1, \dots, c_n) \in k^n$. By Cauchy's theorem there exists exactly one $y = (y_1, \dots, y_n) \in \mathscr{O}^n$ which satisfies (1.3) and the initial condition $y(0) = c$. Since obviously $g_i(t) := \exp tD(X_i)|_{X=c}$ satisfies $g_i(0) = c_i$ for all i, it suffices to prove that $\dot{g}_i(t) = P_i(g_1(t), \dots, g_n(t))$ for all i.

(ii) Put

$$\mathscr{G}_i(t) := \exp tD(X_i).$$

So $\mathscr{G}_i(t) \in k[X][[t]]$ and it follows readily that $\dot{\mathscr{G}}_i(t) = \exp tD(DX_i) = \exp tD(P_i)$. Since $\exp tD$ is a ringhomomorphism from $k[X]$ to $k[X][[t]]$ we get

$$\begin{aligned} \exp tD(P_i) &= \exp tD(P_i(X_1, \dots, X_n)) \\ &= P_i(\exp tD(X_1), \dots, \exp tD(X_n)) = P_i(\mathscr{G}_1(t), \dots, \mathscr{G}_n(t)). \end{aligned}$$

So $\dot{\mathscr{G}}_i(t) = P_i(\mathscr{G}_1(t), \dots, \mathscr{G}_n(t))$. Finally substituting $X = c$ we find that $\dot{g}_i(t) = P_i(g_1(t), \dots, g_n(t))$ for all i, as desired. □

To conclude this section we give a new formula which describes the inverse of a polynomial automorphism of $\mathbb{C}^n$.

Let $F: \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map satisfying $F(0) = 0$ and $\det JF \in C^*$. Define derivations $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ on $C[X]$ by the formula

$$\begin{pmatrix} \partial/\partial F_1 \\ \vdots \\ \partial/\partial F_n \end{pmatrix} = ((JF)^{-1})^T \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \tag{1.5}$$

One readily verifies that $\frac{\partial}{\partial F_i}(F_j) = \delta_{ij}$ for all i, j and it is well-known (cf. [22]) that the derivations $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ commute pairwise.

Since in particular $\det JF(0) \in \mathbb{C}^*$, the local inversion theorem implies that there exist $\mathscr{G}_1, \dots, \mathscr{G}_n$ in $\mathbb{C}[[X]] := \mathbb{C}[[X_1, \dots, X_n]]$ such that $F(\mathscr{G}_1, \dots, \mathscr{G}_n) = X$ and $\mathscr{G}_i(0) = 0$ for all i. Introduce $n+1$ new variables $t, V_1, \dots, V_n$, put

$V := (V_1, \dots, V_n)^T$, $k := \mathbb{C}(V_1, \dots, V_n)$ and $\mathscr{O} := k[[t]]$. Then the element $g_V(t) := (\mathscr{G}_1(tV), \dots, \mathscr{G}_n(tV)) \in \mathscr{O}^n$ satisfies $F(g_V(t)) = tV$. Differentiating this equation with respect to t gives

$$(JF)(g_V(t))\dot{g}_V(t) = V.$$

Consequently

$$\dot{g}_V(t) = (JF)^{-1}(g_V(t))V.$$

Since $\mathscr{G}_i(0) = 0$ for all i we get that $g_V(0) = 0$. So by Proposition 1.4 we obtain

$$(g_V)_i(t) = \exp tD(X_i)|_{X=0}, \quad \text{for all } i,$$

where D is the derivation on $k[X]$ given by the formula

$$\begin{aligned} D &= ((JF)^{-1}V)^T \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ &= (V_1, \dots, V_n)((JF)^{-1})^T \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} = \sum V_i \frac{\partial}{\partial F_i} \quad \text{(by (1.5))}. \end{aligned}$$

Now observe that $\mathscr{G}(V) = g_V(1)$. So we conclude that $\mathscr{G}_i(V) = \exp D(X_i)|_{X=0}$. Summarizing we have proved

Theorem 1.6. *Let $F: \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $F(0) = 0$ and $\det JF \in \mathbb{C}^*$. Then the formal inverse $\mathscr{G} = (\mathscr{G}_1, \dots, \mathscr{G}_n)$ is given by the formula*

$$\mathscr{G}_i(V) = \exp D(X_i)|_{X=0}$$

where D is the derivation $D = \sum V_i \frac{\partial}{\partial F_i}$ on $\mathbb{C}(V_1, \dots, V_n)[X]$.

