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FINITE DIMENSIONALITY OF IRREDUCIBLE UNITARY REPRESENTATIONS OF COMPACT QUANTUM GROUPS

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ABSTRACT. In this paper, we study the representations of Hopf C^*-algebras; the main result is that every irreducible left unitary representation of a Hopf C^*-algebra with a Haar measure is finite dimensional. To prove this result, we first study the comodule structure of the space of Hilbert-Schmidt operators; then we use this comodule structure to show that every irreducible left unitary representation of a Hopf C^*-algebra with a Haar measure is finite dimensional.

1. INTRODUCTION

It is well known that, for compact groups, every irreducible unitary representation is finite dimensional. For a simple proof about this result, we refer to [N]. In this paper, we will generalize this result to Hopf C^*-algebras with Haar measures; namely, we will prove that every irreducible left unitary representation of a Hopf C^*-algebra with a Haar measure is finite dimensional. To prove this result, we first study the comodule structure of Hilbert-Schmidt operators; then we use the result about Hilbert-Schmidt operators to show that every irreducible left unitary representation is finite dimensional.

In earlier work in this direction, Woronowicz [W] proved that every irreducible unitary representation of a compact matrix quantum group is finite dimensional; the author [Q] showed that for a Hopf C^*-algebra with the Peter-Weyl property, every irreducible unitary representation is finite dimensional. The previous approach depends heavily on the Peter-Weyl property. Here we are going to generalize these results; the method we use is elementary, which does not use the Peter-Weyl property.

Before we turn to the contents of the paper, let us recall the definitions of Hopf C^*-algebras, representations, and Haar measures of Hopf C^*-algebras.

Let A be a C^*-algebra with a dense *-subalgebra $\mathscr{A}$ and $\Phi\colon A \to A \otimes A$ a

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C^*-homomorphism. We say that (A, Φ) is a Hopf C^*-algebra if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & A \otimes A \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi\otimes I} \\ A \otimes A & \xrightarrow{I\otimes\Phi} & A \otimes A \otimes A \end{array}$$

commutes and $\Phi(\mathscr{A}) \subset \mathscr{A} \otimes \mathscr{A}$, where Φ is the comultiplication of (A, Φ). If A is a von Neumann algebra, we call A a Hopf-von Neumann algebra. By an involution of (A, Φ), we mean a $*$-anti-isomorphism $k: \mathscr{A} \to \mathscr{A}$ such that the diagram

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{k} & \mathscr{A} & \xrightarrow{\Phi} & \mathscr{A} \otimes \mathscr{A} \\ \downarrow{\scriptstyle \Phi} & & & & {\scriptstyle \tau}\downarrow \\ \mathscr{A} \otimes \mathscr{A} & & \xrightarrow{k\otimes k} & & \mathscr{A} \otimes \mathscr{A} \end{array}$$

commutes where $\tau: A \otimes A \to A \otimes A$ is the flip automorphism and $\tau(a \otimes b) = b \otimes a$, $\forall a, b \in A$.

We say that $e: \mathscr{A} \to C$ is a counit of A if e is a C^*-homomorphism and the following diagram commutes:

$$\begin{array}{ccccc} \mathscr{A} \otimes C & \longrightarrow & \mathscr{A} & \longrightarrow & C \otimes \mathscr{A} \\ & {\scriptstyle I\otimes e}\nwarrow & {\scriptstyle \Phi}\downarrow & \nearrow{\scriptstyle e\otimes I} & \\ & & \mathscr{A} \otimes \mathscr{A} & & \end{array}$$

We say that (A, Φ, k, e) is a compact quantum group if the following diagrams commute:

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{e} & C & \xrightarrow{i} & \mathscr{A} \\ {\scriptstyle \Phi}\downarrow & & & & \downarrow{\scriptstyle m} \\ \mathscr{A} \otimes \mathscr{A} & & \xrightarrow{k\otimes I} & & \mathscr{A} \otimes \mathscr{A} \end{array}$$

