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EQUIVARIANT MAPS FOR HOMOLOGY REPRESENTATIONS

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ABSTRACT. If Y is a homotopy representation of the finite group G of order n and X is a finite G-CW complex such that, for each subgroup H of G, $H_*(X^H; \mathbb{Z}_n) = H_*(Y^H; \mathbb{Z}_n)$ then there exists a G-map $f: X \to Y$ such that $f_*^H: H_*(X^H; \mathbb{Z}_n) \to H_*(Y^H; \mathbb{Z}_n)$ is an isomorphism for each subgroup H.

INTRODUCTION

Let G be a finite group of order n. A *homotopy representation* [tD4] of G is a finite G-CW complex Y such that, for each $H \leq G$, Y^H is an $n(H)$-dimensional complex, homotopy equivalent to a sphere $S^{n(H)}$ for some integer $n(H) \geq -1$ (-1 signifies that Y^H is empty). Natural examples of homotopy representations are the unit spheres of orthogonal representations. From the viewpoint of the function defined on subgroups of G by the numbers $n(H)$ (the *dimension function*), if G is a p-group these are the only examples [DH, tD4]. For general G, homotopy representations can deviate greatly from the linear situation, either in terms of dimension functions or linking of fixed point sets (even p-groups have homotopy representations with nonstandard linking of fixed sets) [tD3].

Suppose that X is a finite G-CW complex such that, for each $H \leq G$, X^H has the $\mathbb{Z}_n$-homology of a sphere (where G has order n). Such an X is called a *$\mathbb{Z}_n$-homology representation* of G (compare [tD4]). If X and Y are $\mathbb{Z}_n$-homology representations of G such that, for each $H \leq G$, X^H and Y^H have the same $\mathbb{Z}_n$-homology then X and Y are *$\mathbb{Z}_n$-homology equivalent.* Suppose now that X is a $\mathbb{Z}_n$-homology representation of G which is homology equivalent to a homotopy representation Y of G. In this situation we obtain

Theorem. *There exists a G-map $\phi: X \to Y$ such that, for any $H \leq G$,*

$$\phi_*^H: H_*(X^H; \mathbb{Z}_n) \to H_*(Y^H; \mathbb{Z}_n)$$

is an isomorphism.

In particular, let G be a p-group. If X is a $\mathbb{Z}_p$-homology representation of G then by [DH, tD4] there exists an orthogonal representation V of G such that X is homology equivalent to $S(V)$.

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Corollary. *Let G be a p-group and X a finite G-CW complex. If X is a $\mathbb{Z}_p$-homology sphere then there is a G-map $\phi\colon X \to S(V)$ such that $\phi_*^H\colon H_*(X^H; \mathbb{Z}_p) \to H_*(S(V)^H; \mathbb{Z}_p)$ is an isomorphism for all $H \leq G$.*

This corollary improves on [DH], where the existence of a stable map was shown.

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In this section we will prove the theorem, modulo several auxiliary results which are proved in §2. If $H \leq G$ with normaliser NH, as is standard, we denote NH/H by WH.