Corollary 1.7 (cf. [22, 13]). *Let $F: \mathbb{C}^n \to \mathbb{C}^n$ with $\det JF \in \mathbb{C}^*$ and $F(0) = 0$. Then F is invertible if and only if all $\frac{\partial}{\partial F_i}$ are locally nilpotent on $\mathbb{C}[X]$.*

Proof. If F is invertible then $\mathbb{C}[X] = \mathbb{C}[F]$ which implies that each $\frac{\partial}{\partial F_i}$ is locally nilpotent on $\mathbb{C}[X]$. Conversely, if all $\frac{\partial}{\partial F_i}$ are locally nilpotent on $\mathbb{C}[X]$, then D is locally nilpotent on $\mathbb{C}(V)[X]$, since the $\frac{\partial}{\partial F_i}$ commute pairwise. So by Theorem 1.6 the formal inverse is polynomial, so F is invertible. □

2. Polynomial flows

Let V be a C^1-vector field on $\mathbb{R}^n$ and consider the initial value problem

$$\dot{y} = V(y), \qquad y(0) = x \in \mathbb{R}^n. \tag{2.1}$$

The solution (flow) φ of (2.1) is said to be a *polynomial flow*, and the vector field V a *polynomial flow vector field*, abbreviated p-f vector field, if φ depends polynomially on the initial condition, i.e., $\varphi_t(x) := \varphi(t, x)$ is polynomial in x for each t where the solution is defined. Polynomial flows where discussed by Meisters in [16] and first studied in [4] by Bass and Meisters. They proved, amongst other things the following results:

(i) If V is a p-f vector field (so it is C^1), then it is polynomial; i.e., all its component functions V_i are polynomial.

(ii) The solutions are complete, i.e., they are defined for all $t \in \mathbb{R}$.

(iii) There exist an integer d and real analytic functions $a_\alpha : \mathbb{R} \to \mathbb{R}^n$ such that $\varphi(t, x) = \sum_{|\alpha| \le d} a_\alpha(t) x^\alpha$.

Using these results, the completeness result (ii) was extended by Coomes in [7] and [9]; he showed

(iv) If V is a p-f vector field, then its flow extends to a holomorphic function on $\mathbb{C}^{n+1}$ satisfying the group property. We call it an entire flow.

This result in turn was improved in a paper of Coomes and Zurkowski [10]; they showed

(v) A polynomial flow of any non-linear n-dimensional system is the solution of a linear ODE, of some sufficient high order.

This result is based on the following beautiful theorem of Coomes and Zurkowski [10, Theorem 3.1].

(vi) Let $V = (V_1, \dots, V_n)$ be a polynomial vector field on $\mathbb{R}^n$ and let $D_V := \sum V_i \partial_i$ be the corresponding derivation on $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$. Then V is a p-f vector field if and only if D_V is a locally finite derivation on $\mathbb{R}[X]$.

The aim of this section is to give simple proofs of the results (iv), (v), and (vi) based on properties (i), (ii), (iii), and Proposition 1.4. From now on we assume that V is polynomial (by (i) this is no restriction) and we write D instead of D_V. First we prove one half of the Coomes-Zurkowski theorem.

Proposition 2.2. *If V is a p-f vector field, then D is locally finite.*

Proof. Let $x \in \mathbb{R}^n$. There exists exactly one element in $\mathbb{R}[[t]]^n$ which satisfies (2.1). Since both $\varphi(t, x) = \sum_{|\alpha| \le d} a_\alpha(t) x^d$ and $\exp tD(X)|_{X=x}$ belong to $\mathbb{R}[[t]]^n$ and satisfy (2.1) we conclude that these series are equal, for all $x \in \mathbb{R}^n$, hence $\sum_{|\alpha| \le d} a_\alpha(t) X^\alpha = \sum_{k \ge 0} (t^k/k!) D^k(X)$. Consequently $\deg_X D^k(X_i) \le d$ for all i and k, which implies that D is locally finite. □

Now suppose that D is locally finite and let $1 \le i \le n$. Then there exists an integer N such that the components of the vector $b = (X_i, D(X_i), \dots, D^{N-1}(X_i))$ form a $\mathbb{C}$-basis of $W := \sum_{p \ge 0} \mathbb{C} D^p(X_i)$.

Observe that $D(W) \subset W$, so we can consider the restriction of D to W which we also denote by D. Put $A_i := \mathrm{Mat}(D, (b))$ and for $x \in \mathbb{C}^n$ let $b(x)$ denote the vector b evaluated at the point x.

Proposition 2.3. *If D is locally finite, then $\varphi_i(t, x)$ is the first component of the vector $\exp(tA_i^T) b(x)^T$. In particular φ is a polynomial flow.*

Proof. For each $k \ge 1$ $\mathrm{Mat}(D^k, (b)) = A_i^k$. So the first column of A_i^k gives the coefficients of $D^k(X_i)$ with respect to the basis (b) of W. So $D^k(X_i)|_{X=x}$ equals the first component of $(A_i^T)^k b(x)^T$. Now using that $\varphi_i(t, x) = \sum (1/k!) t^k D^k(X_i)|_{X=x}$ the proposition follows.