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{e} & C & \xrightarrow{i} & \mathscr{A} \\ {\scriptstyle \Phi}\downarrow & & & & \downarrow{\scriptstyle m} \\ \mathscr{A} \otimes \mathscr{A} & & \xrightarrow{I\otimes k} & & \mathscr{A} \otimes \mathscr{A} \end{array}$$

Let A be a coalgebra and M a linear space. Let $\psi: M \to A \otimes M$ be a linear map which makes the following diagrams commute:

$$\begin{array}{ccc} C \otimes M & \xleftarrow{e\otimes I} & A \otimes M \\ & \searrow^{\sim} & \uparrow \psi \\ & & M \end{array}$$

$$\begin{array}{ccc} A \otimes A \otimes M & \xleftarrow{I\otimes\psi} & A \otimes M \\ \uparrow \Delta\otimes I & & \uparrow \psi \\ A \otimes M & \xleftarrow{\psi} & M \end{array}$$

The pair (M, ψ) is called a unital left A-comodule; if only the second diagram commutes, we call (M, ψ) a left A-comodule, and ψ is said to be its structure map. A right A-comodule can be defined similarly. For more information about comodules, we refer to [A].

Let M be a left A-comodule with structure map ψ; a subspace $M_1 \subset M$ is said to be left invariant if $\psi(M_1) \subset A \otimes M_1$. We say that M is an irreducible left A-comodule if M is the only nonzero left invariant subspace of M.

Now let (A, Φ) be a Hopf C^*-algebra with unit and $\mathscr{A}$ the dense *-subalgebra of A. We say that V is a left A-comodule if it is a left $\mathscr{A}$-comodule which is defined as above. Suppose that V is a finite-dimensional left A-comodule with structure map $\psi: V \to \mathscr{A} \otimes V$. If $\{e_i\}_{i=1}^n$ is a basis of V and

$$\psi(e_i) = \sum_{k=1}^{n} a_{ik} \otimes e_k,$$

the matrix (a_{ij}) is called the coefficient matrix of ψ with respect to $\{e_i\}_{i=1}^n$. Then the comodule property implies that

$$\Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}.$$

Now suppose that V is a left A-comodule, which is also a Hilbert space, with structure map $L: V \to \mathscr{A} \otimes V$ and inner product $(\cdot, \cdot): V \times V \to C$. We can extend $(\cdot, \cdot)$ to $\langle \cdot, \cdot \rangle: (\mathscr{A} \otimes V) \times (\mathscr{A} \otimes V) \to \mathscr{A}$ as

$$\langle a \otimes x, b \otimes y \rangle = ab^*(x, y), \quad \forall a, b \in \mathscr{A}, \ x, y \in V.$$

A left unitary representation π of A on a Hilbert space H is a comodule map from H into $\mathscr{A} \otimes H$ such that

$$\langle \pi(x), \pi(y) \rangle = \langle x, y \rangle, \quad \forall x, y \in H.$$

Let (A, Φ, k, e) be a Hopf C^*-algebra and σ a positive linear functional. We say that σ is a left Haar measure if, for all $x^* \in A^*$, we have

$$x^* \cdot \sigma = \langle x^*, I \rangle \sigma.$$

Similarly, we can define a right Haar measure.

Note that for a left (right) Haar measure σ, we have $\sigma(k(a)) = \sigma(a)$, $\forall a \in \mathscr{A}$. For a proof of this result, we refer to [W].

Finally, we come to the contents of this paper. In §2 we study the comodule structure of Hilbert-Schmidt operators. The main results in this section are that, for any two Hilbert spaces H_1, H_2, which are left unitary left $\mathscr{A}$-comodules, where $\mathscr{A}$ is the dense *-subalgebra of a Hopf C^*-algebra, the space $\mathrm{Hom}_2(H_1, H_2)$ of Hilbert-Schmidt operators from H_1 to H_2 has a natural right $\mathscr{A}$-comodule structure. Also we give a characterization that under what condition a Hilbert-Schmidt operator is a comodule map. In §3 we first show that the space of Hilbert-Schmidt operators is invariant under the action of a Haar measure. Then we use the results in §2 to show that every irreducible left unitary representation of a Hopf C^*-algebra with a Haar measure is finite dimensional.