Let X be as in the statement of the theorem. Suppose $(H_1), \dots, (H_m)$ are the conjugacy classes of isotropy subgroups of G ordered in such a way that $(H_i) < (H_j)$ (i.e., H_i subconjugate to H_j) implies $j < i$. Put $X_j = \{x \in X \mid (G_x) = (H_i) \text{ for some } i \leq j\}$. Each X_j is a G subspace of X and $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m = X$. We would like to construct a G-map $\phi_m\colon X \to Y$ satisfying the condition of the theorem by induction over the X_i. Let $X_0 = \varnothing$, and suppose we have found $\phi_i\colon X_i \to Y$ such that whenever $(H) = (H_j)$, for some $j \leq i$, $(\phi_i^H)_*\colon H_*(X_i^H; \mathbb{Z}_n) \to H_*(Y^H; \mathbb{Z}_n)$ is an isomorphism. We would like to extend ϕ_i to a G-map $\phi_{i+1}\colon X_{i+1} \to Y$. By [tD2, 8.1.5] such extensions are in bijection with the WH extensions $\phi_{i+1}^H\colon X_{i+1}^H \to Y^H$ of $\phi_i^H\colon X_i^H \to Y^H$ for all H such that $(H) = (H_{i+1})$. Let $H = H_{i+1}$. We look for such an extension $\phi_{i+1}^H\colon X_{i+1}^H = X^H \to Y^H$ which induces a homology isomorphism. If $n(H) = 0$ then X^H has two acyclic components and the definition of ϕ_{i+1}^H is clear. If $n(H) \geq 1$, since Y^H is $(n(H)-1)$-connected, certainly there are equivariant extensions of ϕ_i^H to the $n(H)$-skeleton, $\phi_{i+1}^H\colon (X^H)^{(n(H))} \cup X_i^H \to Y^H$. $\phi_i^M\colon X^M \to Y^M$ induces a $\mathbb{Z}_n$-homology isomorphism whenever $H < M$. It follows using Mayer-Vietoris and Smith theory that there is an extension of ϕ_i^H to the $n(H)$-skeleton which induces a $\mathbb{Z}_n$-homology epimorphism. Obstructions to extending ϕ_{i+1}^H to the $(k+1)$-skeleton $(k \geq n(H))$ lie in the group $H_{WH}^{k+1}(X^H, X_i^H; \tilde{\omega}_k(Y^H))$ (see [B]). Because WH acts freely off X_i^H, this group is the $(k+1)$-cohomology of the cochain complex $\operatorname{Hom}_{\mathbb{Z}[WH]}(C_*(X^H, X_i^H; \mathbb{Z}), \pi_k(Y^H)) = C^*$. By Proposition 2 this cohomology group is torsion-prime to $n = |G|$. By Proposition 3 if θ is an obstruction, say of order d, then there is a WH map $f\colon Y^H \to Y^H$ of degree of power of d such that $f \circ \phi_{i+1}^H$ can be extended to the $(k+1)$-skeleton. Eventually we arrive at a WH map $\phi_{i+1}^H\colon X_{i+1}^H = X^H \to Y^H$, which by construction induces a $\mathbb{Z}_n$-homology isomorphism. Finally ϕ_{i+1}^H yields a G-map $\phi_{i+1}\colon X_{i+1} \to Y$ by defining $\phi_{i+1}(x) = g\phi_{i+1}^{H_{i+1}}(g^{-1}x)$ if $x \in gX^{H_{i+1}} = X^{gH_{i+1}g^{-1}}$ [tD2, 8.1.5].

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In this section we prove the several results used in §1. We begin with a lemma needed later to establish Proposition 2.

Lemma 1. *Let X be a $\mathbb{Z}_n$-homology representation of the finite group G of order n which is homology equivalent to the homotopy representation Y. Suppose that*

$H \le G$ and Ω is a collection of subgroups of G which contain H. Then

$$H_*\left(X^H, \bigcup_\Omega X^M; \mathbb{Z}_n\right) = 0, \qquad * > n(H).$$

Proof. Use induction on $n(H)$ and $|\Omega|$. So suppose $M_0 \in \Omega$ has $n(M_0) < n(H)$. Set $\Omega_0 = \Omega - \{M_0\}$. Throughout we use $\mathbb{Z}_n$-coefficients. By Mayer-Vietoris,

$$(*)\qquad \begin{aligned} &H_1(X^H, X^{M_0}) \oplus H_1\left(X^H, \bigcup_{\Omega_0} X^M\right) \\ &\to H_i\left(X^H, \bigcup_{\Omega} X^M\right) \to H_{i-1}\left(X^H, \bigcup_{\Omega_0} X^{MM_0}\right). \end{aligned}$$

It follows immediately by induction that $H_i(X^H, \bigcup_\Omega X^M; \mathbb{Z}_n)$ is zero for $i > n(H)+1$. For $i = n(H)+1$, by induction we have

$$(**)\qquad \begin{aligned} &0 \to H_{n(H)+1}\left(X^H, \bigcup_{\Omega} X^M\right) \to H_{n(H)}\left(X^H, \bigcup_{\Omega} X^{MM_0}\right) \\ &\to H_{n(H)}((X^H, X^{M_0}) \oplus H_{n(H)}\left(X^H, \bigcup_{\Omega_0} X^M\right). \end{aligned}$$

