

ON A MINIMAL NORMAL DILATION PROBLEM

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ABSTRACT. By combining the theory of disintegration of measures with certain approximation theorems in several complex variables, we derive some positive results concerning a minimal normal dilation problem for operator tuples whose coordinates are special functions of subnormal operators as well as their adjoints.

1. PRELIMINARIES

All the Hilbert spaces occurring in this paper are complex and separable. For a Hilbert space \mathcal{H} , $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on \mathcal{H} . Any operator tuple discussed is assumed to consist of commuting bounded linear operators, though this may not always be stated explicitly. Let $S = (S_1, \dots, S_m)$ and $T = (T_1, \dots, T_m)$ be two operator tuples such that $S_i \in \mathcal{B}(\mathcal{H})$ and $T_i \in \mathcal{B}(\mathcal{H})$ for all i , where \mathcal{H} is a Hilbert space containing \mathcal{K} as a closed subspace. We say that S dilates to T if $S_1^{\alpha_1} S_2^{\alpha_2} \dots S_m^{\alpha_m} = P(\mathcal{K}, \mathcal{H}) T_1^{\alpha_1} T_2^{\alpha_2} \dots T_m^{\alpha_m} / \mathcal{K}$ for all nonnegative integers α_i , where $P(\mathcal{K}, \mathcal{H})$ denotes the orthogonal projection of \mathcal{K} onto \mathcal{H} . If, in addition, \mathcal{K} is invariant for each T_i , then S is said to extend to T . If S dilates (extends) to T and each T_i is normal, then T will be referred to as a normal dilation (extension) of S . Also, T will be called a minimal normal dilation of S if T is a normal dilation of S and \mathcal{K} does not have a proper closed subspace containing \mathcal{K} and reducing each T_i . An operator tuple S admitting a normal extension is said to be subnormal, and among all the normal extensions of S there is a 'minimal normal extension', which is unique up to unitary equivalence [1].

Suppose N is a normal extension of a subnormal tuple S . Using the spectral theory for commuting normals, one may consider $\varphi(N) \equiv (\varphi_1(N), \dots, \varphi_k(N))$, where $\varphi = (\varphi_1, \dots, \varphi_k)$ is a tuple of bounded Borel functions on the joint spectrum of N . In certain special cases to be discussed, $\varphi(N)$ turns out to be a normal dilation of

$$P(\mathcal{K}, \mathcal{H})\varphi(N)/\mathcal{H} \equiv (P(\mathcal{K}, \mathcal{H})\varphi_1(N)/\mathcal{H}, \dots, P(\mathcal{K}, \mathcal{H})\varphi_k(N)/\mathcal{H}).$$

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In such a case, we raise the following question: Is $\varphi(N)$ a minimal normal dilation of $P(\mathcal{H}, \mathcal{H})\varphi(N)/\mathcal{H}$? We will provide an answer to this question using the theory of disintegration of measures. Specific applications follow by invoking some highly nontrivial approximation theorems in several complex variables. The arguments in this paper are entirely motivated by those in [C2], and a knowledge of the first three sections of [C2] is more or less essential for an understanding of this paper.

If $S = (S_1, \dots, S_m)$ is a subnormal tuple with a normal extension $N = (N_1, \dots, N_m)$ ($S_i \in \mathcal{B}(\mathcal{H}), N_i \in \mathcal{B}(\mathcal{H})$), then

$$\begin{aligned} &S_1^{*p_1} S_2^{*p_2} \dots S_m^{*p_m} S_1^{q_1} S_2^{q_2} \dots S_m^{q_m} \\ &= P(\mathcal{H}, \mathcal{H}) N_1^{*p_1} N_2^{*p_2} \dots N_m^{*p_m} N_1^{q_1} N_2^{q_2} \dots N_m^{q_m} / \mathcal{H} \end{aligned}$$

for all nonnegative integers p_i and q_i . Suppose $I \cup J$, with $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_t\}$, is a partition of $\{1, 2, \dots, m\}$. By $P(S; I, J)$ we will indicate a polynomial of the type

$$\sum a_{p_1 p_2 \dots p_s q_1 q_2 \dots q_t} S_{i_1}^{*p_1} S_{i_2}^{*p_2} \dots S_{i_s}^{*p_s} S_{j_1}^{q_1} S_{j_2}^{q_2} \dots S_{j_t}^{q_t}.$$