2. Comodule structure for Hilbert-Schmidt operators

In this section, we are going to endow a comodule structure to the space of Hilbert-Schmidt operators between two Hilbert spaces which are left unitary comodules of a Hopf C^*-algebra.

Let H_1, H_2 be Hilbert spaces. An operator $T \in B(H_1, H_2)$ is said to be a Hilbert-Schmidt operator if, for one orthonormal basis $(e_i^1) \subset H_1$, $\sum_i \|Te_i^1\|^2 < \infty$. Let $\mathrm{Hom}_2(H_1, H_2)$ denote the space of all Hilbert-Schmidt operators from H_1 to H_2. For every $T \in H_2(H_2, H_2)$, let

$$\|T\|_2^2 = \sum_i \|Te_i^1\|^2.$$

It is well known that $(\mathrm{Hom}_2(H_1, H_2), \|\cdot\|_2)$ is a Hilbert space with the inner product given by

$$\langle T_1, T_2 \rangle = \sum_i \langle T_1 e_i^1, T_2 e_i^1 \rangle.$$

Now let (A, Φ, k, e) be a Hopf C^*-algebra with a dense *-subalgebra $\mathscr{A}$. Suppose that H_1, H_2 are also left unitary $\mathscr{A}$-comodules with structure maps ψ_1 and ψ_2. Fix orthonormal bases $\{e_i^1\}, \{e_j^2\}$ for H_1, H_2 respectively. Suppose that

$$\psi_1(e_i^1) = \sum_k a_{ik}^1 \otimes e_k^1, \qquad \psi_2(e_j^2) = \sum_s a_{js}^2 \otimes e_s^2.$$

Let H_1' be the complex conjugate of H_1. Then $\{e_i^1\}$ also form an orthonormal basis for H_1'. It is straightforward to verify that

$$\psi_1'(e_i^1) = \sum_k e_k^1 \otimes a_{ki}^1$$

gives H_1' a right $\mathscr{A}$-comodule structure.

Since $\mathrm{Hom}_2(H_1, H_2) \cong H_1' \hat{\otimes} H_2$ (projective tensor product), define

$$\psi_{H_1, H_2}\colon \mathrm{Hom}_2(H_1, H_2) \to \mathrm{Hom}_2(H_1, H_2) \otimes \mathscr{A}$$

as

$$\psi_{H_1, H_2}(e_i^1 \otimes e_j^2) = \sum_{k,s} (e_k^1 \otimes e_s^2) \otimes a_{ki}^1 k(a_{js}^2).$$

Proposition 2.1. *With the notation as above,* $(\mathrm{Hom}_2(H_1, H_2), \psi_{H_1,H_2})$ *is a right* $\mathscr{A}$*-comodule with structure map* ψ_{H_1,H_2}.

Proof. We only need to verify that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_2(H_1, H_2) & \xrightarrow{\psi_{H_1,H_2}} & \mathrm{Hom}_2(H_1, H_2)\otimes\mathscr{A} \\ \Big\downarrow{\scriptstyle \psi_{H_1,H_2}} & & \Big\downarrow{\scriptstyle \psi_{H_1,H_2}\otimes I} \\ \mathrm{Hom}_2(H_1, H_2)\otimes\mathscr{A} & \xrightarrow{I\otimes\Phi} & \mathrm{Hom}_2(H_1, H_2)\otimes\mathscr{A}\otimes\mathscr{A} \end{array}$$