Since $n(M_0) < n(H)$, by induction again $H_{n(H)}(X^{M_0}, \bigcup_{\Omega_0} X^{MM_0}) = 0$. So the inclusion $H_{n(H)}\left(X^H, \bigcup_{\Omega_0} X^{MM_0}\right) \to H_{n(H)}(X^H, X^{M_0})$ is a monomorphism as is seen by looking at the triple $(X^H, X^{M_0}, \bigcup_{\Omega_0} X^{MM_0})$. Therefore, the rightmost map in $(**)$ is a monomorphism, so $H_{n(H)+1}(X^H, \bigcup_\Omega X^M) = 0$.

On the other hand if every $M \in \Omega$ has $n(M) = n(H)$, let $M_0 \in \Omega$ be arbitrary. If $M_1, M_2 \in \Omega$ then $n(M_1) = n(M_2) = n(H)$. Now X is homology equivalent to the homotopy representation Y. Apply relative Mayer-Vietoris to the pairs (Y^H, Y^{M_1}) and (Y^H, Y^{M_2}). One has

$$0 \to H_{n(H)+1}(Y^H, Y^{M_1} \cup Y^{M_2}) \to H_{n(H)}(Y^H, Y^{M_1M_2}) \to 0.$$

Since Y^H has no cells in dimensions higher than $n(H)$,

$$H_{n(H)+1}(Y^H, Y^{M_1} \cup Y^{M_2}) = 0.$$

This shows that $n(M_1M_2) = n(H)$, where M_1M_2 is the subgroup generated by M_1, M_2. Now from $(*)$ (with M_0 arbitrary such that $n(M_0) = n(H)$) it follows by induction on $|\Omega|$ that $H_i(X^H, \bigcup_\Omega X^M) = 0$ for $i > n(H)$. □

Proposition 2. *Suppose that X is a $\mathbb{Z}_n$-homology representation of G which is homology equivalent to a homotopy representation Y of G, where $Y^H \simeq S^{n(H)}$ for all $H \le G$. Let $k \ge n(H)$, and set*

$$C^* = \mathrm{Hom}_{\mathbb{Z}[WH]}(C_*(X^H, X_i^H; \mathbb{Z}), \pi_k(Y^H)).$$

Then $H^{k+1}(C^*)$ is torsion-prime to $n = |G|$.

Proof. $C_*(X^H, X_i^H; \mathbb{Z})$ is a free $\mathbb{Z}[WH]$ chain complex. Let $p|n$. By [CE, pp. 30, 3′] the cochain complex $\mathrm{Hom}_{\mathbb{Z}[WH]}(C_*(X^H, X_i^H, \mathbb{Z}), \mathbb{Z}_p)$ is chain isomorphic to $\mathrm{Hom}_{\mathbb{Z}_p[WH]}(C_*(X^H, X_i^H; \mathbb{Z}_p), \mathbb{Z}_p)$. For the moment suppose that the cohomology of the latter cochain complex is zero in dimensions larger than k. For an abelian group A, let A_p denote the subgroup of elements of order p. One has the two short exact sequences,

$$0 \to \pi_k(Y^H)_p \xrightarrow{j} \pi_k(Y^H) \to p\pi_k(Y^H) \to 0,$$
$$0 \to p\pi_k(Y^H) \xrightarrow{i} \pi_k(Y^H) \xrightarrow{\pi} \pi_k(Y^H) \otimes \mathbb{Z}_p \to 0.$$

These induce exact sequences in cohomology. To see that $H^{k+1}(C^*)$ is torsion-prime to p, it suffices to know that the cochain complexes

$$\mathrm{Hom}_{\mathbb{Z}[WH]}(C_*(X^H, X_i^H, \mathbb{Z}), \pi_k(Y^H)_p)$$

and

$$\mathrm{Hom}_{\mathbb{Z}[WH]}(C_*(X^H, X_i^H, \mathbb{Z}), \pi_k(Y^H) \otimes \mathbb{Z}_p)$$