Recall that an operator tuple $S = (S_1, \dots, S_m)$ is said to be doubly commuting if $S_i^* S_j = S_j S_i^*$ (and, of course, $S_i S_j = S_j S_i$) for $i \neq j$. The multiplication tuples on the tensor products of functional Hilbert spaces, such as the Hardy space or the Bergman space of the unit polydisk, are typical examples of doubly commuting subnormal tuples. It is well known that if S_1, \dots, S_m are doubly commuting subnormals on a Hilbert space, then $S = (S_1, \dots, S_m)$ is (jointly) subnormal [I]. If S is a tuple of doubly commuting subnormals with a normal extension N , then it is easy to see that, for any partition $I \cup J$ of $\{1, \dots, m\}$ and any positive integer k , $(P_1(N; I, J), \dots, P_k(N; I, J))$ is a normal dilation of $(P_1(S; I, J), \dots, P_k(S; I, J))$, where each P_i indicates a polynomial of the type mentioned previously. (For example, with $m = 3$, $k = 1$, $I = \{1, 3\}$, and $J = \{2\}$, $N_3^* N_2 + N_1^{*2}$ is a normal dilation of $S_3^* S_2 + S_1^{*2}$.) The appearance of the adjoints of subnormal operators in the above expressions necessitates an extension of some of the theory developed in [C2], which is the topic of the next section. While the applications in §3 are somewhat specialized, the results in §2 are stated in such a way as to allow for the possibility of more general applications in the future.

2. DISINTEGRATION OF MEASURES

We begin by quoting the following theorem from [C2].

Theorem 1. *If $N = (N_1, \dots, N_m)$ is a commuting tuple of normal operators on a Hilbert space \mathcal{H} , then there is a compact metric space X , bounded Borel functions χ_1, \dots, χ_m on X , and a regular Borel measure μ on X such that each N_j is unitarily equivalent to multiplication by χ_j on $L^2(\mu)$.*

Using the notation of Theorem 1, if S is a subnormal tuple with a normal extension N , then each S_j is defined by multiplication by χ_j on some closed subspace \mathcal{H} of $L^2(\mu)$ ($\cong \mathcal{H}$). We will be exclusively concerned with the case of χ_j being the coordinate functions z_j . Thus, if $\varphi_1, \dots, \varphi_k$ are bounded Borel functions on X and $\varphi(N)$ is a normal dilation of $\varphi(S) \equiv P(\mathcal{H}, \mathcal{H})\varphi(N)/\mathcal{H}$,

then $\varphi(N)$ is a minimal normal dilation of $\varphi(S)$ if and only if

$$(1) \quad L^2(\mu) = V\{\overline{\varphi}_1^{q_1} \overline{\varphi}_1^{q_2} \cdots \overline{\varphi}_k^{q_k} \varphi_1^{p_1} \varphi_2^{p_2} \cdots \varphi_k^{p_k} h : \\ h \in \mathcal{H} ; p_i, q_i \text{ are nonnegative integers}\},$$

where V denotes the closed linear span in the norm of $L^2(\mu)$. Note that if the space \mathcal{H} is invariant for each $\varphi_i(N)$, then (1) is equivalent to requiring $\varphi(N)$ to be a minimal normal extension of $\varphi(S) \equiv \varphi(N)|_{\mathcal{H}}$. (In case the functions φ_i involve the conjugates $\overline{\varphi}_j$, the requirement that \mathcal{H} be invariant for each $\varphi_i(N)$ is rather stringent.) Checking the validity of (1) in specific cases involves two steps, namely:

- (1) invoking an appropriate approximation theorem in several complex variables (involving the conjugate functions $\overline{\varphi}_j$) and
- (2) an appeal to Theorem 3 (or Corollary 1).