Corollary 2.4. *If D is locally finite, then*

$$\varphi(t, x) = \sum_{i=1}^{p} \sum_{j=0}^{q_i} a_{ij}(x) t^j e^{\lambda_i t}$$

for some positive integer p, some nonnegative integers q_i, some complex numbers λ_i, and some polynomial functions a_{ij}. In particular a polynomial flow is entire and satisfies a linear ODE.

Proof. We only show the last statement (since the rest is an obvious consequence of the theory of systems of linear differential equations and Proposition 2.3). Put $d := \max_{i,j} \deg_x a_{ij}$ and $q := \max q_{ij}$. Let W be the $\mathbb{C}$-vector space generated by the functions $t^j e^{\lambda_i t} x^\alpha$, $|\alpha| \le d$, $j \le q$. Observe that $\frac{d}{dt}W \subset W$ and $\varphi \in W$. So $(\frac{d}{dt})^i \varphi \in W$ for all i. Since W is finite dimensional, the set of vectors φ, $\frac{d}{dt}\varphi$, $(\frac{d}{dt})^2\varphi, \ldots$ is dependent. So there exists $N \in \mathbb{N}$ and $c_i \in \mathbb{C}$ such that

$$\left(\frac{d}{dt}\right)^N \varphi + c_{N-1}\left(\frac{d}{dt}\right)^{N-1} \varphi + \cdots + c_1 \frac{d}{dt}\varphi + c_0\varphi = 0$$

which concludes the proof. □

The proof of Proposition 2.3 gives an effective way to compute all solutions of (2.1) in case V is a p-f vector field. This leads to

Question 1 (cf. Question 1 and 9 in [17]). Given a vector field V, how can we decide if V is a p-f vector field?

In case $n = 2$ this question was solved in [13] algorithmically. However the case $n \ge 3$ remains open. A positive answer to Question 1 follows from the following: let $D = \sum V_i \partial_i$ be locally finite. Put $d := \max_i \deg V_i$.

Question 2 (cf. Question 8 in [17]). Can one give an estimate for the dimension of the vector space $\sum_{k \ge 0} \mathbb{C} D^k(X_i)$ in terms of n and d?

3. Eulerian systems of ordinary differential equations

First we recall some of the definitions of [2]. Let $l \in \mathbb{N}$, $l \ge 1$, and denote by $\mathscr{O}_l := \mathbb{C}[[X_1, \ldots, X_l]]$ the ring of formal power series in the variables $X_1, \ldots, X_l$. A map $F: \mathscr{O}_l^n \to \mathscr{O}_l^m$ is called a polynomial differentiable map of order d if F is of the form $F(y) = (F_1(y), \ldots, F_m(y))$, where $y = (y_1, \ldots, y_n) \in \mathscr{O}^n$ and each $F_i(y)$ is a polynomial expression in the unknown functions $y_1, \ldots, y_n$ and their partial derivatives up to order d; the coefficients of the polynomial expressions belong to $\mathbb{C}[X_1, \ldots, X_l]$. To such a polynomial differentiable map and a point $p \in \mathbb{C}[X_1, \ldots, X_l]^n$ we define the linearization of F at p. This is the linear map from $\mathscr{O}_l^n$ to $\mathscr{O}_l^m$, denoted $F'(p)$, defined (in an analogous way as in an elementary calculus course) as follows:

$$F(p+h) - F(p) = F'(p)(h) + \text{nonlinear terms in } h, \quad \text{for all } h \in \mathscr{O}_l^n.$$

Furthermore, a polynomial differentiable map $F: \mathscr{O}_l^n \to \mathscr{O}_l^m$ is called *Eulerian* if for every $q \in \mathbb{C}[X_1, \ldots, X_l]^m$ every formal solution $g \in \mathscr{O}_l^n$ of the equation $F(g) = q$ belongs to $\mathbb{C}[X_1, \ldots, X_l]^n$. The main conjecture stated in [2] is

Eulerian conjecture. If $F: \mathscr{O}_l^n \to \mathscr{O}_l^m$ is a polynomial differentiable map such that $F'(p)$ is Eulerian for all $p \in \mathbb{C}[X_1, \ldots, X_l]^n$, then F is Eulerian.

From now on $\mathscr{O} := \mathbb{C}[[t]]$ the ring of formal power series in one variable. Let $P_1, \ldots, P_n$ be elements of the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]$.