have zero cohomology in dimensions $k+2$ and $k+1$, respectively. But this follows by induction on the order of the groups $\pi_k(Y^H)_p$, $\pi_k(Y^H) \otimes \mathbb{Z}_p$ and the fact, now to be established, that $\mathrm{Hom}_{\mathbb{Z}_p[WH]}(C_*(X^H, X_i^H; \mathbb{Z}_p), \mathbb{Z}_p)$ has zero cohomology in dimensions greater than k. Let M be the cellular dimension of X^H. Then one has

$$(***) \qquad \begin{aligned} &0 \to C_M(X^H, X_i^H; \mathbb{Z}_p) \xrightarrow{\partial_M} \cdots \to C_{k+1}(X^H, X_i^H; \mathbb{Z}_p) \\ &\xrightarrow{\partial_{k+1}} C_k(X^H, X_i^H; \mathbb{Z}_p) \twoheadrightarrow \frac{C_k(X^H, X_i^H; \mathbb{Z}_p)}{\mathrm{Im}\,\partial_{k+1}}. \end{aligned}$$

This sequence is exact by Lemma 1. For $j \geq k+1$ the jth cohomology is $\mathrm{Ext}^{j-k}_{\mathbb{Z}_p[WH]}((C_k(X^H, X_i^H; \mathbb{Z}_p)/\mathrm{Im}\,\partial_{k+1}, \mathbb{Z}_p)$. This is zero since the $\mathbb{Z}_p[WH]$-module $(C_k(X^H, X_i^H; \mathbb{Z}_p)/\mathrm{Im}\,\partial_{k+1}$ is projective. For, by [R], projective over $\mathbb{Z}_p[WH]$ is the same as injective over $\mathbb{Z}_p[WH]$. So in $(***)$ $C_m(X^H, X_i^H; \mathbb{Z}_p)$ being projective is injective. Hence $\mathrm{Im}\,\partial_{M-1} = \ker \partial_{M-2}$ is projective, and hence injective. So $\mathrm{Im}\,\partial_{M-2}$ is projective and so on. Finally, the module $(C_k(X^H, X_i^H; \mathbb{Z}_p))/\mathrm{Im}\,\partial_{k+1}$ is projective. □

Proposition 3. *Suppose that* $\theta \in H^{k+1}_{\mathbb{Z}[WH]}(X^H, X_i^H; \tilde{\omega}_k(Y^H))$ *is an obstruction to extending the WH map* $\phi^H_{i+1}: X_i^H \cup (X^H)^{(k)} \to Y^H$ *to the* $(k+1)$*-skeleton of* X^H. *If* θ *has order* d, *there exists a* WH *map* $f: Y^H \to Y^H$ *of degree a power of* d *such that* $f \circ \phi^H_{i+1}$ *can be extended.*

Proof. Given an element $\alpha \in \pi_k(Y^H) = \pi_k(S^{n(H)})$ of order d, one would like to multiply α by d in following α by a map $f: Y^H \to Y^H$ of degree d. However, $f \circ \alpha$ might not be $d\alpha$ since composition is not left additive (but is right additive) [W, Chapter 11]. However, if $f: Y^H \to Y^H$ is a map of degree a suitable power k of d then $f \circ \alpha = k\alpha$. For by [W, Chapter 11, Theorem 8.9] if $n(H)$ is odd and $k \equiv 0$ or $1 \pmod 4$ one has $(k) \circ \alpha = k\alpha$, where (k) denotes a map of degree k. If $n(H)$ is even and $k \equiv 0$, $+1$, or -1 $\pmod 9$ *and* $k \equiv 0$ or $1 \pmod 4$ then $(k) \circ \alpha = k\alpha + \binom{k}{2}(\alpha + (-i) \circ \alpha)$, where i is the identity map on $S^{n(H)}$. Because composition is right additive, $d(\alpha + (-i) \circ \alpha) = 0$. So if $d | \binom{k}{2}$ then $(k) \circ \alpha = k\alpha$. One can check that $k = d^6$ works for all cases.

Finally according to [tD1, Theorem 4.11] one can find a WH equivariant map $f: Y^H \to Y^H$ of degree k which is the identity on Y_i^H so long as $k \equiv 1$

(mod $|WH|$). But since d is relatively prime to $|WH|$, $(d^6)^j \equiv 1$ (mod $|WH)$ for some j. So an WH map of degree d^{6j} may be used. □

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