The following theorem on disintegration of measures was used crucially in [C2] and is restated here for ready reference.

Theorem 2 [B]. *Suppose X and Z are compact metric spaces and μ is a positive regular Borel measure on X . If $\varphi : X \rightarrow Z$ is a Borel map, then there exists a unique mapping $\xi \rightarrow \lambda_\xi$ of Z into the set $M(X)$ of regular positive Borel measures on X satisfying*

- (a) λ_ξ is a probability measure supported on $\varphi^{-1}(\xi)$ for all ξ in $\varphi(X)$.
- (b) For any continuous function f on X , $\xi \rightarrow \int_X f d\lambda_\xi$ is a $(\mu \circ \varphi^{-1})$ -integrable function on Z .
- (c) For any continuous function f on X

$$\int f d\mu = \int \left[\int f d\lambda_\xi \right] d(\mu \circ \varphi^{-1})(\xi).$$

It is important to note the following particular consequence of Theorem 2:

$$(2) \quad \int |g|^2 d\mu = \int \left[\int_{\varphi^{-1}(\xi)} |g|^2 d\lambda_\xi \right] d(\mu \circ \varphi^{-1})(\xi) \quad \text{for any } g \text{ in } L^2(\mu).$$

The following theorem is an extension of [C2, Theorem 3.5].

Theorem 3. *Suppose S is a subnormal tuple of operators on a Hilbert space \mathcal{H} with a normal extension N of operators on \mathcal{H} and $\varphi = (\varphi_1, \dots, \varphi_k)$ is a k -tuple of bounded Borel functions on the joint spectrum of N with $\varphi(N)$ being a normal dilation of $\varphi(S) \equiv P(\mathcal{H}, \mathcal{H})\varphi(N)|_{\mathcal{H}}$. With μ as in Theorem 1, let $\xi \rightarrow \lambda_\xi$ be the disintegration of μ with respect to φ as described in Theorem 2; and let $\mathcal{H}_\xi = L^2(\lambda_\xi)$ -closure of $\{h/\varphi^{-1}(\xi) : h \in \mathcal{H}_0\}$, where \mathcal{H}_0 is some dense subset of \mathcal{H} . Then $\varphi(N)$ is a minimal normal dilation of $\varphi(S)$ if and only if*

$$L^2(\mu) = \{f \in L^2(\mu) : f/\varphi^{-1}(\xi) \in \mathcal{H}_\xi \text{ a.e. } [\mu \circ \varphi^{-1}]\}.$$

Proof. In view of (1), it suffices to show that $\mathcal{H}' \equiv \{f \in L^2(\mu) : f/\varphi^{-1}(\xi) \in \mathcal{H}_\xi \text{ a.e. } [\mu \circ \varphi^{-1}]\}$ equals

$$\mathcal{H}'' \equiv V\{\overline{\varphi}_1^{q_1} \overline{\varphi}_2^{q_2} \cdots \overline{\varphi}_k^{q_k} \varphi_1^{p_1} \varphi_2^{p_2} \cdots \varphi_k^{p_k} h : \\ h \in \mathcal{H} ; p_i, q_i \text{ are nonnegative integers}\}.$$

That \mathcal{H}' is closed and $\mathcal{H}' \supset \mathcal{H}$ are easy consequences of (2). (The inclusion corresponding to $\mathcal{H}' \supset \mathcal{H}$ seems to have been tacitly assumed in [C2].) The inclusion $\mathcal{H}'' \subset \mathcal{H}'$ is justified using the same argument as in [C2, Theorem 3.5]. To prove that \mathcal{H}'' actually equals \mathcal{H}' , one chooses $f \in \mathcal{H}'$ such that f is orthogonal to \mathcal{H}'' and shows that f is zero. For this one starts off by considering the inner product $\langle \overline{\varphi}_1^{q_1} \cdots \overline{\varphi}_k^{q_k} \varphi_1^{p_1} \cdots \varphi_k^{p_k} h, f \rangle$ instead of $\langle \varphi_1^{p_1} \cdots \varphi_k^{p_k} h, f \rangle$ as in [C2, Theorem 3.5]. The rest of the argument remains essentially the same as there. \square