To such a set of polynomials we associate the polynomial differentiable map $F\colon \mathscr{O}^n \to \mathscr{O}^n$ defined by

$$(3.0) \qquad F(y_1, \dots, y_n) = \begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{pmatrix} - \begin{pmatrix} P_1(y_1, \dots, y_n) \\ \vdots \\ P_n(y_1, \dots, y_n) \end{pmatrix}, \quad \text{for all } y_i \in \mathscr{O}.$$

One readily verifies that for each $p \in \mathbb{C}[t]^n$ the linearization of F at p is the map $F'(p)\colon \mathscr{O}^n \to \mathscr{O}^n$ given by

$$(3.1) \qquad F'(p)(h_1, \dots, h_n) = \begin{pmatrix} \dot{h}_1 \\ \vdots \\ \dot{h}_n \end{pmatrix} - (JP)(p) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \quad \text{for all } h_i \in \mathscr{O}.$$

Proposition 3.2. *The following two statements are equivalent:*

(i) $F^{-1}(\{0\}) \subset \mathbb{C}[t]^n$; *i.e., every formal solution of* $F(y) = 0$ *belongs to* $\mathbb{C}[t]^n$.

(ii) $D_p := \sum P_i \partial_i$ *is a locally nilpotent derivation of* $\mathbb{C}[X]$.

Proof. (ii) $\to$ (i) follows from Proposition 1.4.

(i) $\to$ (ii). From (i) and Proposition 1.4 we deduce that $\exp tD_p(X_i)|_{X=c}$ belongs to $\mathbb{C}[t]^n$ for all $c \in \mathbb{C}^n$ and all i. Put $D := D_p$. Let $1 \le i \le n$. Writing $\exp tD(X_i) = \sum_{k=0}^{\infty} a_k(X)t^k$ it follows that for each $c \in \mathbb{C}^n$ $\sum a_k(c)t^k$ is a polynomial in t. Hence there exists an integer $N(c) \ge 0$ such that $a_k(c) = 0$ for all $k \ge N(c)$. *Claim*: there exists $N \in \mathbb{N}$ such that $a_k(X) = 0$ for all $k \ge N$. If this is not the case we would have an infinite sequence of nonzero polynomials $a_{k_1}, a_{k_2}, \dots$ with the property that each $c \in \mathbb{C}^n$ is zero of all a_{k_i} with $k_i \ge N(c)$. In other words $\mathbb{C}^n \subset \bigcup_{i=1}^{\infty} a_{k_i}^{-1}(0)$. So $\mathbb{C}^n$ is contained in a countable union of hypersurfaces, contradicting Baire's theorem. So the claim is proved.

(iii) Observe that $a_k(X) = \frac{1}{k!}D^k(X_i)$, so $D^k(X_i) = 0$ for all $k \ge N$, which implies that D is locally nilpotent. □

Now we are able to reformulate in terms of locally nilpotent derivations the property that the map $F\colon \mathscr{O}^n \to \mathscr{O}^n$ (defined above) is Eulerian. Therefore let $q \in \mathbb{C}[t]^n$ and consider the system

$$(3.3) \qquad \begin{aligned} \dot{y}_1 - P_1(y_1, \dots, y_n) &= q_1(t), \\ \vdots \qquad & \quad \vdots \\ \dot{y}_n - P_n(y_1, \dots, y_n) &= q_n(t). \end{aligned}$$

Define $\widetilde{P}_i := P_i(X_1, \dots, X_n) + q_i(X_{n+1}) \in \mathbb{C}[X_1, \dots, X_n, X_{n+1}]$ for all $1 \le i \le n$, $\widetilde{P}_{n+1} := 1$, and $\widetilde{F}\colon \mathscr{O}^{n+1} \to \mathscr{O}^{n+1}$ by the formula

$$(3.4) \qquad \widetilde{F}(y_1, \dots, y_{n+1}) = \begin{pmatrix} \dot{y}_1 - \widetilde{P}_1(y_1, \dots, y_{n+1}) \\ \vdots \\ \dot{y}_{n+1} - \widetilde{P}_{n+1}(y_1, \dots, y_{n+1}) \end{pmatrix}.$$

If $(y_1, \dots, y_n)$ satisfies (3.3) and $y_{n+1}(t) := t$, then $(y_1, \dots, y_n, y_{n+1})$ satisfies $\widetilde{F}(y_1, \dots, y_{n+1}) = 0$. Conversely, if $(y_1, \dots, y_{n+1})$ satisfies $\widetilde{F}(y_1, \dots, y_{n+1})$