In fact, for $e_i^1\otimes e_j^2$, we have

$$\begin{aligned}(\psi_{H_1,H_2}\otimes I)\psi_{H_1,H_2}(e_i^1\otimes e_j^2) &= \sum_{k,s}\psi_{H_1,H_2}(e_k^1\otimes e_s^2)\otimes a_{ki}^1 k(a_{js}^2) \\ &= \sum_{k,s}\sum_{p,q}(e_p^1\otimes e_q^2)\otimes a_{pk}^1 k(a_{sq}^2)\otimes a_{ki}^1 k(a_{js}^2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(I\otimes\Phi)\psi_{H_1,H_2}(e_i^1\otimes e_j^2) &= \sum_{k,s}(e_k^1\otimes e_s^2)\Phi(a_{ki}^1 k(a_{js}^2)) \\ &= \sum_{k,s}\sum_{p,q}(e_p^1\otimes e_q^2)\otimes a_{pk}^1 k(a_{sq}^2)\otimes a_{ki}^1 k(a_{js}^2).\end{aligned}$$

This completes the proof.

For $T\in\mathrm{Hom}_2(H_1, H_2)$, $v\in H_1$, we have

$$(\psi_{H_1,H_2}T)(v) = \tau(m\otimes I)(I\otimes k\otimes I)(I\otimes\psi_2)(I\otimes T)(\psi_1(v)).$$

The main result of this section is

Theorem 2.2. *With the above assumption and for* $T\in\mathrm{Hom}_2(H_1, H_2)$*, the following are equivalent:*

(1) T *is a left* $\mathscr{A}$*-comodule map,*
(2) T *is a right* A^**-module map,*
(3) $\psi_{H_1,H_2}(T) = T\otimes I$,
(4) $x^*\cdot T = x^*(I)T$, $\forall x^*\in A^*$.

Proof. The equivalence between (1) and (2) and (3) and (4) is well known. So we only need to prove the equivalence of (1) and (3).

(1) $\Rightarrow$ (3). Suppose that T is a left $\mathscr{A}$-comodule map. Then

$$\begin{array}{ccc} H_1 & \xrightarrow{T} & H_2 \\ \Big\downarrow{\scriptstyle \psi_1} & & \Big\downarrow{\scriptstyle \psi_2} \\ \mathscr{A}\otimes H_1 & \xrightarrow{I\otimes T} & \mathscr{A}\otimes H_2 \end{array}$$

commutes. So, for any $e_i^1\in H_1$, we have

$$\psi_2(Te_i^1) = (I\otimes T)\psi_1(e_i^1),$$

i.e.,

$$\psi_2(Te_i^1) = \sum_k (I \otimes T)(a_{ik}^1 \otimes e_k^1) = \sum_k a_{ik}^1 \otimes Te_k^1,$$

$$\begin{aligned}(\psi_{H_1,H_2}T)(e_i^1) &= \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2)(I \otimes T)\psi_1(e_i^1) \\ &= \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2)\left(\sum_k a_{ik}^1 \otimes Te_k^1\right) \\ &= \tau(m \otimes I)(I \otimes k \otimes I)\left(\sum_{k,s} a_{ik}^1 \otimes a_{ks}^1 \otimes Te_s^1\right) \\ &= \sum_{k,s} Te_s^1 \otimes a_{ik}^1 k(a_{ks}^1) = \sum_s Te_s^1 \otimes e(a_{is}^1) \\ &= T\left(\sum_s e(a_{is}^1)e_s^1\right) \otimes I = T(e_i^1) \otimes I.\end{aligned}$$

So $\psi_{H_1,H_2}(T) = T \otimes I$.