Corollary 1. *In the notation of Theorem 3, $\varphi(N)$ is a minimal normal dilation of $\varphi(S)$ if and only if $\mathcal{H}_\xi = L^2(\lambda_\xi)$ a.e. $[\mu \circ \varphi^{-1}]$.*

Remark 1. When N is a minimal normal extension of S and φ belongs to the algebra $\mathcal{B}(S)^{(k)}$ as defined in [C2], Theorem 3 in effect reduces to [C2, Theorem 3.5]. The assumption about the minimality of N in [C2] is not always essential. It is, however, handy in certain cases such as when φ_j are rational functions with poles off the joint Taylor spectrum of S . (For the particular case mentioned here, interpreting $\varphi(N)$ poses no problem in view of the spectral inclusion property for subnormal tuples [P].) In all of the examples discussed below, the normal extensions N of S turn out to be minimal in view of the Stone-Weierstrass theorem.

3. APPLICATIONS

If X is a compact subset of \mathbb{C}^m and f is a continuous function from X into \mathbb{C}^m , then it is easy to see that X is homeomorphic to $Y = \{(z, f(z)): z \in X\}$, the graph of X , under the map $z \rightarrow (z, f(z))$. It follows that $C(X)$ is isometrically isomorphic to $C(Y)$. This observation will be helpful in interpreting the conclusions of some of the theorems quoted below, which are slight reformulations of those stated originally. Observe that if polynomials are uniformly dense in $C(Y)$, then so are they in $C(Y')$ for any closed subset Y' of Y . If μ is a compactly supported regular Borel measure in \mathbb{C}^m , then $P^2(\mu)$ denotes the closure of polynomials in $L^2(\mu)$. If f is a sufficiently smooth function, then, writing $z = x + iy$, f_z stands for $\frac{1}{2}(\partial f/\partial x - i\partial f/\partial y)$ and $f_{\bar{z}}$ for $\frac{1}{2}(\partial f/\partial x + i\partial f/\partial y)$. The symbol \mathbb{D}^m will denote the closed unit polydisk in \mathbb{C}^m .

If μ is a compactly supported measure in \mathbb{C}^m and $N_i(S_i)$ is multiplication by the coordinate function z_i on $L^2(\mu)$ ($P^2(\mu)$), then $N = (N_1, \dots, N_m)$ ($S = (S_1, \dots, S_m)$) will be referred to as the multiplication tuple on $L^2(\mu)$ ($P^2(\mu)$). Theorem 4 is due to Hörmander and Wermer (see also [W1, W2]), and the rest of the theorems are due to Preskenis (see also [O'FP]). For $z = (z_1, \dots, z_m)$ in \mathbb{C}^m , $|z|$ stands for $\sqrt{|z_1|^2 + \dots + |z_m|^2}$.

Theorem 4 [HW]. *Let X be a compact subset of \mathbb{C}^m , and let Ω be an open neighbourhood of X . Also let $R = (R_1, \dots, R_m)$ be a vector-valued function defined on Ω . Assume there exists a positive $k < 1$ such that $|R(z) - R(z')| \leq k|z - z'|$ if $z, z' \in \Omega$. If each R_j is an $(m+2)$ -times continuously differentiable function, then the polynomials in z_1, \dots, z_m and $\bar{z}_1 + R_1, \dots, \bar{z}_m + R_m$ are dense in $C(X)$.*