$= 0$ then $y_{n+1} = t + c$ for some $c \in \mathbb{C}$ and then $(y_1, \dots, y_n)$ satisfies $\dot{y}_i = P_i(y_1, \dots, y_n) + q_i(t + c)$ for all $1 \le i \le n$. Consequently, F is Eulerian if and only if for every $q_1, \dots, q_n \in \mathbb{C}[X_{n+1}]$ every formal solution of

$$\dot{y}_1 = P_1(y_1, \dots, y_n) + q_1(t), \dots, \dot{y}_n = P_n(y_1, \dots, y_n) + q_n(t), \dot{y}_{n+1} = 1$$

is polynomial. By Proposition 3.2 this is equivalent to the statement that for every $q_1, \dots, q_n \in \mathbb{C}[X_{n+1}]$ the derivation

$$\partial_{n+1} + \sum_{i=1}^{n}(P_i(X_1, \dots, X_n) + q_i(X_{n+1}))\partial_i$$

is locally nilpotent. Summarizing we have proved

Theorem 3.5. *Let $F: \mathscr{O}^n \to \mathscr{O}^n$ as above. Then the following two statements are equivalent*:

(i) *F is Eulerian*

(ii) *For every n-tuple $q_1(X_{n+1}), \dots, q_n(X_{n+1})$ in $\mathbb{C}[X_{n+1}]$ the derivation*

$$\partial_{n+1} + \sum_{i=1}^{n}(P_i + q_i(X_{n+1}))\partial_i$$

is locally nilpotent.

Corollary 3.6. *The Eulerian conjecture for the polynomial differentiable maps described in* (3.0) *can be expressed in terms of locally nilpotent derivations.*

Remark 3.7. From Theorem 3.5 it follows that if F is Eulerian, then D_p is locally nilpotent (take all $q_i = 0$ and observe that $D_p + \partial_{n+1}$ is locally nilpotent on $\mathbb{C}[X_1, \dots, X_{n+1}]$ if and only if D_p is locally nilpotent on $\mathbb{C}[X_1, \dots, X_n]$).

4. $\mathscr{G}_a$-ACTIONS

Although the results of this section are not new, we have decided to add this section since $\mathscr{G}_a$-actions form another subject where our slogan described in the introduction can be applied. We will discuss two types of questions: the first type deals with triangulability questions of $\mathscr{G}_a$-actions on affine n-space and the second with computability of the invariant rings of such actions. For more facts concerning algebraic group actions on affine space we refer the reader to the excellent survey paper of Kraft [15] and the paper [26] of Snow.

Throughout this section k denotes an algebraically closed field of characteristic zero.

Let $\varphi: k \times k^n \to k^n$ be a $\mathscr{G}_a$-action on $\mathbb{A}^n$ i.e. a polynomial morphism satisfying $\varphi(0, x) = x$ and $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$ for all $x \in k^n$ and all s, t in k. Let φ^* be the induced ringhomomorphism from $k[X]$ to $k[X, T]$. Then it is well known that there exists a uniquely determined locally nilpotent derivation D on $k[X]$ such that $\varphi^*(f) = \exp TD(f)$ for all $f \in k[X]$. For every $t \in \mathscr{G}_a(= k)$ the map φ^* induces a polynomial automorphism $\varphi_t^*: k[X] \to k[X]$ which satisfies $\varphi_t^*(f) = \exp tD(f)$. The extension of φ_t^* to the quotient field $k(X)$ is also denoted by φ_t^*. Finally recall that a $\mathscr{G}_a$-action φ or $\mathbb{A}^n$ is called *triangulable* if there exists a polynomial automorphism ψ of $k[X]$ such that for all $t \in \mathscr{G}_a$ $\psi^{-1}\varphi_t^*\psi$ is of the form

$$(\lambda_1 X_1, \lambda_2 X_2 + h_1(X_1), \dots, \lambda_n X_n + h_n(X_1, \dots, X_n)),$$

with $\lambda_i \in k^*$ and $h_i(X_1, \dots, X_{i-1}) \in k[X_1, \dots, X_{i-1}]$ for all i.

4.1. **Triangulability problems of $\mathscr{G}_a$-actions on $\mathbb{A}^n$.** In [24] Rentschler showed that every locally nilpotent derivation D on $k[X, Y]$ is conjugate with a derivation of the form $f(Y)\partial_X$ for some $f(Y) \in k[X]$; i.e., there exists a polynomial automorphism ψ of $k[X, Y]$ such that $\psi^{-1}D\psi = f(Y)\partial_X$. It follows immediately that every $\mathscr{G}_a$-action on the affine plane is triangulable. However in [3] Bass gave an example of a $\mathscr{G}_a$-action on $\mathbb{A}^3$ which is not triangulable; take $D = (XZ + Y^2)(X\partial_Y - 2Y\partial_Z)$. Then D is locally nilpotent on $k[X, Y, Z]$, hence determines a $\mathscr{G}_a$-action on $\mathbb{C}^3$ by the formula $\varphi_t = \exp tD$. This action is not triangulable: if φ is triangulable then for some automorphism ψ of $k[X]$ we have $\psi^{-1}\varphi_t^*\psi(X_i) = \lambda_i(t)X_i + h_i(t, X_{i+1}, \dots, X_n)$, where each $\lambda_i(t)$ is a nonvanishing polynomial in t and $\lambda_i(0) = 1$. So $\lambda_i(t) = 1$ for all i. So the fixed point set of $\psi^{-1}\varphi_t^*\psi$ is the set of common zeros of the polynomials $h_i(t, X_{i+1}, \dots, X_n)$. Since the variable X_1 does not appear in these polynomials, this fixed point set is of the form $k \times V$ for some affine algebraic set V. Also observe that the dimension of the fixed point set of φ_t^* is $\leq n-1$ and it contains $W :=$ the zero-set of $XZ + Y^2$. So W is a component of the fixed point set of φ_t^*, hence W is isomorphic to a set of the form $k \times V'$. However this is a contradiction since one readily verifies that W has an isolated singularity, which is impossible for a set of the form $k \times V'$.