(3) $\Rightarrow$ (1). Suppose that $\psi_{H_1,H_2}(T) = T \otimes I$. Then, for any $e_i^1 \in H_1$, we have

$$(\psi_{H_1,H_2}T)(e_i^1) = T(e_i^1) \otimes I.$$

Let $Te_i^1 = \sum_p b_{ip}e_p^2$. So we have

$$\begin{aligned}(\psi_{H_1,H_2}T)(e_i^1) &= \tau(m \otimes I)(I \otimes k \otimes I)(I \otimes \psi_2)\left(\sum_k a_{ik}^1 \otimes Te_k^1\right) \\ &= \tau(m \otimes I)(I \otimes k \otimes I)\left(\sum_k a_{ik}^1 \otimes \psi_2(Te_k^1)\right) \\ &= \sum_{k,p} b_{kp}\tau(m \otimes I)(I \otimes k \otimes I)\left(a_{ik}^1 \otimes \sum_s a_{ps}^2 \otimes e_s^2\right) \\ &= \sum_{k,p,s} b_{kp}e_s^2 \otimes a_{ik}^1 k(a_{ps}^2).\end{aligned}$$

Thus

$$\begin{aligned}&(m \otimes I)(2,3,1)(\psi_2 \otimes I)(\psi_{H_1,H_2}T)(e_i^1) \\ &\quad = (m \otimes I)(2,3,1)\left(\sum_{k,p,s,t} b_{kp}a_{st}^2 \otimes e_t^2 \otimes a_{ik}^1 k(a_{ps}^2)\right) \\ &\quad = (m \otimes I)\left(\sum_{k,p,s,t} b_{kp}a_{ik}^1 k(a_{ps}^2) \otimes a_{st}^2 \otimes e_t^2\right) = \sum_{k,p,s,t} b_{kp}a_{ik}^1 k(a_{ps}^2)a_{st}^2 \otimes e_t^2) \\ &\quad = \sum_{k,p,t} b_{kp}e(a_{pt}^2)a_{ik}^1 \otimes e_t^2 = \sum_{k,p} a_{ik}^1 \otimes b_{kp}e_p^2 = \sum_k a_{ik}^1 \otimes Te_k^1,\end{aligned}$$

where $(2,3,1)$ is the map from $\mathscr{A} \otimes H \otimes \mathscr{A}$ into $\mathscr{A} \otimes \mathscr{A} \otimes H$ such that

$(2, 3, 1)(a \otimes h \otimes b) = b \otimes a \otimes h$, $\forall a, b \in \mathscr{A}$, $h \in H$. On the other hand,

$$(m \otimes I)(2, 3, 1)(\psi_2 \otimes I)(Te_i^1 \otimes I) = (m \otimes I)(2, 3, 1)\left(\sum_p b_{ip}\psi_2(e_p^2) \otimes I\right)$$
$$= \sum_{p,s} b_{ip}a_{ps}^2 \otimes e_s^2 = \psi_2(Te_i^1).$$

Thus we have that $\psi_2(Te_i^1) = \sum_k a_{ik}^1 \otimes Te_k^1$, i.e., T is a comodule map. This completes the proof.

3. Finite dimensionality of irreducible unitary representations

In this section, we are going to show that every irreducible unitary representation of a Hopf C^*-algebra with a Haar measure is finite dimensional. We begin the section with the following result, which is the consequence of Theorem 2.2.

Proposition 3.1. *With the same notation as above, if* $T \in \mathrm{Hom}_2(H_1, H_2)$, *then* $\sigma \cdot T \in \mathrm{Hom}_2(H_1, H_2)$.

Proof. Since $T = \sum_{i,j} b_{ij}e_i^1 \otimes e_j^2$, where $\sum_{i,j}|b_{ij}|^2 < \infty$,

$$\begin{aligned}\sigma \cdot T &= (I \otimes \sigma)\psi_{H_1, H_2}(T)\\ &= (I \otimes \sigma)\left[\sum_{i,j} b_{ij}\psi_{H_1, H_2}(e_i^1 \otimes e_j^2)\right]\\ &= (I \otimes \sigma)\left[\sum_{i,j} b_{ij}\sum_{k,s}(e_k^1 \otimes e_s^2)a_{ki}^1k(a_{js}^2)\right]\\ &= \sum_{k,s}\left[\sum_{i,j} b_{ij}\sigma(a_{ki}^1k(a_{js}^2))\right] e_k^1 \otimes e_s^2.\end{aligned}$$