Proposition 1. *Let $N = (N_1, \dots, N_m)$ be the multiplication tuple on $L^2(\mu)$ and $S = (S_1, \dots, S_m)$ be that on $P^2(\mu)$. Also let R_1, \dots, R_m be analytic*

polynomials such that $R = (R_1, \dots, R_m)$ satisfies the hypotheses of Theorem 4 (with $X = \text{support of } \mu$). If $\varphi = (z_1 + R_1, \dots, z_m + R_m)$, then $\varphi(N)$ is a minimal normal extension of $\varphi(S) \equiv \varphi(N)/P^2(\mu)$ and $\varphi(N)^*$ is a minimal normal dilation of $\varphi(S^*)$.

Remark 2. Statements such as those in Proposition 1 are obvious in view of (1) and an appropriate approximation theorem in several complex variables. Hereafter, we will not dwell on such straightforward applications of (1), though it must be mentioned that corresponding to every theorem mentioned below, a similar result as in Proposition 1 is valid. Our main interest lies in the application of Theorem 3 (or Corollary 1) to derive less obvious results. Note that the conclusion of Theorem 4 can be reformulated as follows: The $2m$ -variable analytic polynomials are uniformly dense in $C\{(z, \bar{z} + R(z)): z \in X\}$.

Proposition 2. For μ supported on \mathbb{D}^{2m} , let $N = (N_1, \dots, N_{2m})$ be the multiplication tuple on $L^2(\mu)$ and $S = (S_1, \dots, S_{2m})$ be that on $P^2(\mu)$, with S being doubly commuting. Let $\varphi_j = z_{j+m} - (\bar{z}_j + R_j)$ ($j = 1, \dots, m$), where $R = (R_1, \dots, R_m)$ satisfies the hypotheses of Theorem 4 with $X = \mathbb{D}^m$ and each R_j is just a polynomial in \bar{z}_j . Then $\varphi(N)$ is a minimal normal dilation of $\varphi(S) \equiv P(L^2(\mu), P^2(\mu))\varphi(N)|P^2(\mu)$.

Proof. Let $\mathcal{K}_0 \subset P^2(\mu)$ be the (dense) set of $2m$ -variable P^2 analytic polynomials with complex rational coefficients. By Corollary 1 it suffices to show that $L^2(\lambda_\xi) = \mathcal{K}_\xi$ a.e. $[\mu \circ \varphi^{-1}]$. ($\xi = (\xi_1, \dots, \xi_m) \in \varphi(\text{support of } \mu) \subset \mathbb{C}^m$.) Now $\mathcal{K}_\xi = L^2(\lambda_\xi)$ -closure of $\{h/\varphi^{-1}(\xi): h \in \mathcal{K}_0\}$, and λ_ξ is supported on

$$\begin{aligned} \varphi^{-1}(\xi) &= \{z = (z_1, \dots, z_{2m}) \in \text{support of } \mu: \varphi_j(z) = \xi_j\} \\ &= \{z \in \text{support of } \mu: z_{j+m} = \bar{z}_j + R_j + \xi_j\}. \end{aligned}$$

Since $R = (R_1 + \xi_1, \dots, R_m + \xi_m)$ satisfies the hypotheses of Theorem 4, it follows that $\{h/\varphi^{-1}(\xi): h \in \mathcal{K}_0\}$ is dense in $C(\varphi^{-1}(\xi))$ in the uniform norm and hence dense in $L^2(\lambda_\xi)$ for every ξ in $\varphi(\text{support of } \mu)$. \square

Remark 3. For $X = \mathbb{D}$ the hypotheses of Theorem 4 can be somewhat weakened; specifically, it suffices to assume that $|R(z) - R(z')| < |z - z'|$ for $z, z' \in \mathbb{D}$ with $z \neq z'$ (see [W1]). Letting $R_1(z_1) = \bar{z}_1^2/2$, it follows in particular that $N_2 - (N_1^* + N_1^{*2}/2)$ is a minimal normal dilation of $S_2 - (S_1^* + S_1^{*2}/2)$, where $S = (S_1, S_2)$ is the multiplication pair on the Hardy (or Bergman) space of the closed unit bidisk \mathbb{D}^2 and $N = (N_1, N_2)$ is the multiplication pair on $L^2(\mu)$, μ being the product arc-length measure on the distinguished boundary of \mathbb{D}^2 (or μ being the product area measure on \mathbb{D}^2).