This argument, which is due to Popov in [23], where nontriangulable $\mathscr{G}_a$-actions on $\mathbb{A}^n$ are constructed for every $n \geq 3$, has led Snow in [26] to the following question.

Question 3. Is the condition that for each t the fixed point set of φ_t^* is isomorphic to a set of the form $k \times V$ the only obstruction for a derivation to be triangulable?

Or even more likely

Question 4. If $D = \sum a_i\partial_i$ is locally nilpotent such that the a_i have no common zero, is D triangulable?

Of course these questions are very important for a good understanding of locally nilpotent derivations, since problems concerning these derivations are often easy to solve in case the derivations are in triangular form.

Instead of asking for a derivation to be triangulable one can pose a weaker question

Question 5 (Bass [3]). Can the action of a unipotent group $\mathscr{G}$ on $\mathbb{A}^n$ be rationally triangularized; i.e., can we write $k(X_1, \dots, X_n) = k(Y_1, \dots, Y_n)$ so that each subfield $k(Y_1, \dots, Y_i)$ is $\mathscr{G}$-invariant?

In [11, 12] Deveney and Finston gave a positive answer to this question for the additive group $\mathscr{G} = \mathscr{G}_a$ in the cases $n = 3$ and $n = 4$. It was also shown in [12] that for the n-dimensional case the field of invariants of a $\mathscr{G}_a$-action becomes rational (i.e., purely transcendental) after the adjunction of one more variable. So this relates Question 5 to the Zariski problem (asking: if a pure transcendental extension $F(U_1, \dots, U_m)$ of F (where F is a finitely generated field extension of $\mathbb{C}$) is isomorphic to a pure transcendental extension of $\mathbb{C}$, is F itself isomorphic to a pure transcendental extension of $\mathbb{C}$?). Since the Zariski problem was solved in the negative in [5], the question of a rational triangulability remains open for $n \geq 5$.

4.2. **An algorithm to compute the invariant ring of a $\mathscr{G}_a$-action on affine space.** Let A be a finitely generated commutative k-algebra without zero-divisors and D a nonzero locally nilpotent derivation on A. Then the *invariant ring* of the corresponding $\mathscr{G}_a$-action on $\operatorname{Spec} A$ is the set of elements $a \in A$ such that $\varphi_t^*(a) = a$ for all $t \in \mathscr{G}_a$, where φ_t^* is the ringhomomorphism $\exp tD$. It is not difficult to verify that this invariant ring is equal to $\ker(D, A)$.

Question 6. Is $\ker D$ a k-algebra of finite type?

It was recently shown by Deveney and Finston that the answer is no in general. The gave a counterexample for $A = k[X_1, \dots, X_7]$ thereby furnishing a new counterexample to Hilbert 14 (cf. [20] and [21] for more details on Hilbert 14). Also some positive results concerning Question 6 are known; for example every linear action of $\mathscr{G}_a$ on $\mathbb{A}^n$ has a finitely generated invariant ring [27], and if A is a normal affine variety of dimension ≤ 3 then $\ker D$ is of finite type (this result is due to Zariski [28]).

The question we consider in this section is the following: Can we compute $\ker(D, A)$ in case it is a finitely generated k-algebra? This question was solved completely by the author in [14]. To avoid some technicalities (which can be overcome by using results from the theory of Gröbner bases) we only explain the algorithm of [14] for the case $A = k[X_1, \dots, X_n]$. The starting point is the following property [14, Proposition 1.4].

Proposition 4.3.[1] *Let $A = k[a_0, a_1, \dots, a_m]$ be a finitely generated k-algebra and D a locally nilpotent derivation on A satisfying $D(a_0) = 1$. Then* $\ker D = k[a_{ij}; i \geq 1, \ j \geq 0]$, *where a_{ij} is the coefficient of t^j in*

$$\sum_{p\geq 0} \frac{1}{p!}(t - a_0)^p D^p(a_i). \tag{4.4}$$

Now we show how this result is used to compute $\ker(D, k[X])$ in four steps.