But

$$\sum_{k,s}\sum_{i,j}|b_{ij}|^2|\sigma(a_{ki}^1k(a_{js}))|^2 \le \sum_{k,s}\sum_{i,j}|b_{ij}|^2\sigma(a_{ki}^1(a_{ki}^1)^*)\sigma(a_{js}^2(a_{js}^2)^*)$$
$$= \sum_{i,j}|b_{ij}|^2\sigma\left(\sum_k a_{ki}^1(a_{ki}^1)^*\right)\sigma\left(\sum_s a_{js}^2(a_{js}^2)^*\right) = \sum_{i,j}|b_{i,j}|^2,$$

since H_1, H_2 are unitary $\mathscr{A}$-comodules. Hence $\sigma \cdot T \in \mathrm{Hom}_2(H_1, H_2)$. This finishes the proof.

Lemma 3.2. *If* σ *is a left Haar measure on* $\mathscr{A}$, *then, for any* $T \in \mathrm{Hom}_2(H_1, H_2)$, $\sigma \cdot T$ *is a left* $\mathscr{A}$*-comodule map.*

Proof. It is a direct consequence of Theorem 2.2 and the definition of a Haar measure.

Theorem 3.3. *If* (A, Φ, k, e) *is a Hopf* C^**-algebra with a dense subalgebra* $\mathscr{A}$ *and a Haar measure* σ, *then every irreducible left unitary representation is finite dimensional.*

Proof. Suppose that H is an irreducible left unitary $\mathscr{A}$-comodule with structure map π. Choose an orthonormal basis $\{e_i\}_{i\in I}$ for H. Let $\pi(e_i) = \sum_j a_{ij} \otimes e_j$. Then, for any $i, j, p, q \in I$, we have

$$\begin{aligned}\langle e_i, \sigma \cdot (e_p \otimes e_q) e_j\rangle &= \langle e_i, (I \otimes \sigma)\psi_{H,H}(e_p \otimes e_q) e_j\rangle \\ &= \left\langle e_i, (I \otimes \sigma)\left[\sum_{k,s}(e_k \otimes e_s) a_{pk} k(a_{qs})\right] e_j\right\rangle \\ &= \sum_{k,s} \sigma(a_{pk} k(a_{qs}))\langle e_j, e_k\rangle\langle e_s, e_i\rangle \\ &= \sigma(a_{pj} k(a_{qi})).\end{aligned}$$

Since π is a left unitary representation, there exist $i, j, p, q \in I$ such that $\sigma(a_{pj}k(a_{qi})) \neq 0$. Also, $e_p \otimes e_q \in \mathrm{Hom}_2(H, H)$, so $\sigma \cdot (e_p \otimes e_q) \in \mathrm{Hom}_2(H, H)$. Because $\sigma \cdot (e_p \otimes e_q)$ is a left comodule map and H is irreducible, there exists $0 \neq \lambda \in C$ such that

$$\sigma \cdot (e_p \otimes e_q) = \lambda \cdot I.$$

Thus we get that $I \in \mathrm{Hom}_2(H, H)$. This implies that $\dim H < \infty$. This finishes the proof.

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References

[A] E. Abe, *Hopf algebras*, Cambridge Univ. Press, London and New York, 1980.

[D] V. Drinfeld, *Quantum groups*, Proc. Internat. Congr. Math. (Berkeley), Amer. Math. Soc., Providence, RI, 1986, pp. 798–820.

[N] L. Nachbin, *On the finite dimensionality of every irreducible unitary representation of a compact group*, Proc. Amer. Math. Soc. **12** (1961), 11–12.

[HR] E. Hewitt and K. Ross, *Abstract harmonic analysis*. II, Springer-Verlag, Berlin, 1970.

[L] R. G. Larson, *Characters of Hopf algebras*, J. Algebra **17** (1971), 352–368.

[Q] X. Quan, *Representations of Hopf C^*-algebras*, preprint.

[W] S. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.

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