Theorem 5 [P1]. Let $f \in C(\mathbb{D})$ be such that

- (i) f is differentiable on the interior of \mathbb{D} ,
- (ii) $\text{Re } f_{\bar{z}} \geq |f_z|$ on the interior of \mathbb{D} , and
- (iii) $f^{-1}(f(a))$ is countable for each a in \mathbb{D} .

Then the two-variable analytic polynomials are dense in $C(\{(z, f(z)): z \in \mathbb{D}\})$.

Proposition 3. For μ supported on \mathbb{D}^2 , let $N = (N_1, N_2)$ be the multiplication pair on $L^2(\mu)$ and let $S = (S_1, S_2)$ be that on $P^2(\mu)$, with S being doubly commuting. If $f(z)$ is a polynomial in \bar{z} such that $\text{Re } f_{\bar{z}} \geq 0$ on the interior

of \mathbb{D} and $\varphi(z_1, z_2) = z_2 - f(z_1)$, then $\varphi(N)$ is a minimal normal dilation of $\varphi(S)$.

Proof. Let $\mathcal{H}_0 \subset P^2(\mu)$ be the (dense) set of 2-variable analytic polynomials with complex rational coefficients. Note that, for $\xi \in \varphi$ (support of μ), $\varphi^{-1}(\xi) = \{(z_1, z_2) \in \text{support of } \mu : z_2 = f(z_1) + \xi\}$. Clearly, $g(z) = f(z) + \xi$ is a polynomial in \bar{z} satisfying the hypotheses of Theorem 5. Thus $\{h/\varphi^{-1}(\xi) : h \in \mathcal{H}_0\}$ is dense in $C(\varphi^{-1}(\xi))$ and Corollary 1 can be applied. \square

Remark 4. It is interesting to observe that the example mentioned in Remark 3 is also a consequence of Proposition 3 if one chooses $f(z) = \bar{z} + \bar{z}^2/2$. As another example, consider $\varphi(z_1, z_2) = z_2 - \exp(\bar{z}_1)$. Here $f(z) = \exp(\bar{z})$ is not a polynomial but is easily seen to satisfy the hypotheses of Theorem 5, so the conclusion of Proposition 3 still holds if S and N are as in Remark 3.

Theorem 6 [P2]. Let $f = \bar{z}^k \psi(|z|^{2k})$, where ψ is continuous in $[0, 1]$ and differentiable in $(0, 1)$, k is a positive integer, and $|f_{\bar{z}}| > |f_z|$ on (Interior of \mathbb{D}) $- \{0\}$. Then the 2-variable analytic polynomials are uniformly dense in $C(\{z, f(z) : z \in \mathbb{D}\})$.

An operator S is called quasi-normal if S and S^*S commute.

Proposition 4. For μ support on \mathbb{D}^2 , let $N = (N_1, N_2)$ be the multiplication pair on $L^2(\mu)$ and $S = (S_1, S_2)$ be that on $P^2(\mu)$, with S being doubly commuting and S_1 being quasi-normal. If f, ψ , and k are as in Theorem 6 with ψ being a polynomial, and $\varphi(z_1, z_2) = z_2 - f(z_1)$, then $\varphi(N)$ is a minimal normal dilation of $\varphi(S)$.