Step 1. First choose an element a in $k[X]$ such that $D^2(a) = 0$ and $d := D(a) \neq 0$. This is possible since D is locally nilpotent and D is nonzero. Put $A := k[X][d^{-1}]$ and extend D to A. This extension we also denote by D and observe that it is also locally nilpotent. The element $a_0 := d^{-1}a \in A$ satisfies $D(a_0) = 1$. It follows readily that $\ker(D, A) = \ker(D, k[X])[d^{-1}]$ and from this and Proposition 4.3 one concludes: there exists a finitely generated k-subalgebra R_0 of $R := \ker(D, k[X])$ such that $R_0 \subset R \subset R_0[d^{-1}]$. More precisely, $R_0 = k[\dots r_{ij} \dots]$, where the r_{ij} can be computed as follows: for each $1 \leq i \leq n$ let a_{ij} be the coefficient of t^j in $\sum_{p\geq 0} \frac{1}{p!}(t - a_0)D^p(X_i)$. Choose $e_{ij} \in \mathbb{N}$ such that $d^{e_{ij}}a_{ij}$ belongs to $k[X]$. Take $r_{ij} := d^{e_{ij}}a_{ij}$.

Step 2. Define for each $m \geq 1 R_m$ inductively as the k-subalgebra of $k[X]$ generated by the elements $g \in k[X]$ satisfying $dg \in R_{m-1}$. By an induction argument we see that we get an ascending chain of k-subalgebras $R_0 \subset R_1 \subset \cdots \subset R := k[X]$. Now observe that if R is finitely generated over k, say $R = k[I_1, \dots, I_q]$ for some I_j, then $R = R_r$ for some $r \in \mathbb{N}$. (Namely, since $R_0 \subset R[d^{-1}]$ there exists $r \in \mathbb{N}$ with $d^r I_j \in R_0$ for all j. So $d^{r-1}I_j \in R_1$,

[1] In a recent paper of the author (*Locally nilpotent derivations and their applications*, III, J. Pure Appl. Algebra (to appear)) it is shown that $\ker D$ can be generated by $m + 1$ elements.

whence $d^{r-2}I_j \in R_2$, etc. We arrive at $I_j \in R_r$ for all j, implying $R = R_r$.) So to compute R we need to answer the following two questions.

Question 4.5. How can we compute R_m from R_{m-1} for all $m \geq 1$?

Question 4.6. How can we decide if $R_{m-1} = R_m$? (If $R_{m-1} = R_m$ then obviously $R = R_{m-1}$, and we are done.)

The last question is solved by the Membership algorithm of Shannon and Sweedler in [25]. So it remains to consider Question 4.5. This will be done in the next two steps.

Step 3. Let $R_{m-1} = k[F_1, \dots, F_l]$ and denote by I the set of $P \in k[Y] := k[Y_1, \dots, Y_l]$ such that $P(F) := P(F_1, \dots, F_l) \in k[X]d$. Observe that I is an ideal in $k[Y]$, hence it is generated by a finite number of elements $P_1, \dots, P_s$ say (we show below that we can effectively compute such generators of I). So $P_i(F) = f_i d$ for some $f_i \in k[X]$, which by definition belongs to R_m.

Now we show that $R_m = k[F_1, \dots, F_l, f_1, \dots, f_s]$. Therefore it remains to verify "$\subset$". So let $g \in R_m$; i.e., $dg \in R_{m-1}$. Then $dg = P(F)$ for some $P \in k[Y]$. In other words $P(F) \in k[X]d$. So $P = \sum a_i(Y)P_i$ for some $a_i(Y) \in k[Y]$. Consequently $P(F) = \sum a_i(F)P_i(F)$. Since $P_i(F) = f_i d$ and $P(F) = dg$, we get

$$g \in \sum k[F]f_i \subset k[F_1, \dots, F_l, f_1, \dots, f_s].$$

It remains to show that the P_i can be computed effectively:

Step 4. Observe that

$$\begin{aligned} P(F) \in k[X]d &\Leftrightarrow P(Y) \in J := k[X, Y](Y_1 - F_1(X), \dots, Y_l - F_l(X), d(X)) \\ &\Leftrightarrow P(Y) \in k[Y] \cap J. \end{aligned}$$

So the ideal I $(= \{P \in k[Y] | P(F) \in k[X]d\})$ equals $k[Y] \cap J$. Then it is well known (cf. [6]) that if $\mathcal{G}$ is the reduced Gröbner bases of J with respect to an admissible ordering such that $X_i > k[Y]$ for all i, then $\mathcal{G} \cap J$ is a Gröbner basis of $k[Y] \cap J = I$. This completes the description of the algorithm.

References

1. K. Adjamagbo, *On polynomiality or rationality of formal power series and Jacobian conjecture*, Bull. Soc. Math. Belg. **43** (1991), 9–14.
2. K. Adjamagbo and A. van den Essen, *Eulerian systems of partial differential equations and the Jacobian conjecture*, J. Pure Appl. Algebra **74** (1991), 1–15.
3. H. Bass, *A non-triangular action of $\mathcal{G}_a$ on* $\mathbf{A}^3$, J. Pure Appl. Algebra **33** (1984), 1–5.
4. H. Bass and G. Meisters, *Polynomial flows in the plane*, Adv. Math. **55** (1985), 173–208.
5. A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer, *Variétés stablemenet rationnelles non rationnelles*, Ann. of Math. (2) **121** (1985), 283–318.
6. B. Buchberger, *Gröbner bases: An algorithmic method in polynomial ideal theory*, Multidimensional Systems Theory (N. Bose, ed.), Reidel, Dordrecht, 1985.
7. B. Coomes, *Polynomial flows, symmetry groups, and conditions sufficient for injectivity of maps*, Ph.D. thesis, Univ. Nebraska, Lincoln, 1988.
8. ———, *The Lorenz system does not have a polynomial flow*, J. Differential Equations **82** (1989), 386–407.
9. ———, *Polynomial flows on* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **320** (1990), 493–506.

10. B. Coomes and V. Zurkowski, *Linearization of polynomial flows and spectra of derivations*, J. Dynamics Differential Equations **3** (1990), 29–66.

11. J. Deveney and D. Finston, *Fields of $\mathcal{G}_a$-invariants are ruled*, Canad. Bull. Math. (to appear).

12. ———, *Rationally triangulable automorphisms*, J. Pure Appl. Algebra **72** (1991), 1–4.

13. A. van den Essen, *Locally finite and locally nilpotent derivations with applications to polynomial flows and polynomial morphisms*, Proc. Amer. Math. Soc. **116** (1992), 861–871.

14. ———, *An algorithm to compute the invariant ring of a $\mathcal{G}_a$-action on an affine variety*, Report 9202, Catholic Univ., Nijmegen; J. Symbolic Comput. (to appear).

15. H. Kraft, *Algebraic automorphisms of affine space*, Topological Methods in Algebraic Transformation Groups, Birkhäuser, Boston, 1989.

16. G. Meisters, *Jacobian problems in differential equations and algebraic geometry*, Rocky Mountain J. Math. **12** (1982), 679–705.

17. ———, *Polynomial flows on* $\mathbb{R}^n$, Banach Center Publ., vol. 23, PWN, Warsaw, 1989, pp. 9–24.

18. ———, *Inverting polynomial maps of n-space by solving differential equations*, Proc. Conf. Differential and Delay Equations (Iowa State Univ., October 18–19, 1991), World Sci. Publ., Singapore and Teaneck, NJ, 1992.

19. G. Meisters and C. Olech, *A polyflow formulation of the Jacobian conjecture*, Bull. Polish Acad. Sci. Math. **35** (1987), 725–731.

20. M. Nagata, *On the fourteenth problem of Hilbert*, Proc. Internat. Congr. Math. 1958, Cambridge Univ. Press, New York, 1966, pp. 459–462.

21. ———, *Lectures on the fourteenth problem of Hilbert*, Lecture Notes in Math., vol. 31, Tata Institute, Bombay, 1965.

22. P. Nousiainen and M. Sweedler, *Automorphisms of polynomial and power series rings*, J. Pure Appl. Algebra **29** (1983), 93–97.

23. V. Popov, *On actions of $\mathcal{G}_a$ on* $\mathbb{A}^n$, Algebraic Groups (Utrecht, 1986), Lecture Notes in Math., vol. 1271, Springer-Verlag, Berlin, Heidelberg, and New York, 1987.

24. R. Rentschler, *Opèrations du groupe additif sur le plan affine*, C. R. Acad. Sci. Paris **267** (1968), 384–387.

25. D. Shannon and M. Sweedler, *Using Gröbner bases to determine algebra membership, split surjective algebra homomorphisms and determine birational equivalence*, J. Symbolic Comput. **6** (1988), 267–273.

26. D. Snow, *Unipotent actions on affine space*, Topological Methods in Algebraic Transformation Groups, Birkhäuser, Bostcn, 1989.

27. R. Weitzenböck, *Über die Invarianten von Linearen Gruppen*, Acta Math. **58** (1962), 231–293.

28. O. Zariski, *Interpretations algebrico-geometriques du quatorzième problem de Hilbert*, Bull. Sci. Math. (2) **78** (1954), 155–168.

Department of Mathematics, Catholic University Nijmegen, Nijmegen, The Netherlands

E-mail address: essen@sci.kun.nl