Proof. Let $f(z_1) = \bar{z}_1^k(a_0 + a_1|z_1|^{2p_1k} + \dots + a_n|z_1|^{2p_nk})$, where p_i are positive integers. Thus

$$\varphi(N) = N_2 - N_1^{*k}(a_0 + a_1N_1^{*p_1k}N_1^{p_1k} + \dots + a_nN_1^{*p_nk}N_1^{p_nk}).$$

The assumption of quasi normality of S_1 guarantees that

$$(S_i^{*(p_i+1)k}S_i^{p_i k})(S_i^{*(p_j+1)k}S_i^{p_j k}) = S_i^{*(p_i+p_j+2)k}S_i^{(p_j+p_i)k}.$$

(This requires an elementary verification.) Combining this observation with the assumption of ‘‘doubly commuting’’, it follows that $\varphi(N)$ is a normal dilation of $\varphi(S)$. It is clear that the function f in Theorem 6 may be replaced by $f_1 = f + \xi$ without affecting the conclusion there. (Any polynomial in z and $f_1 - \xi$ is a polynomial in z and f_1 .) The reader can now furnish the rest of the argument. \square

Remark 5. If f is as in the proof of Proposition 4, $a_0 = 0, a_i > 0$ for all i (or $a_i < 0$ for all i), and $p_1 < p_2 < \dots < p_n$, then it is easy to check that f satisfies the hypotheses of Theorem 6. Since the multiplication pair $S = (S_1, S_2)$ on the Hardy space of the bidisk is doubly commuting with S_1 (and S_2) being quasi-normal, it is interesting to check Proposition 4 for S . Here each S_i is actually an isometry and the corresponding N_i a unitary; thus the relevant conclusion is that $N_2 - aN_1^{*k}$ is a minimal normal dilation of $S_2 - aS_1^{*k}$ for any nonzero real a and positive integer k . The conclusion of Proposition 4 also holds if S , for example, is the multiplication pair on the tensor product of the Hardy space of \mathbb{D} with the Bergman space of \mathbb{D} (or the Hardy or Bergman space of the annulus $\{z \in \mathbb{C} : r \leq |z| \leq 1\}$ ($r > 0$)).

In order not to obscure the main ideas, we have avoided introducing too many generalities in the statements of propositions. In Proposition 2 one could let $\varphi_j = z_{j+m} - (\bar{z}_j + R_j)$, where R_j is a polynomial in $\bar{z}_{i_1}, \dots, \bar{z}_{i_m}$ being a permutation of $\{1, \dots, m\}$; also, the subtuples (S_1, \dots, S_m) and (S_{m+1}, \dots, S_{2m}) are not required to be doubly commuting there. The choice of functions φ can be generalized to involve coanalytic functions other than polynomials in \bar{z}_j . Also, the frequent assumption that S_i are contractions can be relaxed. There is one strengthening of the results, however, which demands more attention and is

Proposition 5. *In the notation of Proposition 2 and with $m = 1$ (or in the notation of Proposition 3 or 4), $p(\varphi(N))$ is a minimal normal dilation of $p(\varphi(S))$, where p is any one-variable polynomial.*

Proof. Let $\xi \rightarrow \lambda_\xi$ be the disintegration of μ with respect to $p \circ \varphi$. For $\xi \in (p \circ \varphi)(\text{support of } \mu)$, $(p \circ \varphi)^{-1}(\xi) = \{(z_1, z_2) \in \text{support of } \mu : p(\varphi(z_1, z_2)) = \xi\} = \bigcup_{i=1}^n F_i$, where $F_i = \{(z_1, z_2) \in \text{support of } \mu : \varphi(z_1, z_2) = \eta_i\}$, η_1, \dots, η_n being n distinct roots of $p - \xi$. Since our hypotheses guarantee that 2-variable analytic polynomials are uniformly dense in $C(F_i)$, it follows by a well-known theorem from the theory of function algebras (see [L, Chapter 3, Theorem 8]) that these are uniformly dense in $C(\bigcup_{i=1}^n F_i)$ as well. The rest is obvious. \square

The above discussion indicates that the methods introduced in [C2] need more investigation and have interesting operator-theoretic implications besides the ones for the minimal normal extension problem as considered in [C1, C2, D].

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